Marco Colombini – Università di Genova – Italy

Weakly nonlinear analysis of river bed forms

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Colombini, Seminara & Tubino (1987)
Finite-amplitude alternate bars
Journal of Fluid Mechanics 181

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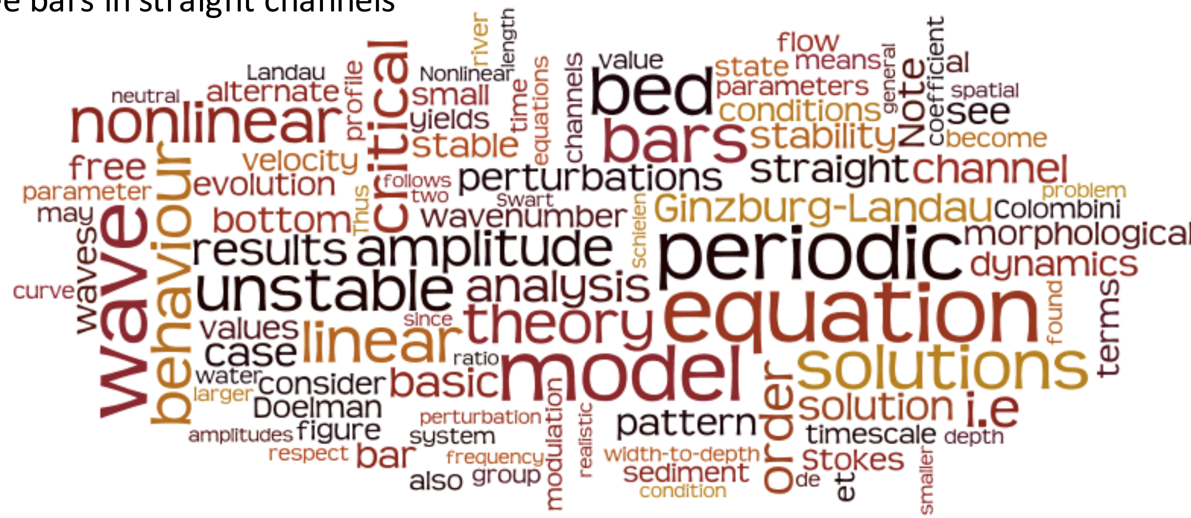
Schielen, Doelman & de Swart (1993)
On the nonlinear dynamics of free bars in straight channels
Journal of Fluid Mechanics 252



Stability and Bifurcations

Stability and Bifurcations

- Bedforms are the result of the instability of the system composed by the flow and by the bed (the container).
- A variety of flow-bed configurations are observed, which correspond to different regimes in the space of physical parameters.
- Bifurcation Theory provides a mathematical tool to determine the regions of the parameter space characteristic of each regime and the corresponding shape of the bedform.
- The stability of a Base State is studied with respect to perturbations of the flow and the bed. Here, the Base State is represented by a steady uniform flow in an infinitely wide channel with active sediment transport.
- Bifurcations is the process whereby a new solution takes over (bifurcates) as the boundary of a stable region in the parameter space is crossed.

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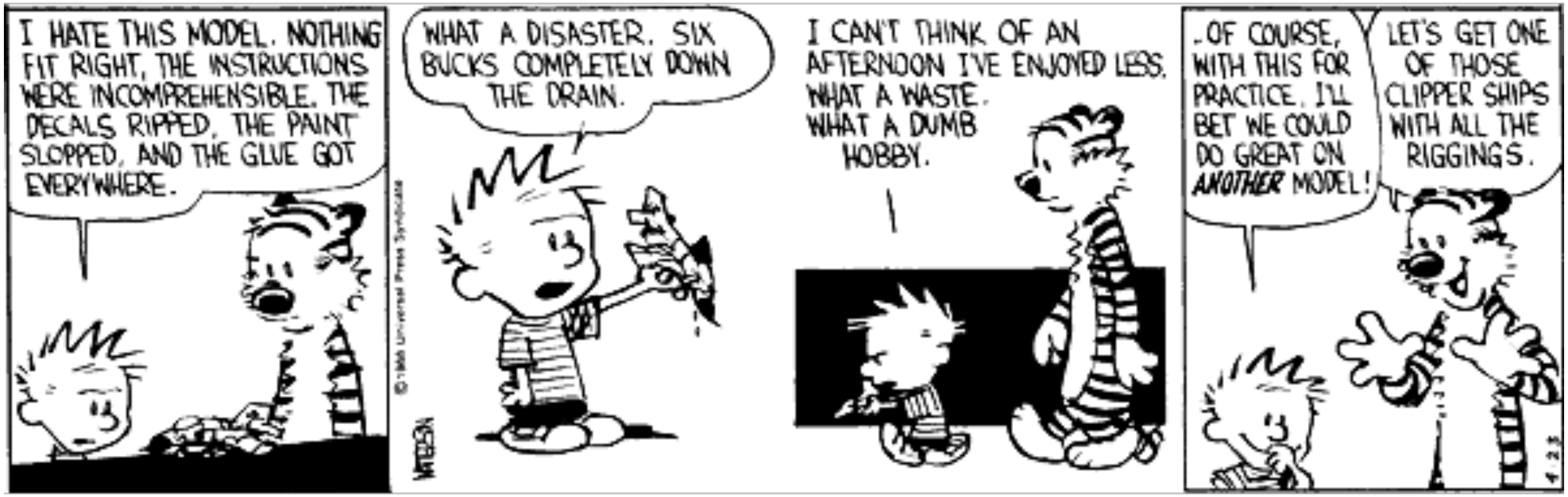
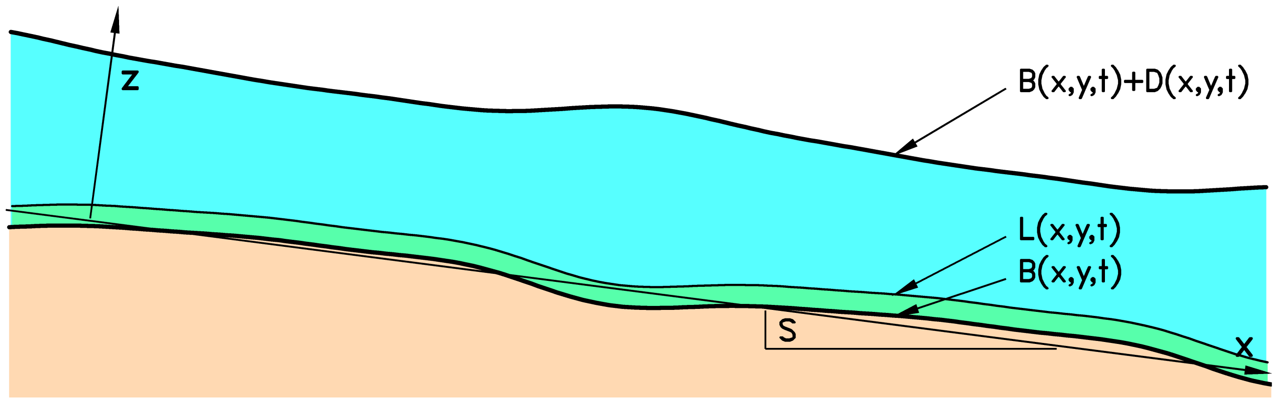
Flow and Sediment Transport Models



Flow and sediment transport model



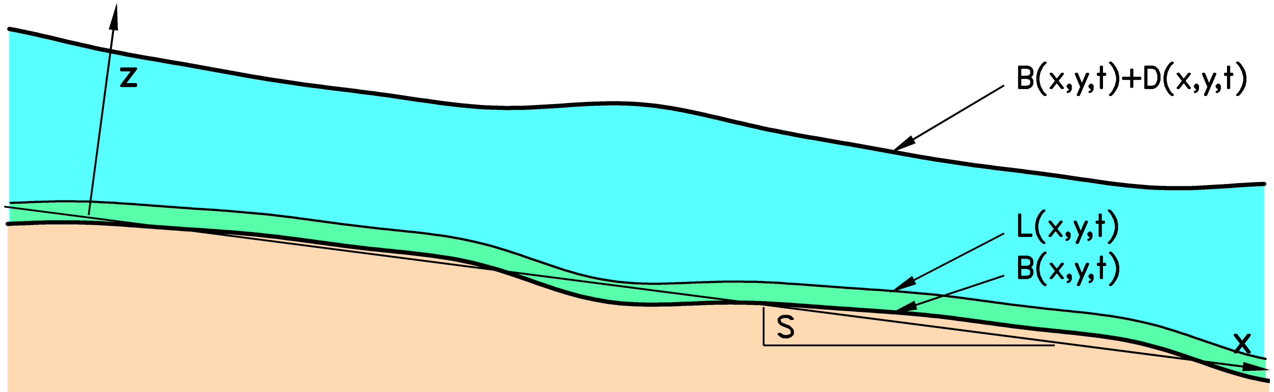
FLOW MODEL

- 2D SHALLOW WATER FLOW MODEL
- EMPIRICAL CLOSURE FOR BED SHEAR STRESS (Chézy conductance coefficient)

SEDIMENT TRANSPORT MODEL

- EQUILIBRIUM MODEL (Exner)
- BEDLOAD ONLY (MPM bedload function)
- CORRECTIONS FOR SEDIMENT WEIGHT (x – Fredsøe, y – Engelund)

Flow and sediment transport model



$$\zeta = \frac{z - B}{D}$$

2D SW EQUATIONS + CONTINUITY (dimensionless with ρ, U_0^*, D_0^*)

$$DU_{,t} + DUU_{,x} + DVU_{,y} = \frac{SD}{Fr^2} - \frac{D}{Fr^2}(B+D)_{,x} - T_x^B + \left[(T_{xx}^R - T_{xx}^D)D\right]_{,x} + \left[(T_{xy}^R - T_{xy}^D)D\right]_{,y}$$

$$DV_{,t} + DUV_{,x} + DVV_{,y} = \qquad - \frac{D}{Fr^2}(B+D)_{,y} - T_y^B + \left[(T_{xy}^R - T_{xy}^D)D\right]_{,x} + \left[(T_{yy}^R - T_{yy}^D)D\right]_{,y}$$

$$D_{,t} + UD_{,x} + VD_{,y} + DU_{,x} + DU_{,y} = 0$$

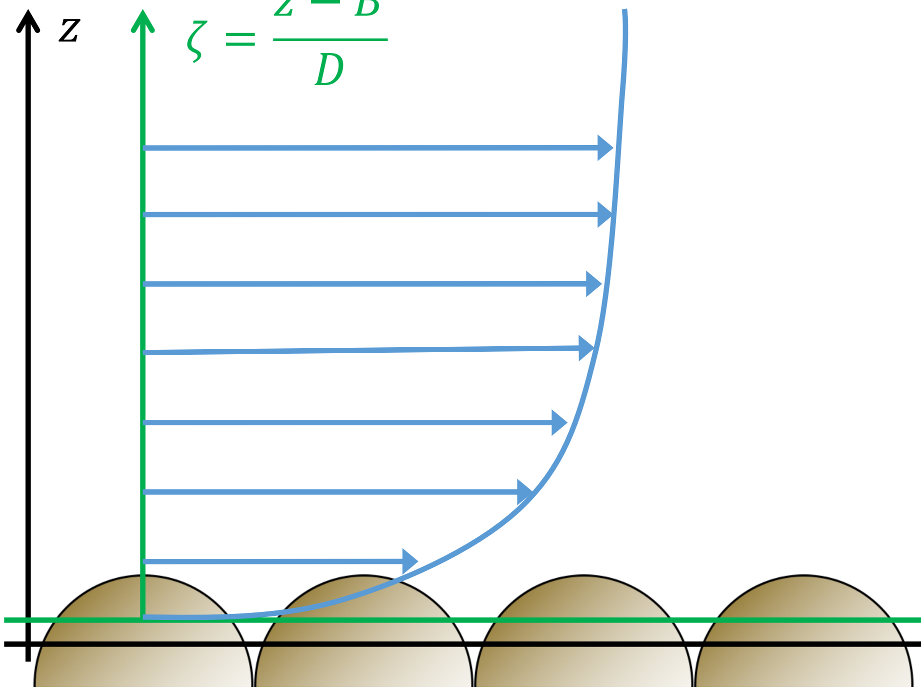
DEPTH-AVERAGING PROCEDURE

$$U = \int_0^1 u\,d\zeta \qquad V = \int_0^1 v\,d\zeta \qquad T_x^B = (\tau_{xz})_{\zeta=0} \qquad T_y^B = (\tau_{yz})_{\zeta=0}$$

$$T_{xx}^R = \int_0^1 \tau_{xx}\,d\zeta \qquad T_{xy}^R = \int_0^1 \tau_{xy}\,d\zeta \qquad T_{yy}^R = \int_0^1 \tau_{yy}\,d\zeta$$

$$T_{xx}^D = \int_0^1 (u-U)^2\,d\zeta \qquad T_{xy}^D = \int_0^1 (u-U)(v-V)\,d\zeta \qquad T_{yy}^D = \int_0^1 (v-V)^2\,d\zeta$$

