Many-Body Quantum Interference on Hypercubes

C. Dittel, R. Keil and G. Weihs, *Quantum Sci. Technol.*, **2**, 015003 (2017)

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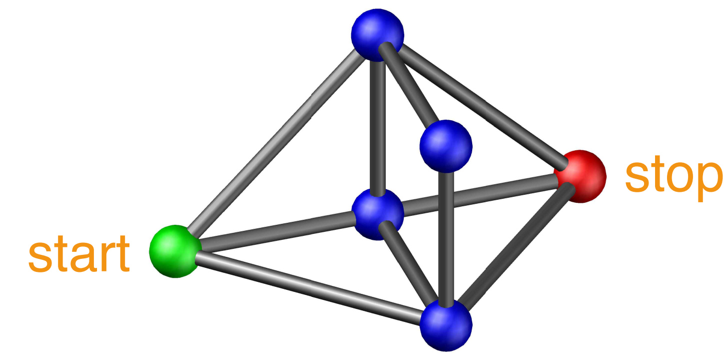
Introduction

Single particle quantum transport:

coherent evolution of a quantum system on a graph

- continuous in time [1]
- (in discrete time steps [2])



feasible for

- algorithmic tools (e.g. search algorithms)
- excitation transfer

Quantum Computation

Quantum Communication

spin chains [3]

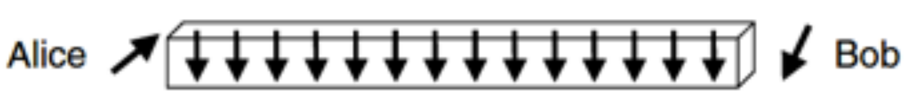
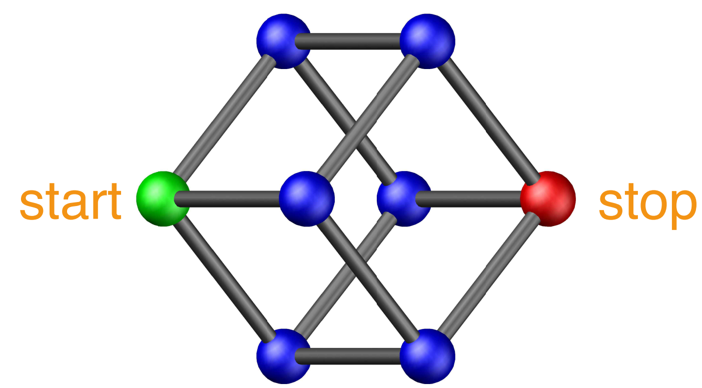


Figure reference: [3]

hypercube graphs [4-8]

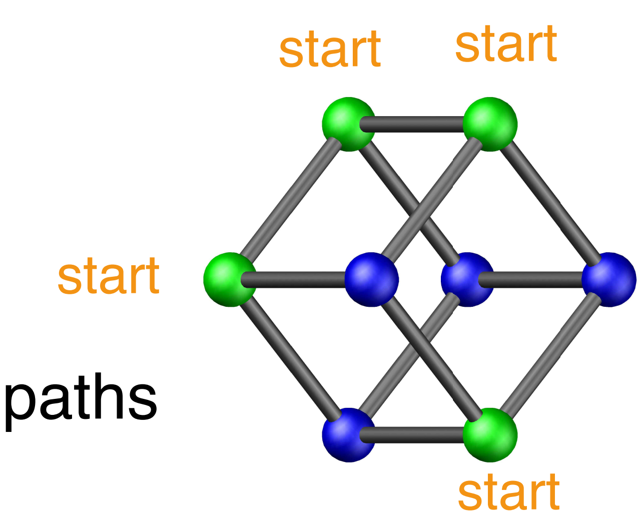


- exp. speed-up of hitting times
- robust under imperfections

[1] Farhi and Gutmann, *Phys. Rev. A,* **58**, 915-928 (1998)
[2] Aharonov et al., *Phys. Rev. A,* **48**, 1687-1690 (1993)
[3] Bose, *Phys. Rev. Lett.,* **91**, 207901 (2003)
[4] Kempe, *Probab. Theory Relat. Fields,* **133**, 215-235 (2005)
[5] Alagic and Russell, *Phys. Rev. A,* **72**, 062304 (2005)
[6] Krovi and Brun, *Phys. Rev. A,* **73**, 032341 (2006)
[7] Makmal et al., *Phys. Rev. A,* **90**, 022314 (2014)
[8] Makmal et al., *Phys. Rev. A,* **93**, 022322 (2016)