Flow and sediment transport model



$\zeta = 0 \qquad z = B_0 = \zeta_0$

$z = 0$

$$u = \frac{U}{\kappa C} \ln\left(\frac{\zeta + \zeta_0}{\zeta_0}\right)$$

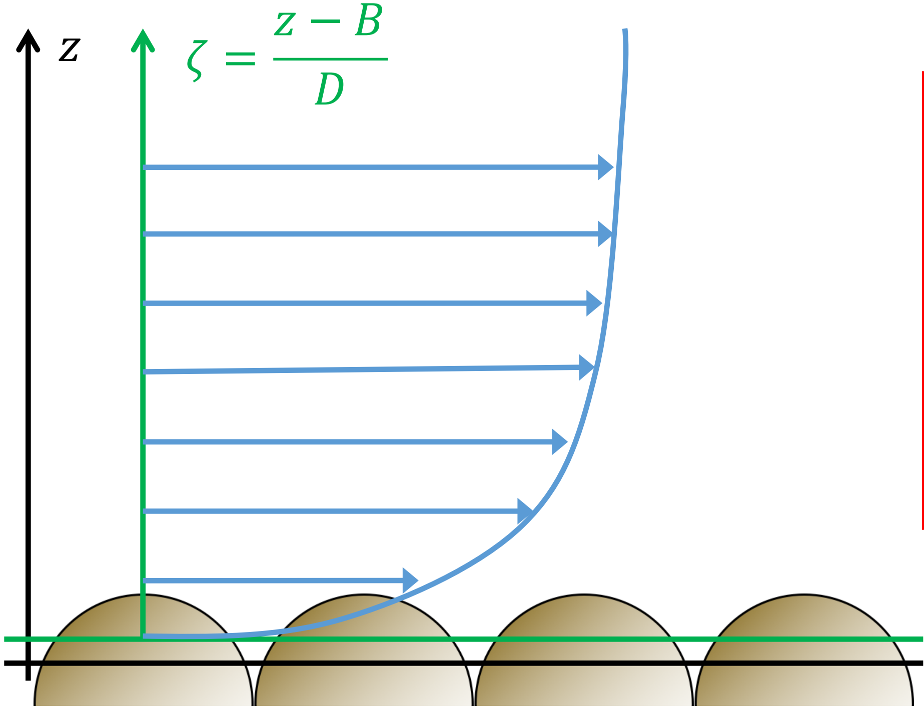
$$\nu_T = \kappa D \frac{U}{C}(\zeta + \zeta_0)(1 - \zeta)$$

$$C = \frac{1}{\kappa} \ln\left(\frac{11.09}{k}\right)$$

$$U = \int_0^1 u d\zeta$$

$$\zeta_0 = \exp(-\kappa C - 1) = \frac{k}{30} = \frac{2.5d}{30} = \frac{d}{12}$$

Flow and sediment transport model



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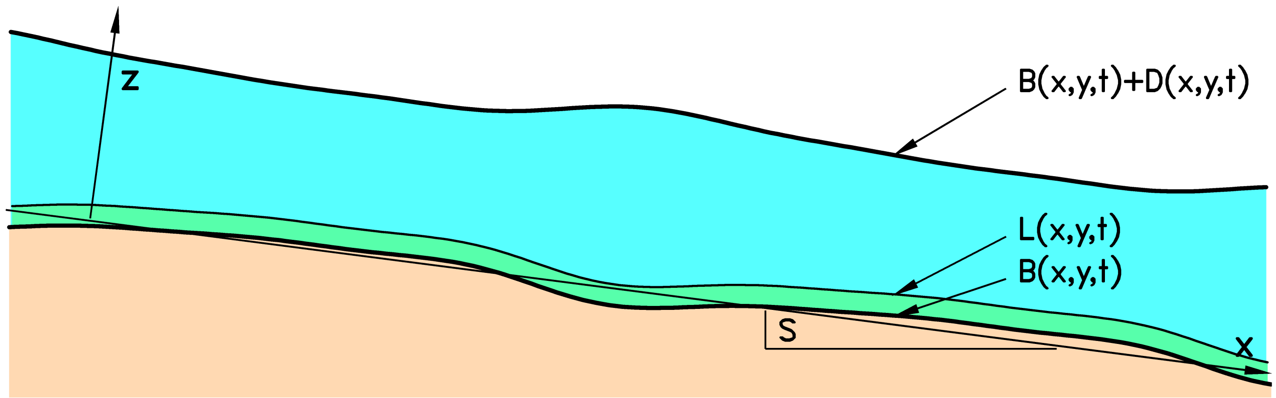
$$T_{xx}^R = \int_0^1 \tau_{xx} d\zeta = \frac{2NDUU_{,x}}{C^2} \qquad N = \frac{1}{6}\left(\kappa C + \frac{1}{6}\right)$$

$$T_{xx}^D = \int_0^1 (u - U)^2 d\zeta = \frac{U^2}{\kappa^2 C^2} \qquad T_x^B = (\tau_{xz})_{\zeta=0} = (\nu_T u_{,z})_{\zeta=0} = \frac{U^2}{C^2}$$



2D SW EQUATIONS + CONTINUITY (dimensionless with ρ, U_0^*, D_0^*)

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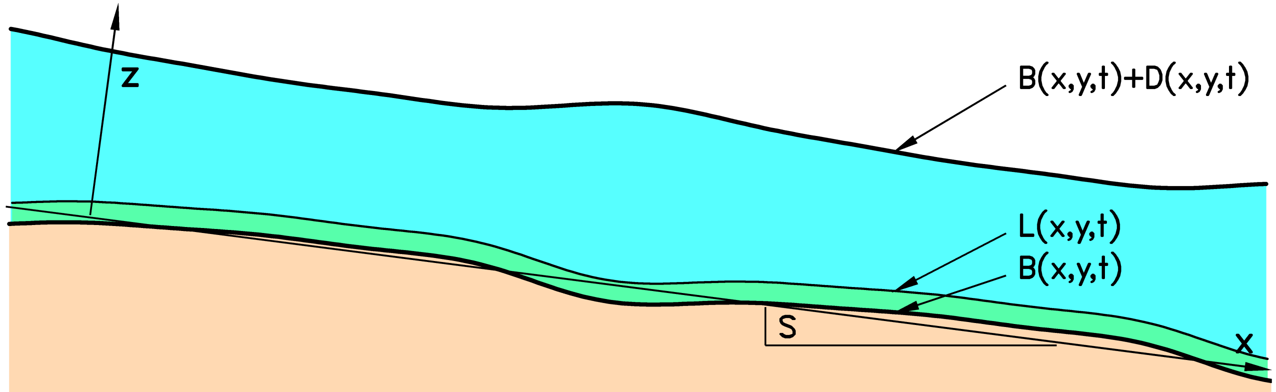
DEPTH-AVERAGING PROCEDURE

$$T_x^B = \frac{\mathcal{U}U}{C^2} \qquad T_y^B = \frac{\mathcal{U}V}{C^2} \qquad \mathcal{U} = \sqrt{U^2 + V^2}$$

$$T_{xx}^R = \frac{2ND\mathcal{U}U_{,x}}{C^2} \qquad T_{xy}^R = \frac{ND\mathcal{U}\left(U_{,y} + V_{,x}\right)}{C^2} \qquad T_{yy}^R = \frac{2ND\mathcal{U}V_{,y}}{C^2}$$

$$T_{xx}^D = \frac{U^2}{\kappa^2 C^2} \qquad T_{xy}^D = \frac{UV}{\kappa^2 C^2} \qquad T_{yy}^D = \frac{V^2}{\kappa^2 C^2}$$

Flow and sediment transport model



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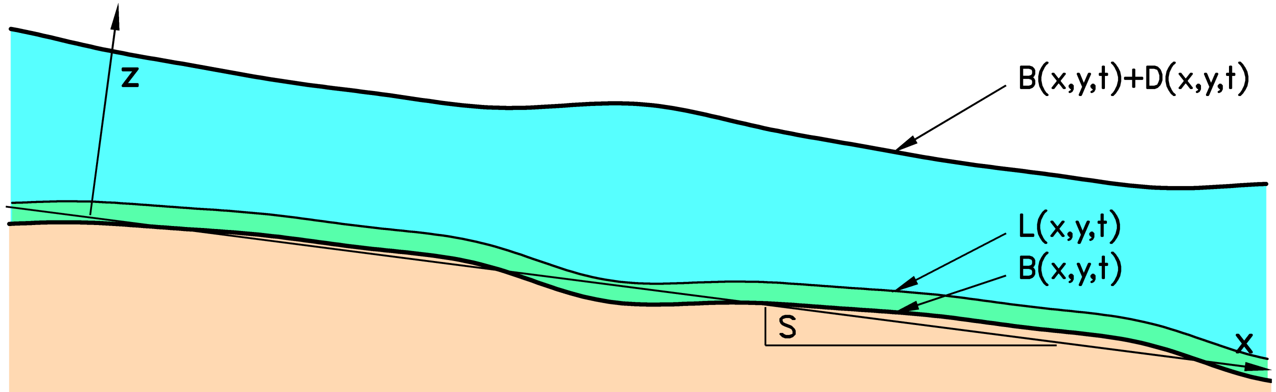

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2D SEDIMENT CONTINUITY (Exner)

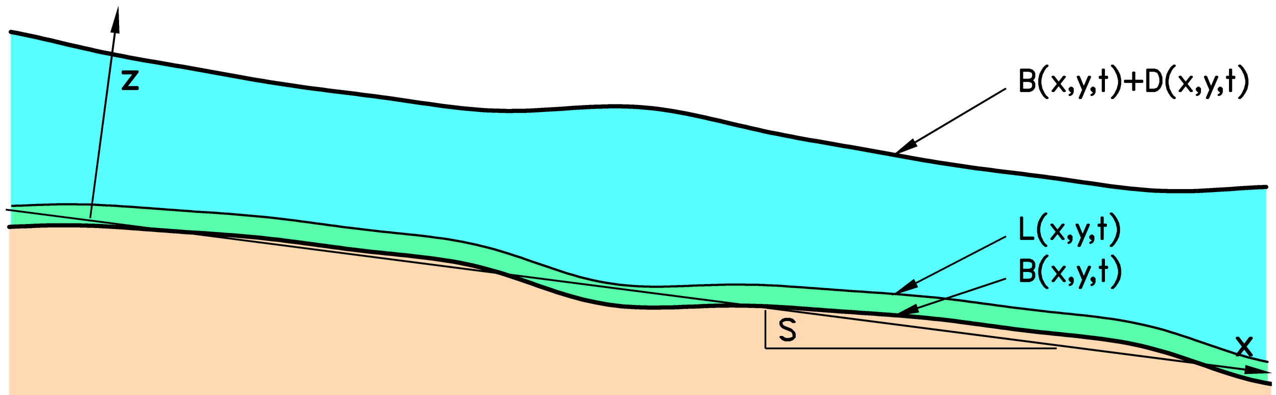
$$B_{,t} + \mathcal{Q}\left(\phi_{x,x} + \phi_{y,y}\right) = 0 \qquad \mathcal{Q} = \frac{\sqrt{(s-1)gd^{*3}}}{U_0^* D_0^* (1-p)} = \frac{\sqrt{(s-1)d^3}}{Fr(1-p)}$$

CORRECTIONS FOR SEDIMENT WEIGHT (x – Fredsøe, y – Engelund)

$$\left(\phi_x, \phi_y\right) = \Phi(\cos\delta\,, \sin\delta) \qquad \sin\delta = \frac{T_y^B}{C^2} - \frac{\mu_y}{\sqrt{\vartheta}} B_{,y} \qquad \Phi = A(\vartheta - \vartheta_c)^{\frac{3}{2}}$$

$$\vartheta - \vartheta_c = \frac{\mathcal{U}^2 Fr^2}{(s-1)dC^2} - \left(\vartheta_{CH} - \mu_y\left(S - B_{,x}\right)\right)$$

Flow and sediment transport model



2D SW EQUATIONS + 2D FLUID & SEDIMENT CONTINUITY EQUATIONS

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$$D_{,t} + UD_{,x} + VD_{,y} + DU_{,x} + DU_{,y} = 0$$

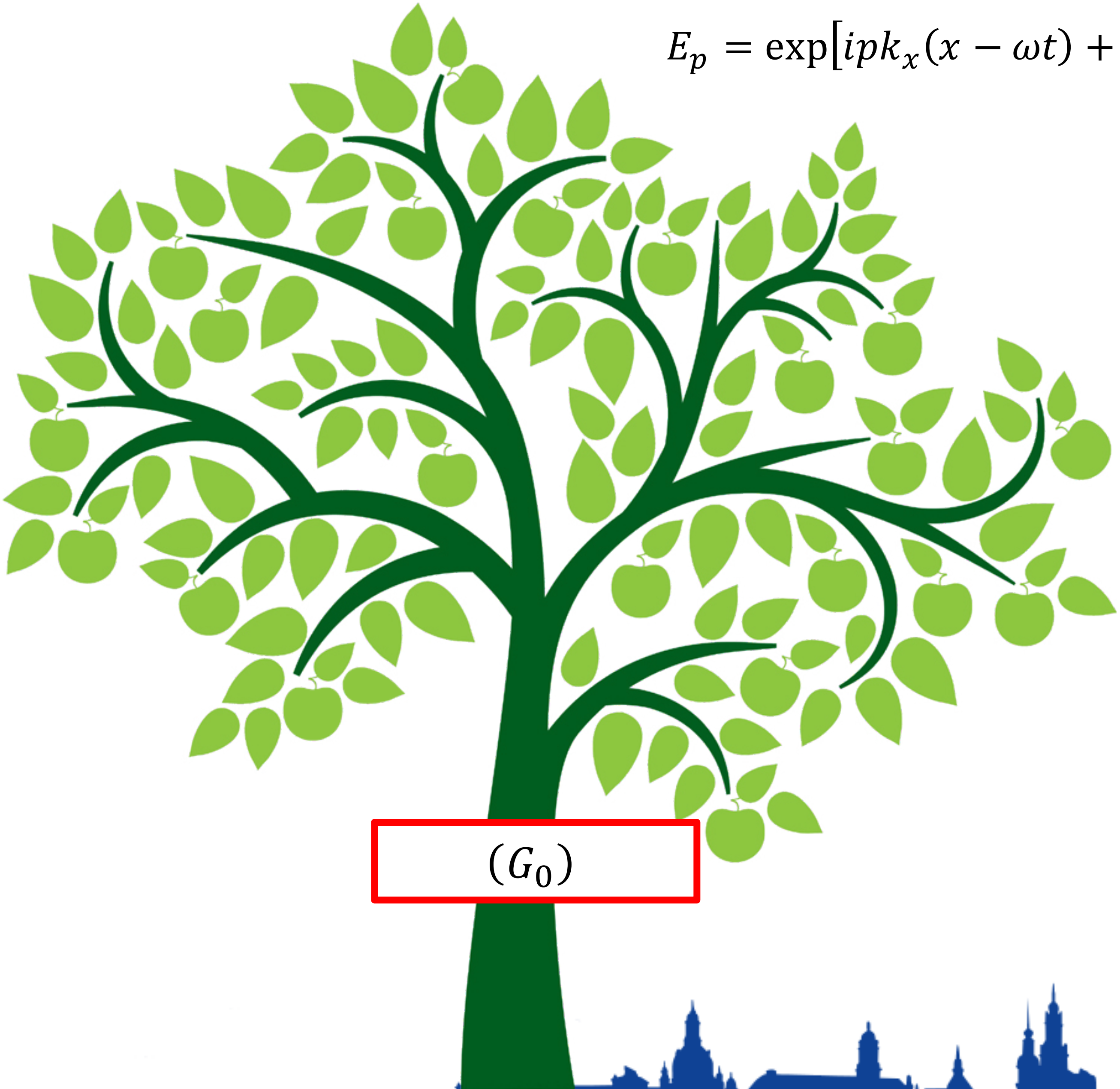
$$B_{,t} + Q(\phi_{x,x} + \phi_{y,y}) = 0$$

+ ALGEBRAIC CLOSURES FOR STRESSES & SEDIMENT TRANSPORT

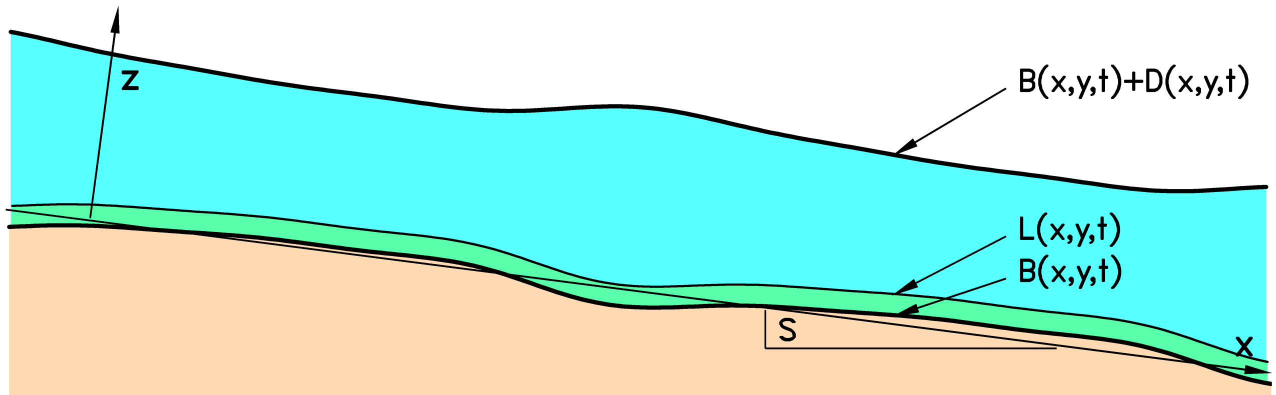
Expansions, Interactions & Cascade processes

Ground floor: base state

$$E_p = \exp\left[ipk_x(x - \omega t) + ipk_y y\right]$$



Ground level: Base state



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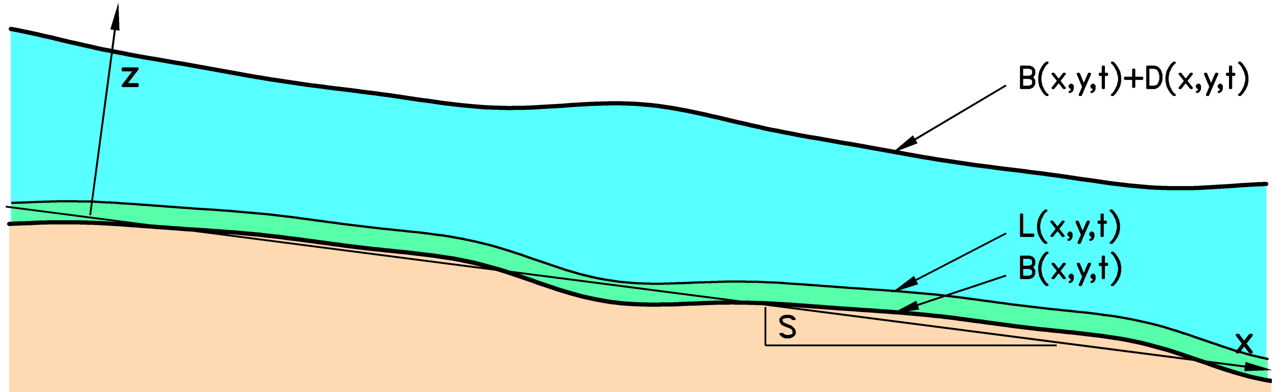

2D SW EQUATIONS + 2D FLUID & SEDIMENT CONTINUITY EQUATIONS

$$= \frac{SD}{Fr^2} \qquad - T_x^B$$

$$=$$

$$= 0$$

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+ ALGEBRAIC CLOSURES FOR STRESSES & SEDIMENT TRANSPORT

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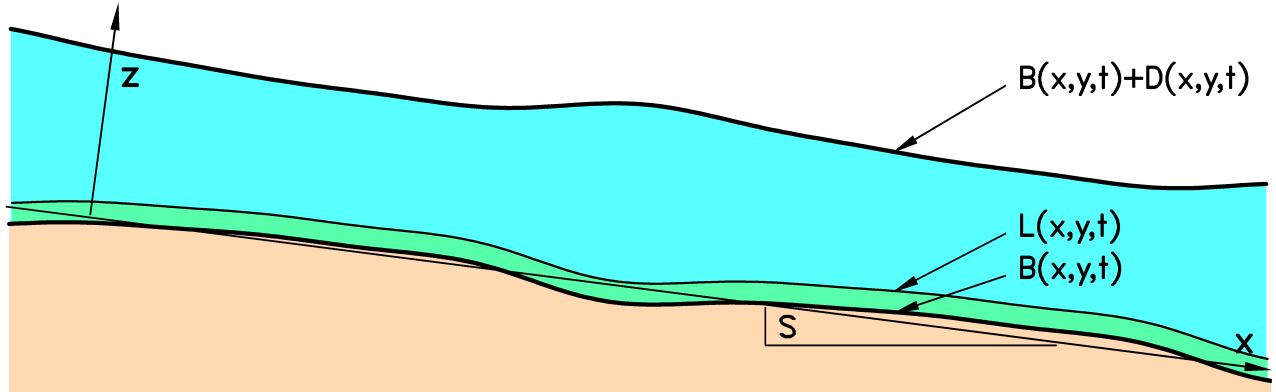