Introduction - Many-Particle Quantum Transport

Quantum transport of multiple indistinguishable particles:



increasing particle number

- interference among a growing number of many-particle paths
- gives rise to intricate evolution scenarios [9]
- exp. demonstrated in two dimensions for

  two [10,11]

  and three [12] particles

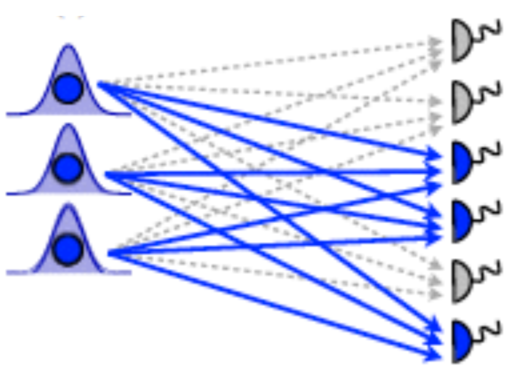


Figure reference: [9]

[9] Tichy, *J. Phys. B: At. Mol. Opt. Phys.* **47**, 103001 (2014)

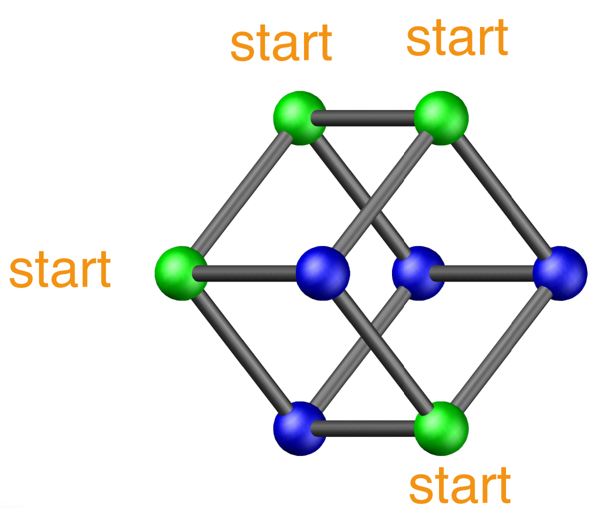
[10] Poulios et al., *Phys. Rev. Lett.,* **112**, 143604 (2014)

[11] Crespi et al., *Phys. Rev. Lett.,* **114**, 090201 (2015)

[12] Spagnolo et al., *Nat. Commun.,* **4**, 1606 (2013)

Introduction - Symmetries

Symmetries simplify the complexity of such evolution scenarios



few such symmetries have been investigated

- discrete Fourier transform [13,14,15]
- Sylvester matrices [16]
- J_x Unitary ⇨ **Talk by Robert Keil on Wednesday**

Here: We investigate the symmetries on hypercube graphs

[13] Tichy et al., *New. J. Phys.*, **14**, 093015 (2012)

[14] Crespi et al., *Nat. Commun.*, **7**, 10469 (2016)

[15] Carolan et al., *Science*, **349**, 6249 (2015)

[17] Crespi, *Phys. Rev. A*, **91**, 013811 (2015)

Outline

Quantum Interference on Hypercubes

- Introduction
- Particle Interference
- Symmetry Suppression Law
- Generalization to Arbitrary Subgraphs
- Summary and Outlook

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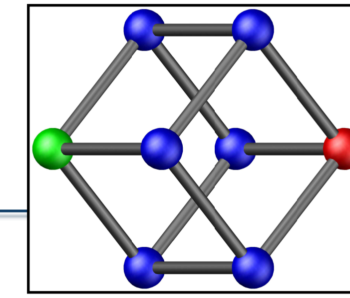
Single-Particle Transition