2D SW EQUATIONS + 2D FLUID & SEDIMENT CONTINUITY EQUATIONS

$$= \frac{SD}{Fr^2} \qquad - T_x^B$$

$$=$$

$$= 0$$

$$= 0$$

$$\boxed{(U_0, V_0, D_0, B_0) = (1, 0, 1, \zeta_0)}$$

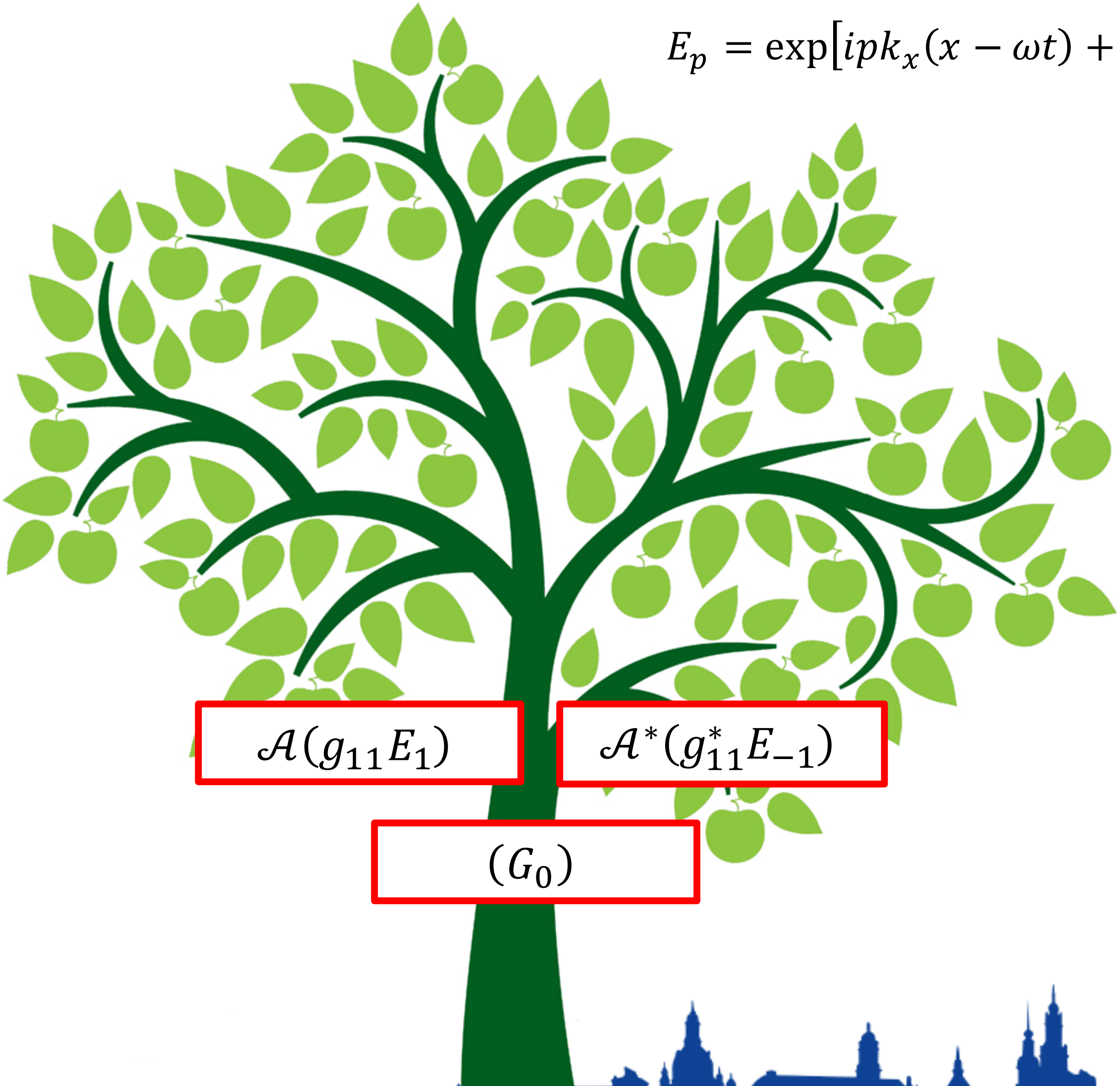
+ ALGEBRAIC CLOSURES FOR STRESSES & SEDIMENT TRANSPORT

$$\boxed{\left(T^B_{x0}, T^B_{y0}, T^R_{xx0}, T^R_{xy0}, T^R_{yy0}, T^D_{xx0}, T^D_{xy0}, T^D_{yy0}\right) = \left(\frac{1}{C^2}, 0, 0, 0, 0, \frac{1}{\kappa^2 C^2}, 0\right)}$$

$$\boxed{(\vartheta_0, \vartheta_{C0}, \Phi_0) = \left(\frac{S}{(s-1)d}, \vartheta_{CH} - \mu_x S, A(\vartheta_0 - \vartheta_{C0})^{\frac{3}{2}}\right)}$$

First floor: linear level

$$E_p = \exp\left[ipk_x(x - \omega t) + ipk_y y\right]$$



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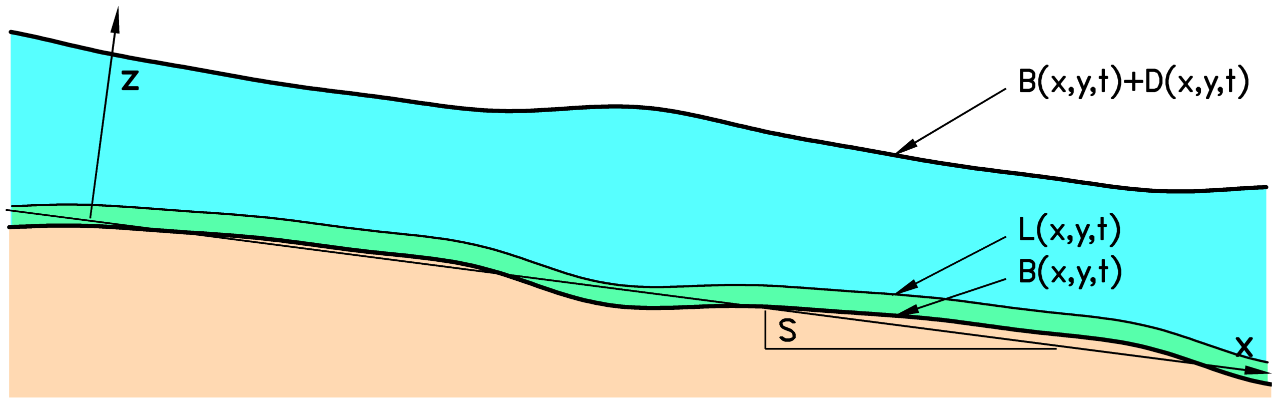

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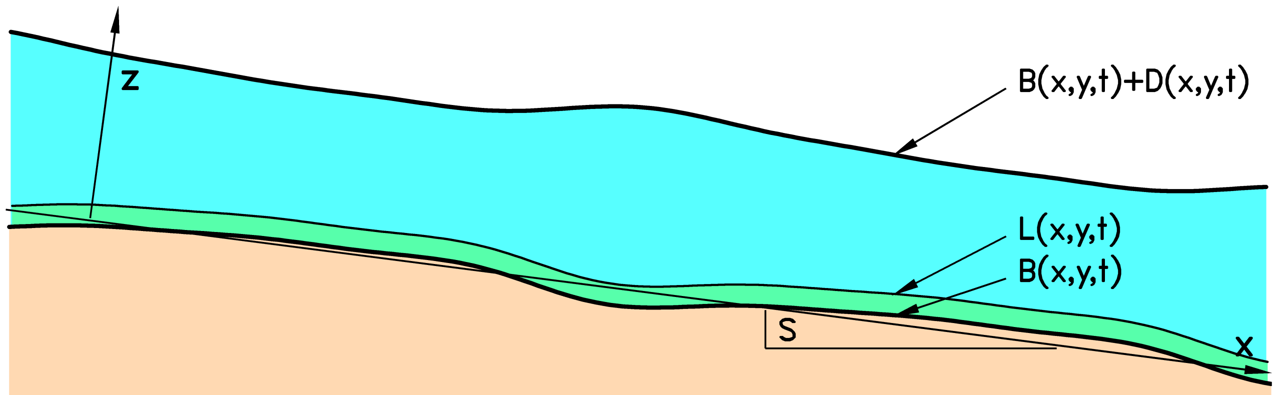

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$$D_{,t} + UD_{,x} + VD_{,y} + DU_{,x} + DU_{,y} = 0$$

$$B_{,t} + Q\left(\phi_{x,x} + \phi_{y,y}\right) = 0$$

$$\boxed{G_{11} = g_{11}\exp\left[ik_x(x - \omega t) + ik_y y\right]}$$

+ ALGEBRAIC CLOSURES FOR STRESSES & SEDIMENT TRANSPORT

$$\boxed{\begin{array}{l} D_0U_{11,t} + D_{11}U_{0,t} + D_0U_0U_{11,x} + D_0U_{11}U_{0,x} + D_{11}U_0U_{0,x} + D_0V_0U_{11,y} + D_0V_{11}U_{0,y} + D_{11}V_0U_{0,y} \\ -ik_x\omega u_{11} \qquad\qquad + ik_x u_{11} \end{array}}$$

$$G_{pq} = g_{pq}\,\exp\left[ipk_x(x-\omega t) + iqk_y y\right]$$

• **LINEAR LEVEL: algebraic eigenvalue problem**

$$(\boldsymbol{A}_{11} - k_x\omega\boldsymbol{I})\cdot\boldsymbol{x}_{11} = 0$$

$$\boldsymbol{A}_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & k_x/Fr^2 \\ a_{21} & a_{22} & k_y/Fr^2 & k_y/Fr^2 \\ k_x & k_y & k_x & 0 \\ \gamma a_{41} & \gamma k_y & 0 & \gamma a_{44} \end{bmatrix} \qquad \boldsymbol{x}_{11} = \begin{bmatrix} u_{11} \\ v_{11} \\ d_{11} \\ b_{11} \end{bmatrix}$$

where:

$$\gamma = Q\Phi_0 \ll 1$$

and a_{ij} depend on the wavenumbers and on base flow quantities

First floor: Linear level

$$G_{pq} = g_{pq} \exp\left[ipk_x(x - \omega t) + iqk_y y\right]$$

- **LINEAR LEVEL: algebraic eigenvalue problem**

$$\det(\boldsymbol{A}_{11} - k_x\omega\boldsymbol{I}) = 0$$

$$\det\begin{bmatrix} a_{11} - k_x\omega & a_{12} & a_{13} & k_x/Fr^2 \\ a_{21} & a_{22} - k_x\omega & k_y/Fr^2 & k_y/Fr^2 \\ k_x & k_y & k_x - k_x\omega & 0 \\ \gamma a_{41} & \gamma k_y & 0 & \gamma a_{44} - k_x\omega \end{bmatrix} = 0$$

which provides a quartic polynomial: **FOUR** eigenvalues, three for the flow, one for the bed;

$$G_{pq} = g_{pq} \exp\left[ipk_x(x - \omega t) + iqk_y y\right]$$

- **LINEAR LEVEL: algebraic eigenvalue problem**

$$\det(\boldsymbol{A}_{11} - k_x \omega \boldsymbol{I}) = 0$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & k_x/Fr^2 \\ a_{21} & a_{22} & k_y/Fr^2 & k_y/Fr^2 \\ k_x & k_y & k_x & 0 \\ \gamma a_{41} & \gamma k_y & 0 & \gamma a_{44} - k_x \omega \end{bmatrix} = 0$$

which provides a quartic polynomial: **FOUR** eigenvalues, three for the flow, one for the bed;

if the quasi-steady hypothesis is made (no time derivatives in flow equations): **ONE** eigenvalue for the bed;

$$k_x \omega = \gamma\left(a_{41}\hat{u}_{11} + k_y \hat{v}_{11} + a_{44}\right)$$

$$G_{pq} = g_{pq} \exp[ipk_x(x-\omega t) + iqk_y y]$$

• LINEAR LEVEL: algebraic eigenvalue problem

$$k_x\omega = \gamma(a_{41}\hat{u}_{11} + k_y\hat{v}_{11} + a_{44})$$

where $\hat{u}_{11}$ and $\hat{v}_{11}$are solutions of the linear nonhomogeneous reduced algebraic system:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & k_y/Fr^2 \\ k_x & k_y & k_x \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_{11} \\ \hat{v}_{11} \\ \hat{d}_{11} \end{bmatrix} = -\begin{bmatrix} k_x/Fr^2 \\ k_y/Fr^2 \\ 0 \end{bmatrix}$$

Obtained from the previous eigensystem by eliminating the last row (Exner equation) and by moving the last column (proportional to b_{11}) to the right hand side.
The solution of this system provides the flow response to a bed perturbation of unitary amplitude.

First floor: Linear level

$$G_{pq} = g_{pq} \exp[ipk_x(x - \omega t) + iqk_y y]$$

- **LINEAR LEVEL: algebraic eigenvalue problem**

$$\Omega = \Omega(k_x, k_y; Fr, C) \qquad \Omega = \Omega(\lambda, \beta; \vartheta, d)$$

DUNE FLAVOUR

$$\lambda = \frac{2\pi W_h^*}{L_x^*} = k_x\,\beta$$

$$\vartheta \cong 0.14\,\frac{Fr^2 e^{\kappa C}}{C^2}$$

BAR FLAVOUR

$$\beta = \frac{W_h^*}{D^*} = \frac{\pi}{2k_y}$$

$$C = \frac{1}{\kappa}\ln\left(\frac{11.09}{2.5d}\right)$$

$$\boxed{\Omega = k_x\omega^i = \gamma\left[A_0 k_x\left(\frac{\hat{T}^{Bi}_{x11}}{T^B_{x0}} - k_x\frac{\mu_x}{\vartheta_0}\right) + k_y\left(\frac{\hat{T}^{Bi}_{y11}}{T^B_{x0}} - k_y\frac{\mu_y}{\sqrt{\vartheta_0}}\right)\right]}$$

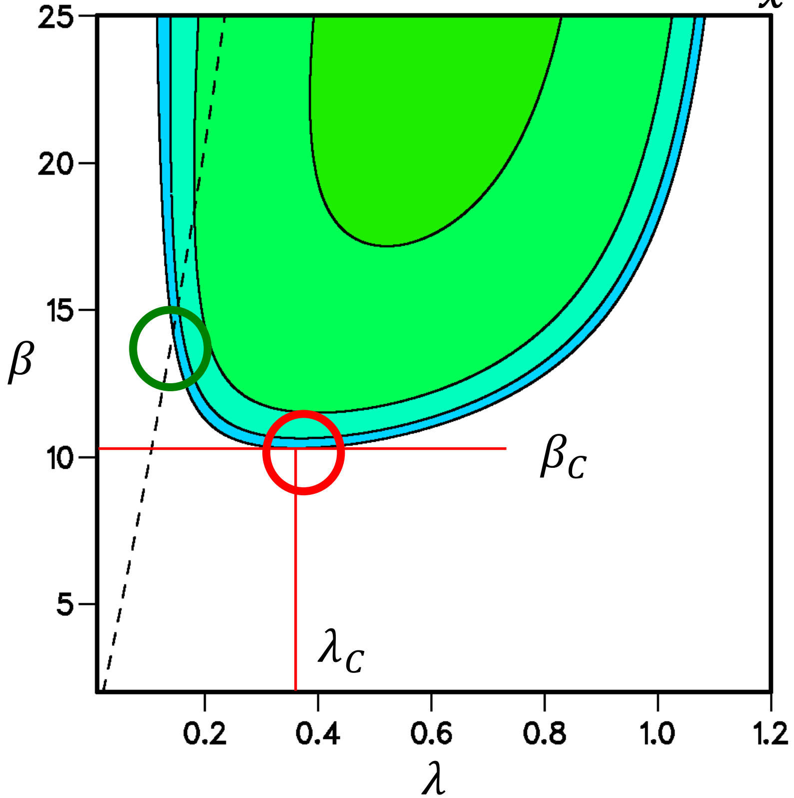
Destabilizing Stabilizing

First floor: Linear level

$$G(x,y,t) = G_0 + \varepsilon g_1 \exp\left[ik_x(x-\omega t) + ik_y y + \Omega t\right] + c.c.$$

- **LINEAR LEVEL: algebraic eigenvalue problem**

$$\Omega = \Omega(\lambda, \beta; \vartheta, C) \quad \lambda = \frac{2\pi W_h^*}{L_x^*} = k_x\,\beta \quad \beta = \frac{W_h^*}{D^*} = \frac{\pi}{2k_y}$$



Resonant conditions - (λ_R, β_R)

Critical conditions - (λ_C, β_C)

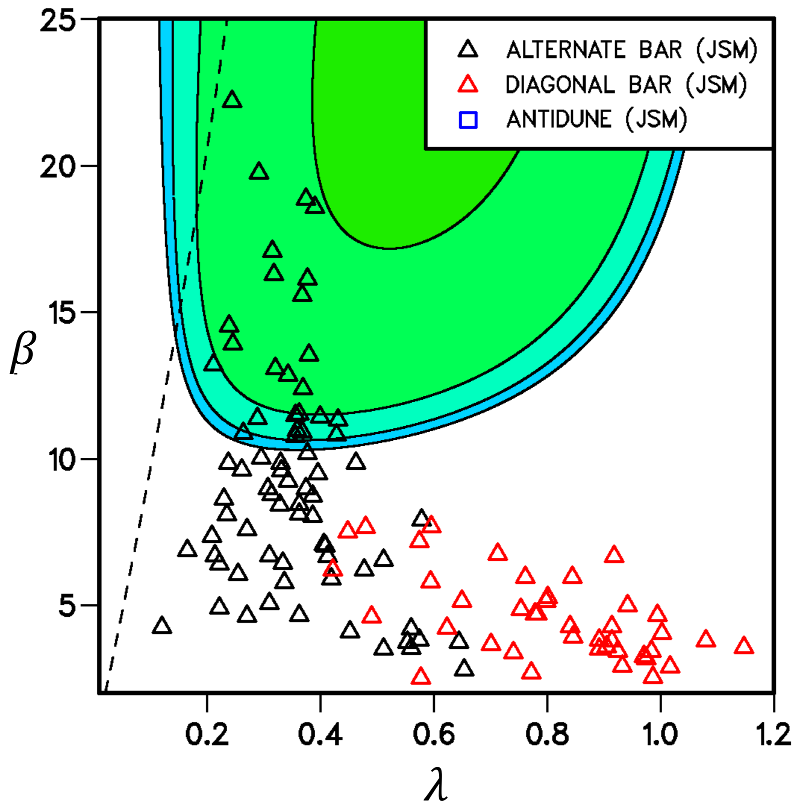
$$\boxed{\beta > \beta_C(\vartheta, C)}$$

Alternate Bars do not form in a narrow channel

First floor: Linear level

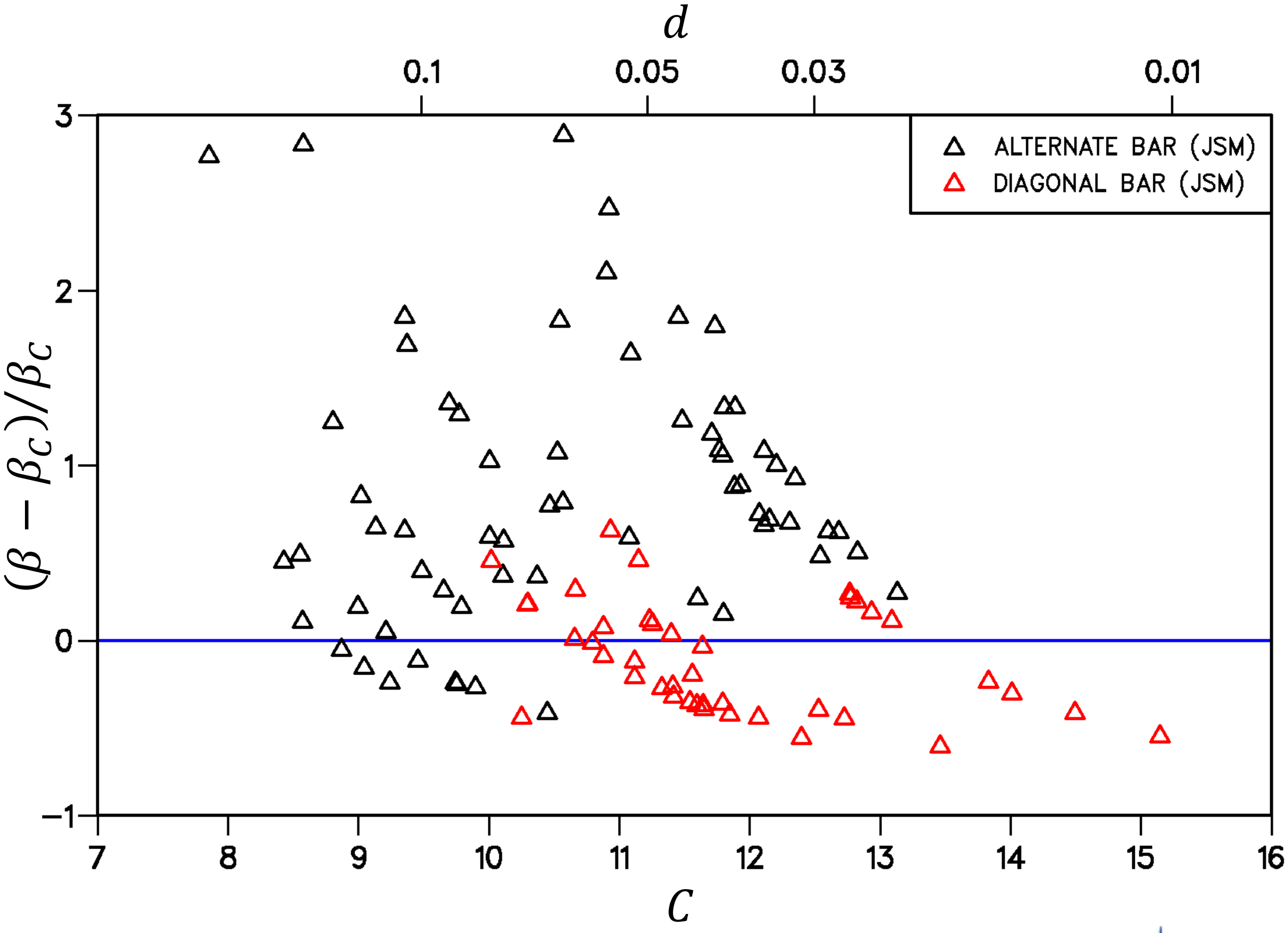
$$\frac{\vartheta}{\vartheta_C} = 3 \quad C = 13$$

$$Fr = 1 \quad d = 0.025$$

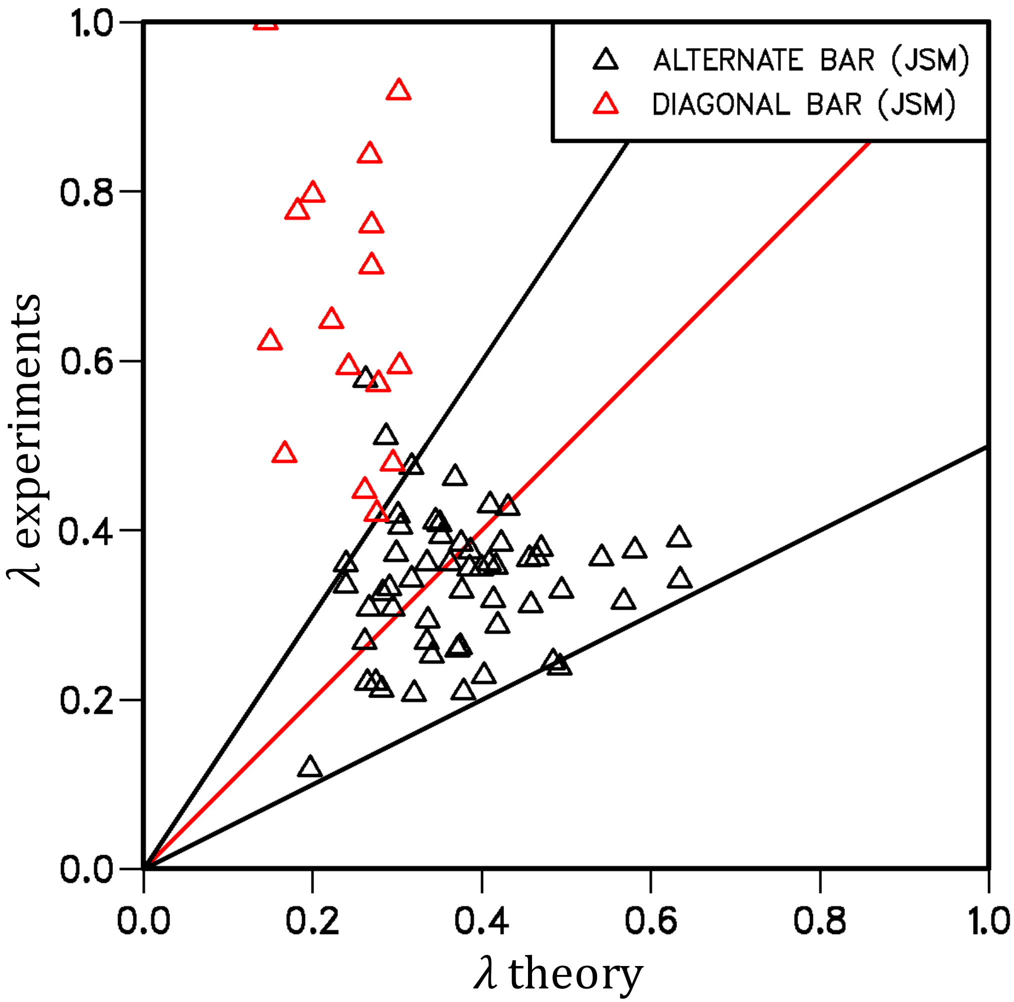


First floor: Linear level

BIFURCATION PARAMETER



WAVELENGTH OF MAXIMUM AMPLIFICATION

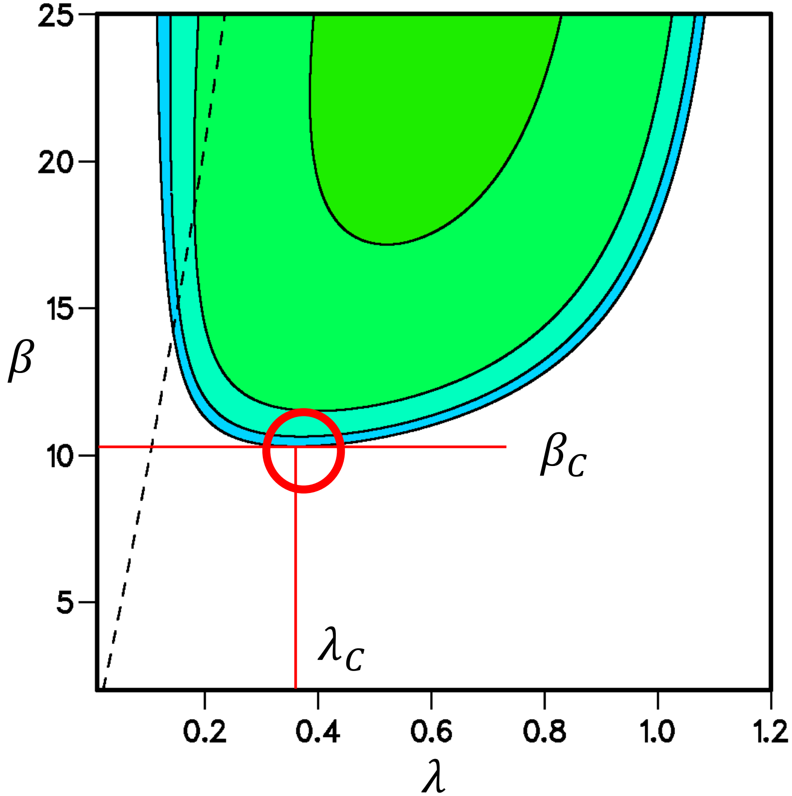


$$G(x,y,t) = G_0 + \varepsilon g_1 \exp\left[ik_x(x-\omega t) + ik_y y + \Omega t\right] + c.c.$$

- **LINEAR LEVEL: algebraic eigenvalue problem**

$$\Omega = \Omega(\lambda, \beta, \vartheta, d)$$

Critical conditions - (k_C, Fr_C)



$$\boxed{\beta > \beta_C(\vartheta, C)}$$

β_C represents the bifurcation point: Alternate Bars do not form below critical conditions

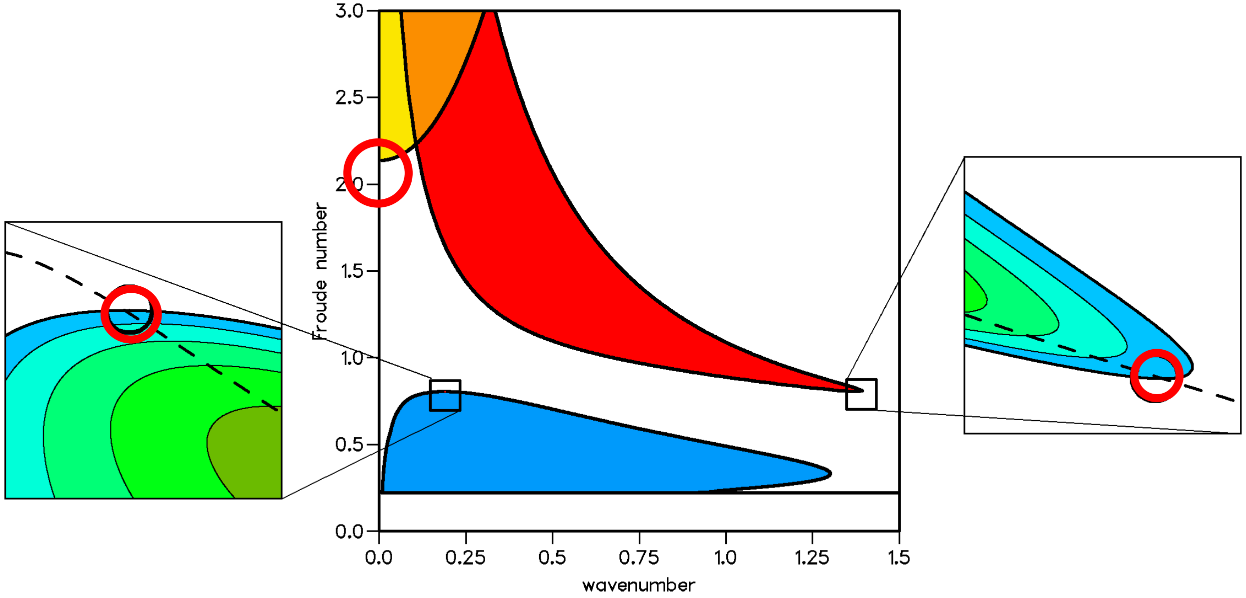
Critical conditions in dune instability

$$G(x,z,t) = G_0(z) + \varepsilon g_1(z)\exp[ik(x-\omega t) + \Omega t] + c.c.$$

- **LINEAR LEVEL: differential eigenvalue problem**

$$\Omega = \Omega(k, Fr, C)$$

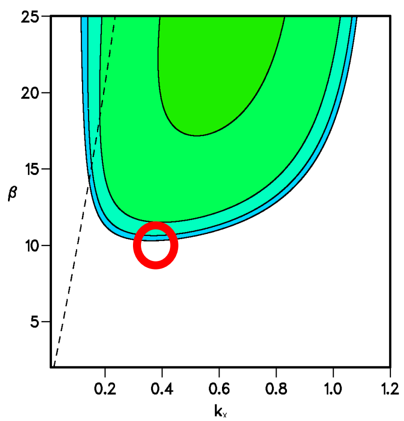
Critical conditions - (k_C, Fr_C)

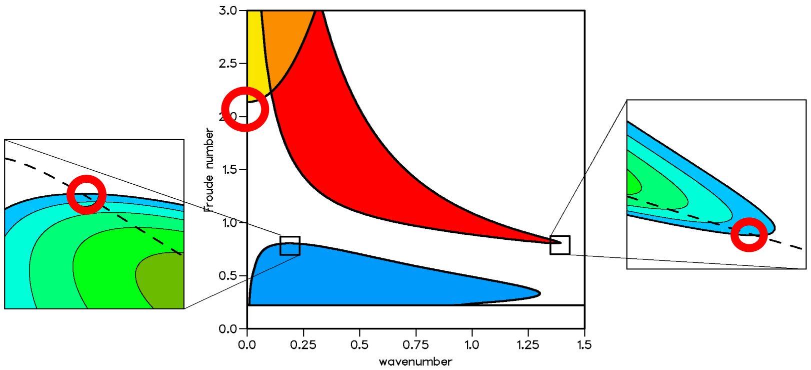


There are several critical values Fr_C that represent different bifurcation points for Dunes, Antidunes and Roll Waves

WNL Analysis

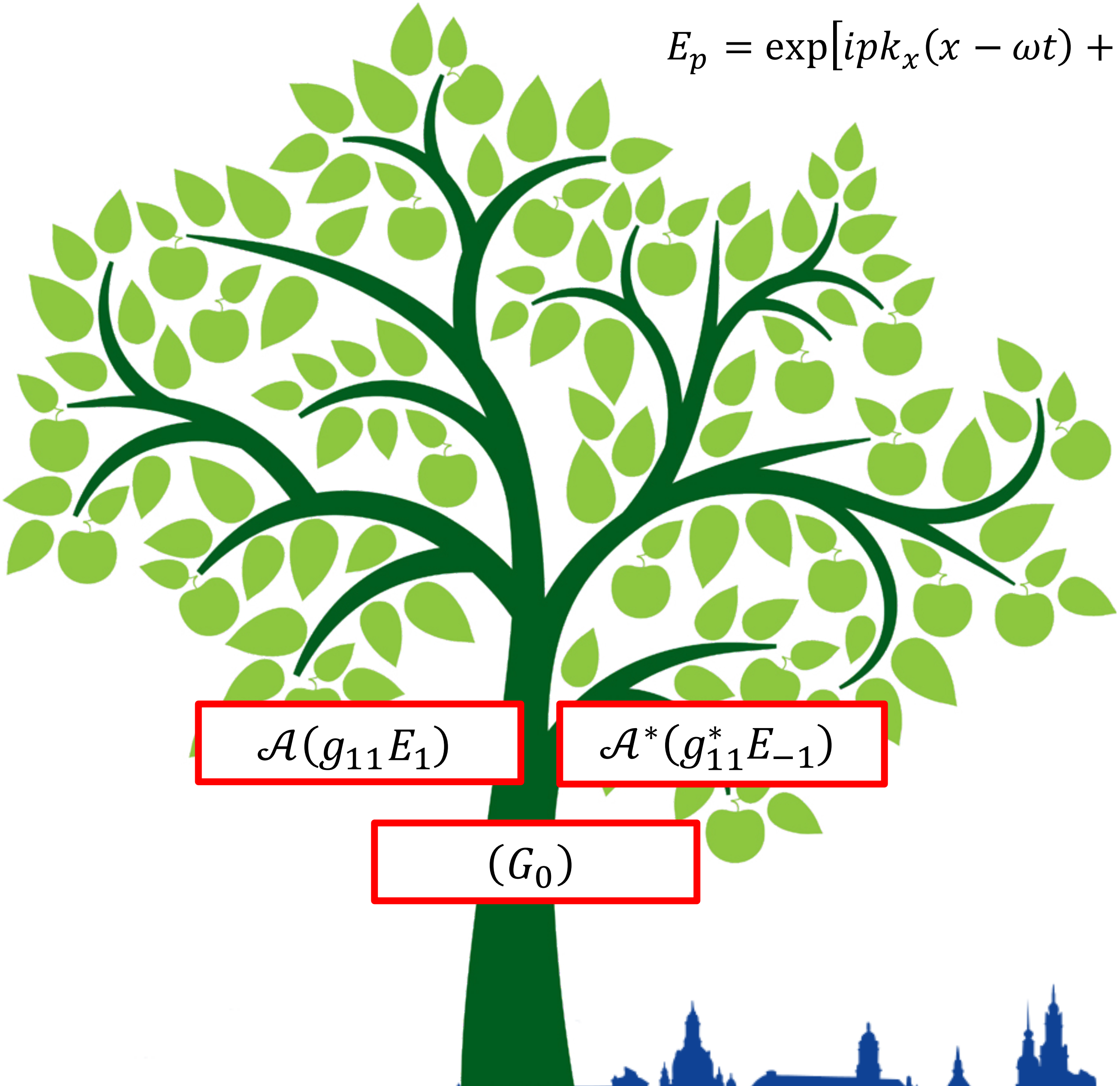
- WNL analysis provides a tool to investigate the neighbourhood of the critical points;
- The perturbation parameter is expanded as $\beta = \beta_C(1+\varepsilon^2)$;
- A *slow* time scale $T = \varepsilon^2 t$ is introduced;
- The amplitude of the perturbation evolves on the slow time scale due to the fact that we slightly exceed the bifurcation point, where the growth rate of the perturbation vanishes;





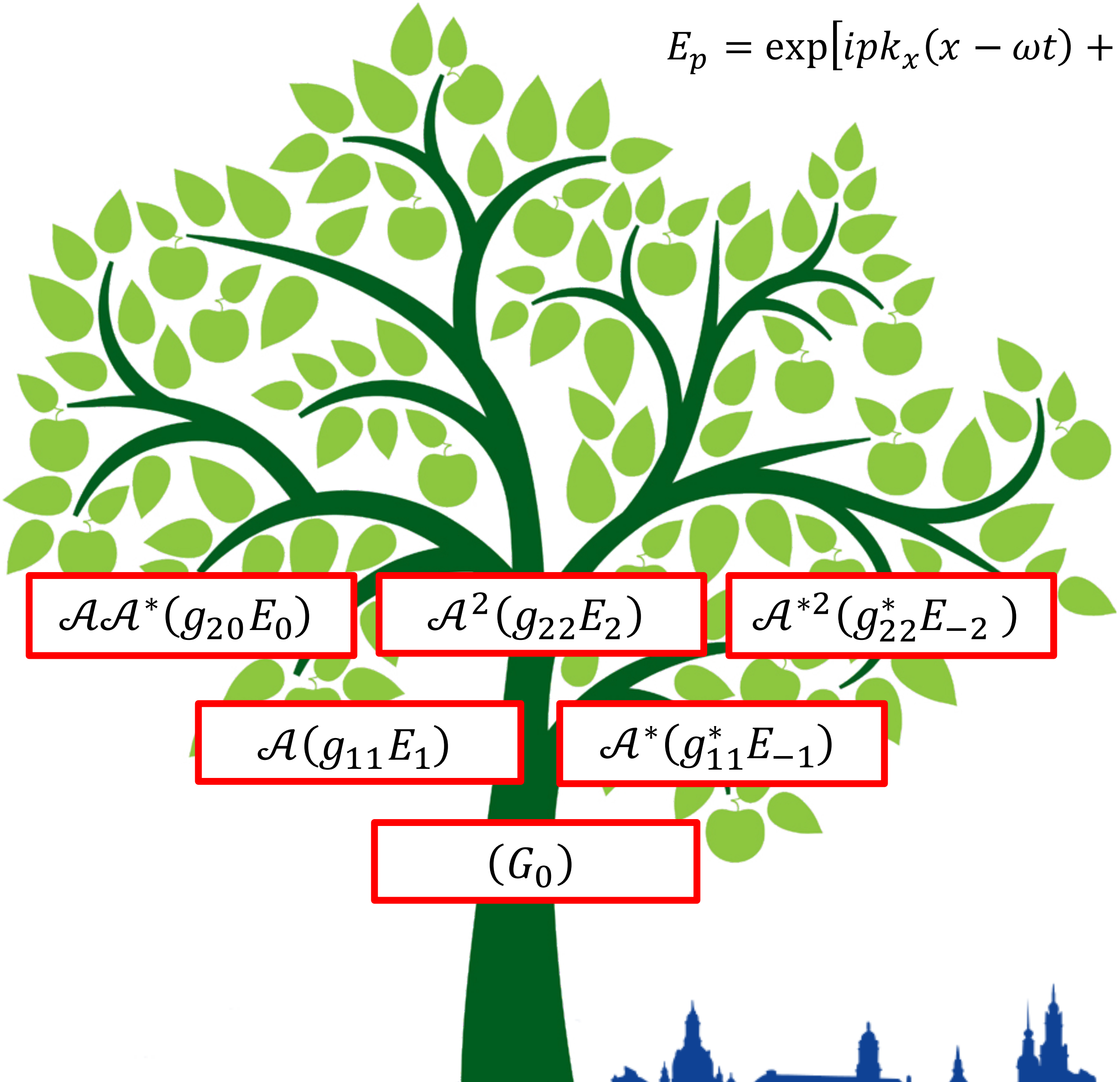
First floor: linear level

$$E_p = \exp[ipk_x(x - \omega t) + ipk_y y]$$



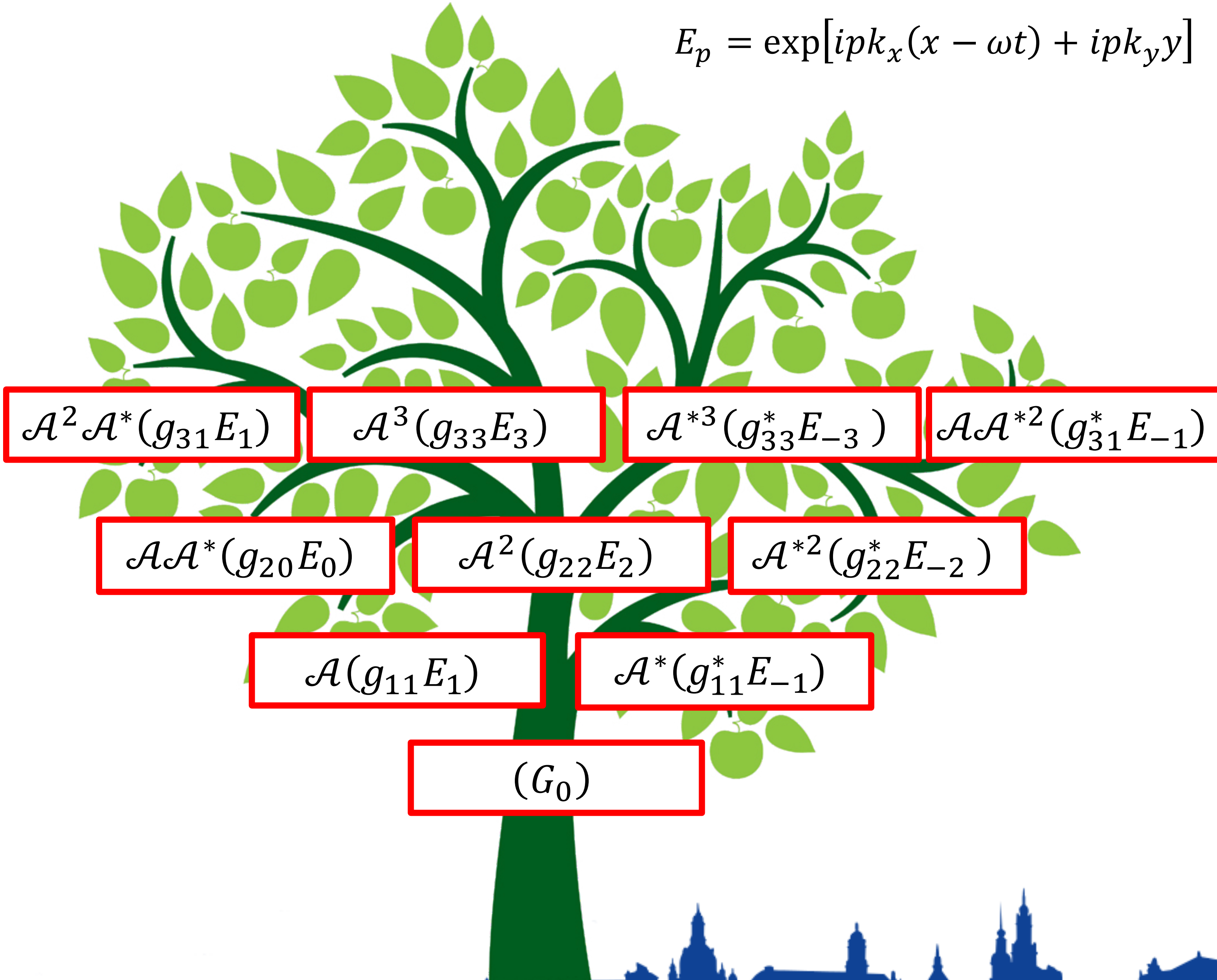
Second floor: quadratic interactions

$$E_p = \exp[ipk_x(x - \omega t) + ipk_y y]$$



Third floor: cubic interactions

$$E_p = \exp\left[ipk_x(x-\omega t) + ipk_y y\right]$$



Third floor: cubic interactions

$$E_p = \exp[ipk_x(x-\omega t) + ipk_y y]$$

Secular terms

$\mathcal{A}^2\mathcal{A}^*(g_{31}E_1)$ | $\mathcal{A}^3(g_{33}E_3)$ | $\mathcal{A}^{*3}(g_{33}^*E_{-3})$ | $\mathcal{A}\mathcal{A}^{*2}(g_{31}^*E_{-1})$

$\mathcal{A}\mathcal{A}^*(g_{20}E_0)$ | $\mathcal{A}^2(g_{22}E_2)$ | $\mathcal{A}^{*2}(g_{22}^*E_{-2})$

$\mathcal{A}(g_{11}E_1)$ | $\mathcal{A}^*(g_{11}^*E_{-1})$

(G_0)

WNL Analysis: Landau-Stuart

Nonlinearity gives rise to interactions between the fundamental and itself which lead to the generation of higher harmonics both in the longitudinal and in the transverse directions. Following this cascade process one finds that the fundamental is reproduced at third order, which leads to the generation of secular terms.

In order to prevent their occurrence the *slow* time dependence of the amplitude of the fundamental must also be forced to produce a contribution at third order.

This provides a solvability condition that yields the Landau-Stuart amplitude equation

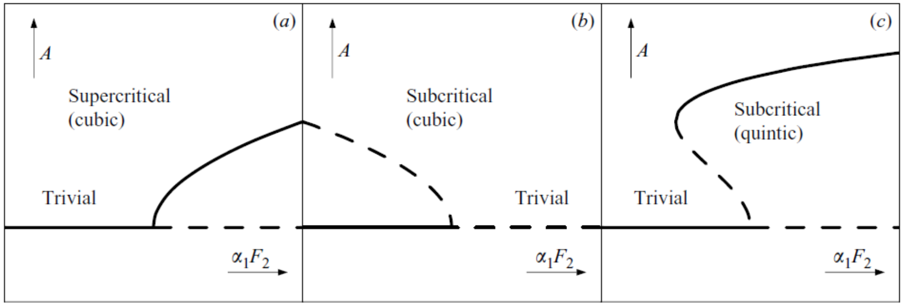
$$\frac{d\mathcal{A}}{dT} = \alpha_1 \mathcal{A} + \alpha_2 \mathcal{A}^2 \mathcal{A}^*$$

WNL Analysis: Landau-Stuart

$$\frac{d\mathcal{A}}{dT} = \alpha_1 \mathcal{A} + \alpha_2 \mathcal{A}^2 \mathcal{A}^*$$

$$\frac{d|\mathcal{A}|^2}{dT} = 2\alpha_1^r |\mathcal{A}|^2 + 2\alpha_2^r |\mathcal{A}|^4$$

α_1^r is always positive (related to the fact that the growth rate increases as $\beta > \beta_C$;
If α_2^r is negative the bifurcation is supercritical;
If α_2^r is positive the bifurcation is subcritical;

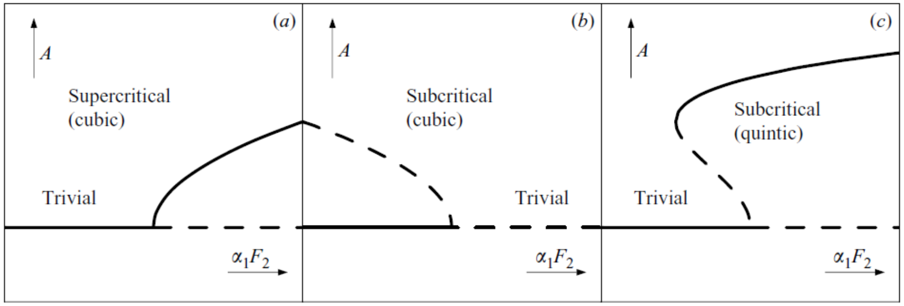


WNL Analysis: Landau-Stuart

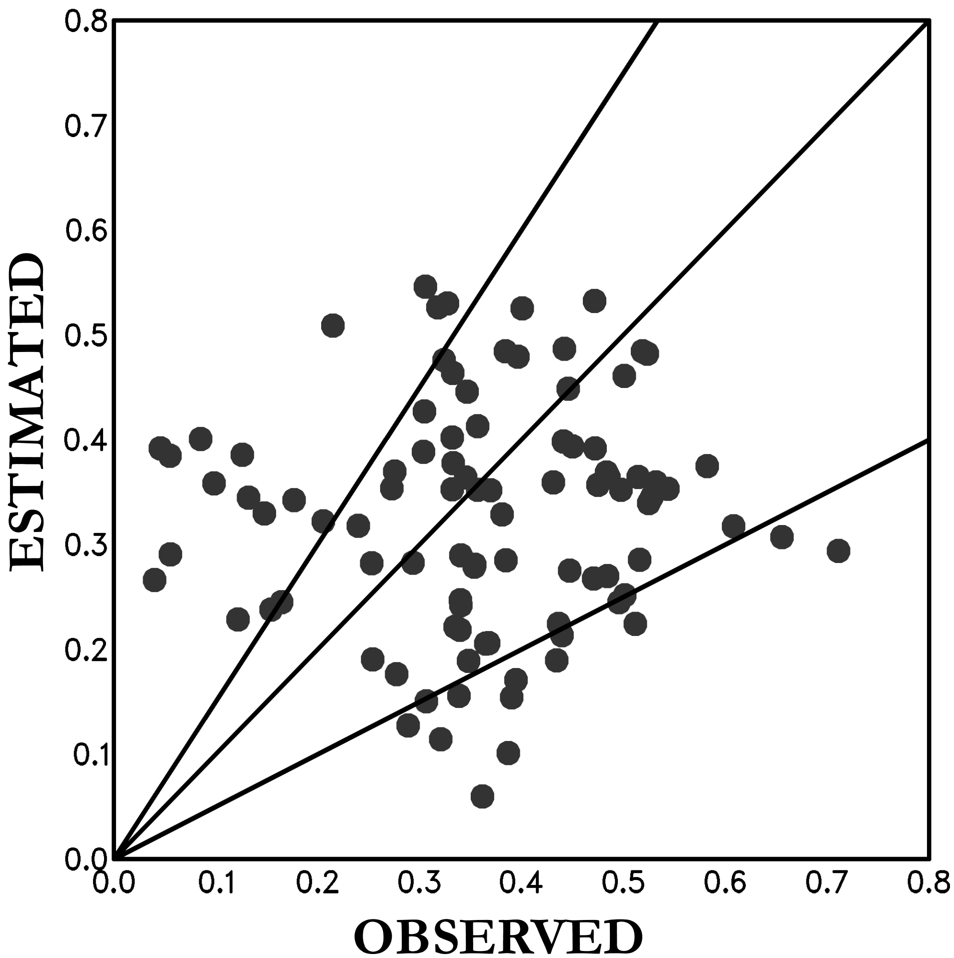
$$\frac{d|\mathcal{A}|^2}{dT} = 2\alpha_1^r|\mathcal{A}|^2 + 2\alpha_2^r|\mathcal{A}|^4 = 0$$

$$|\mathcal{A}_e|^2 = -\frac{\alpha_1^r}{\alpha_2^r}$$

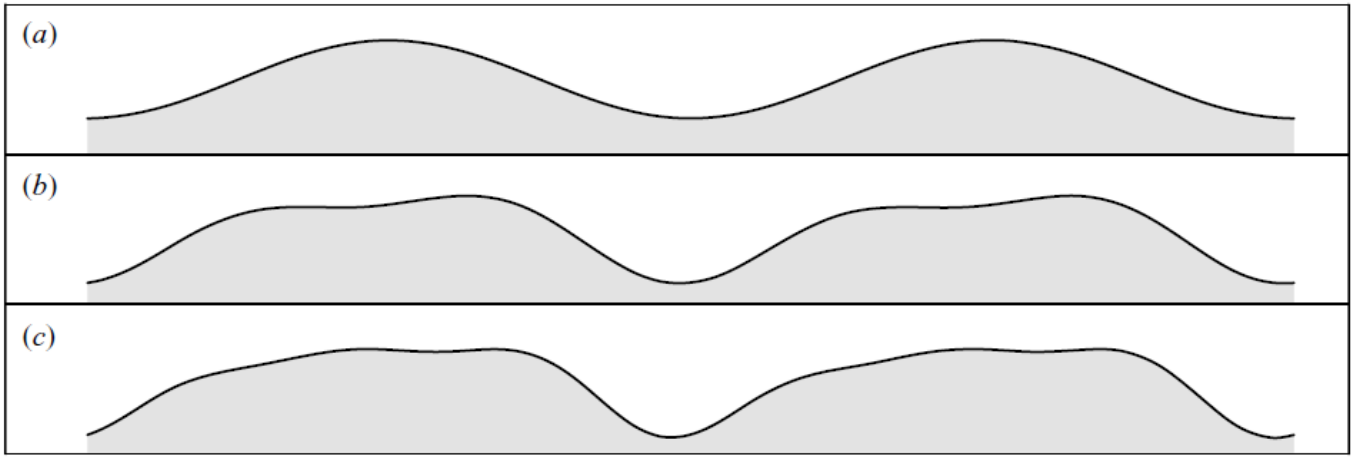
α_1^r is always positive (related to the fact that the growth rate increases as $\beta > \beta_C$;
If α_2^r is negative the bifurcation is supercritical;
If α_2^r is positive the bifurcation is subcritical;
An equilibrium amplitude is reached only if the bifurcation is supercritical

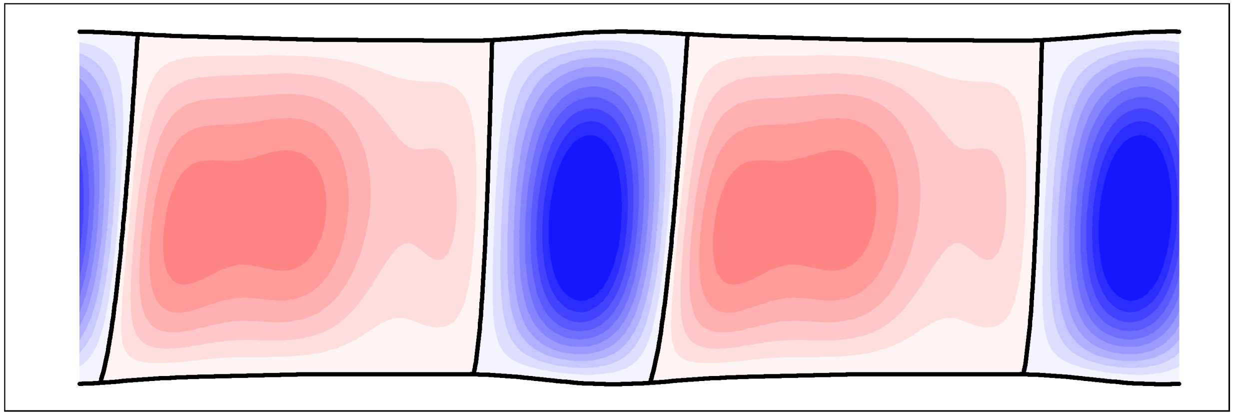


DUNES: EQUILIBRIUM AMPLITUDE

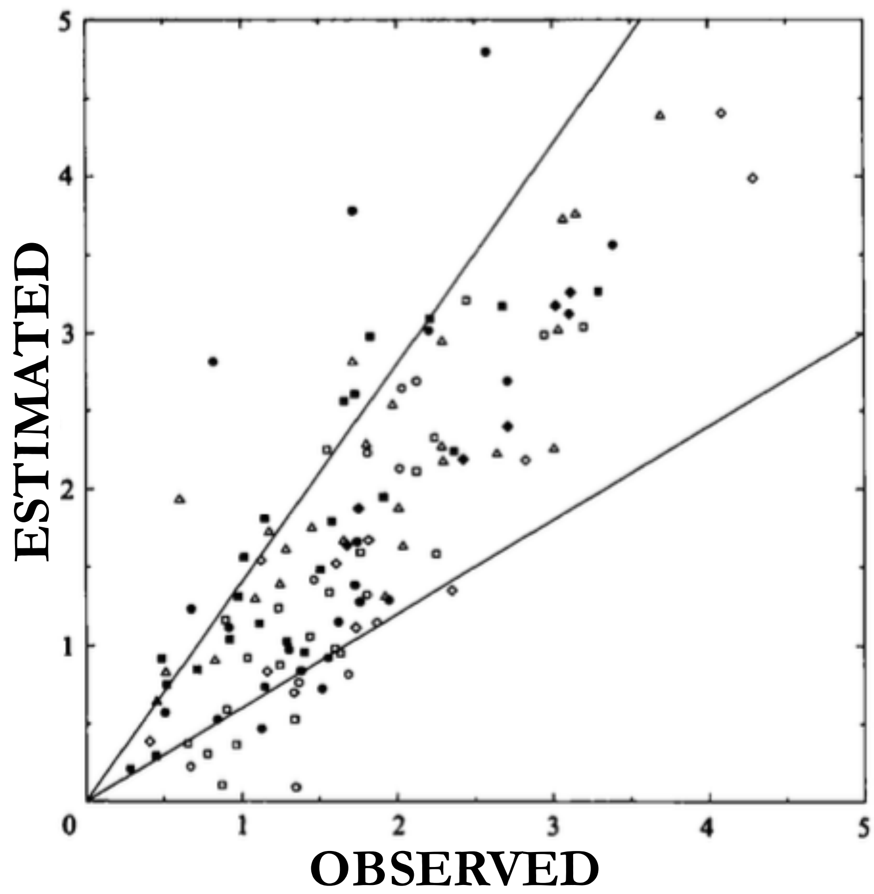


DUNES: EQUILIBRIUM SOLUTION

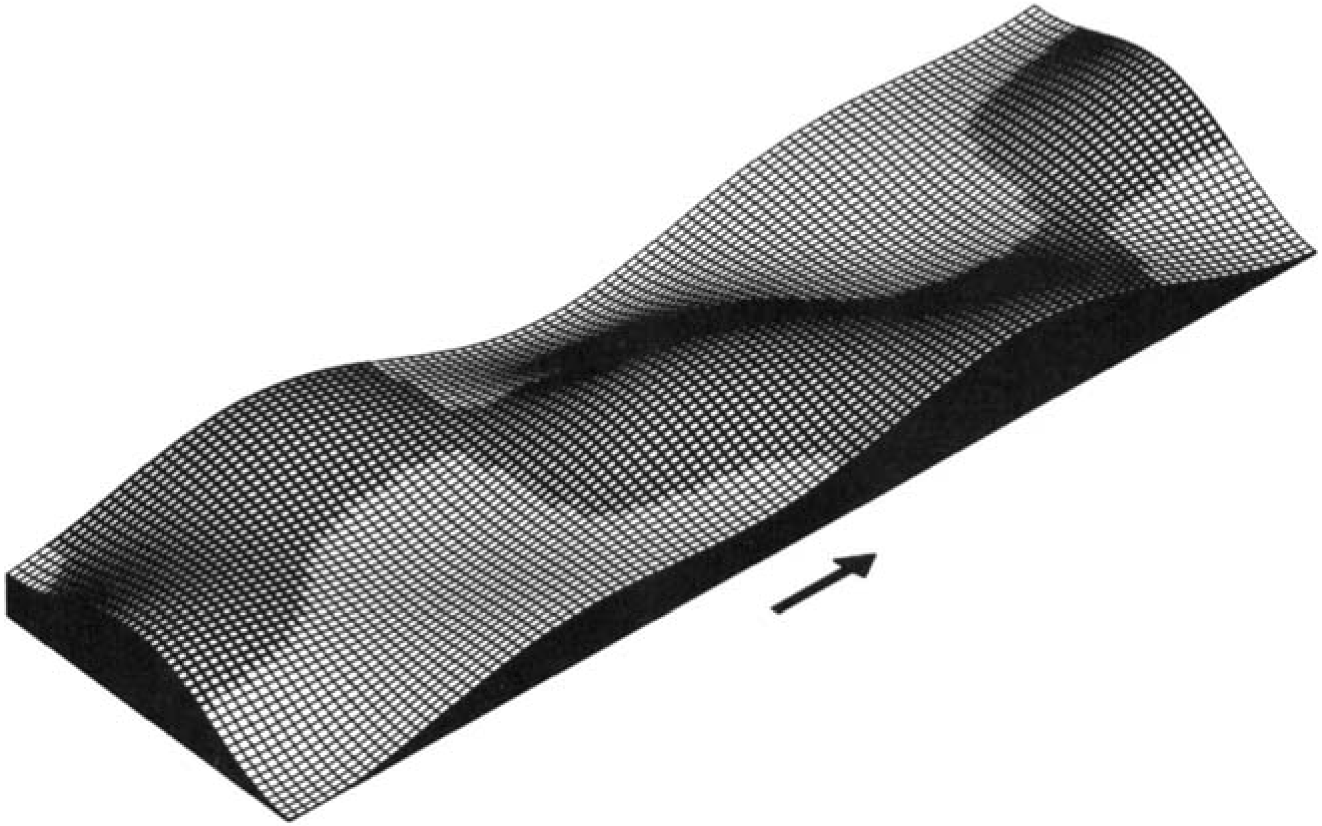




BARS: EQUILIBRIUM AMPLITUDE



BARS: EQUILIBRIUM SOLUTION



Invitation to river stability phenomena
Gary Parker

Summer school on stability of river and coastal forms – Perugia, Italy September 3-14, 1990

Leave nature to its devices and it composes its own poetry. Simple rules interact to give rise to a hierarchy of structures, each nevertheless possessing, manifest or hidden, an internal symmetry that reveals itself to that part of the human mind capable of recognizing beauty. To be a scientist is to listen to the song of nature. To experience the instant when a mist of dissonance lifts to reveal the harmony of a heretofore unexplained phenomenon is to watch the sun rise on a clear day from the top of a mountain.