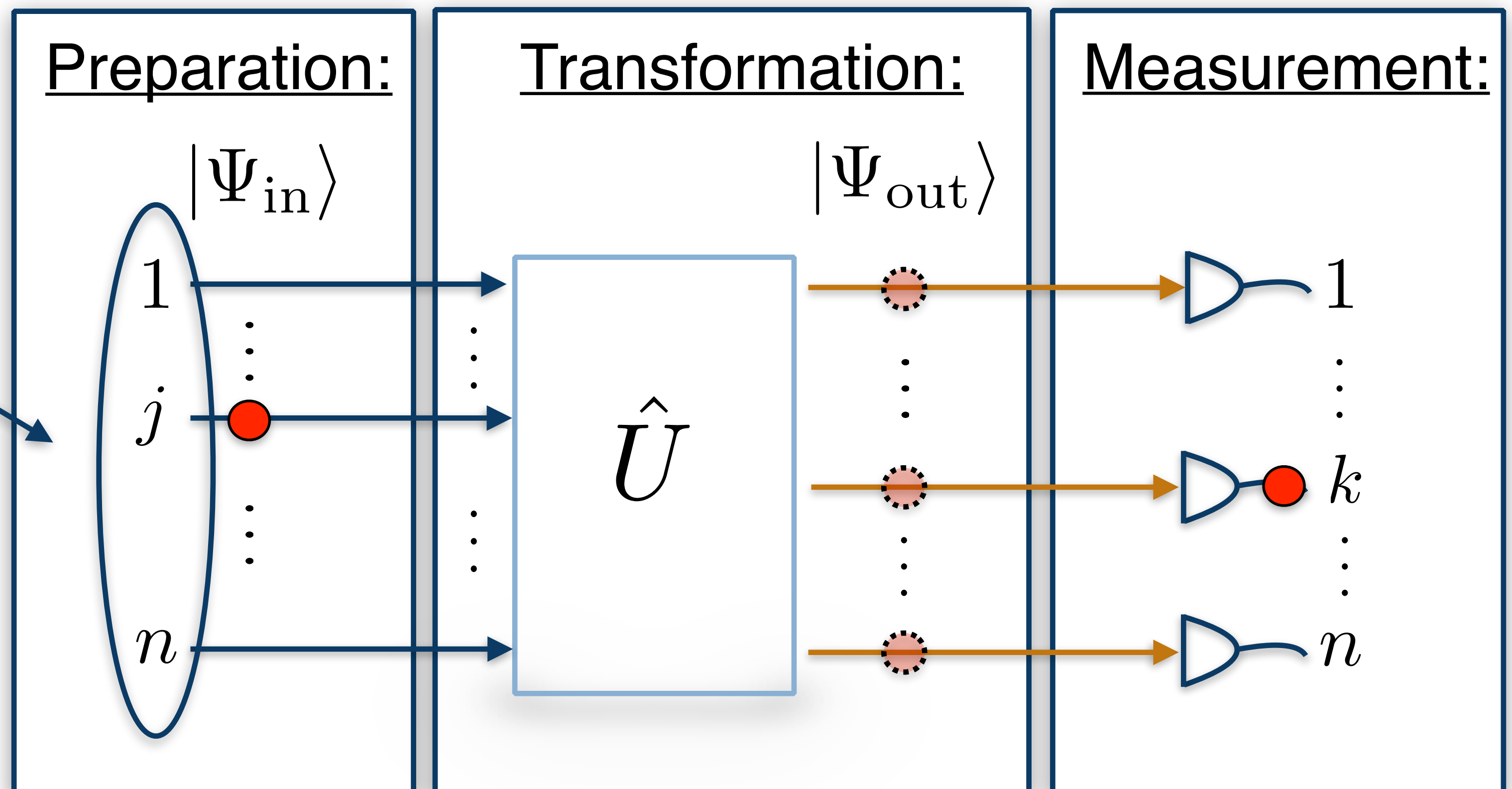


Schrödinger equation: $\hat{\mathcal{H}}|\Psi\rangle = i\hbar\,\partial_t|\Psi\rangle$

$|\Psi_{\text{out}}\rangle = \hat{U}\;|\Psi_{\text{in}}\rangle$ with $\hat{U} = e^{-i\hat{\mathcal{H}}t/\hbar}$

edges of the graph

vertices of the graph

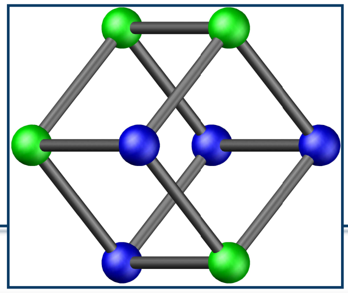


transition amplitude: $\hat{U}_{j,k}$

transition probability: $P_{j,k} = \left|\langle\Psi_k|\hat{U}|\Psi_j\rangle\right|^2 = \left|\hat{U}_{j,k}\right|^2$

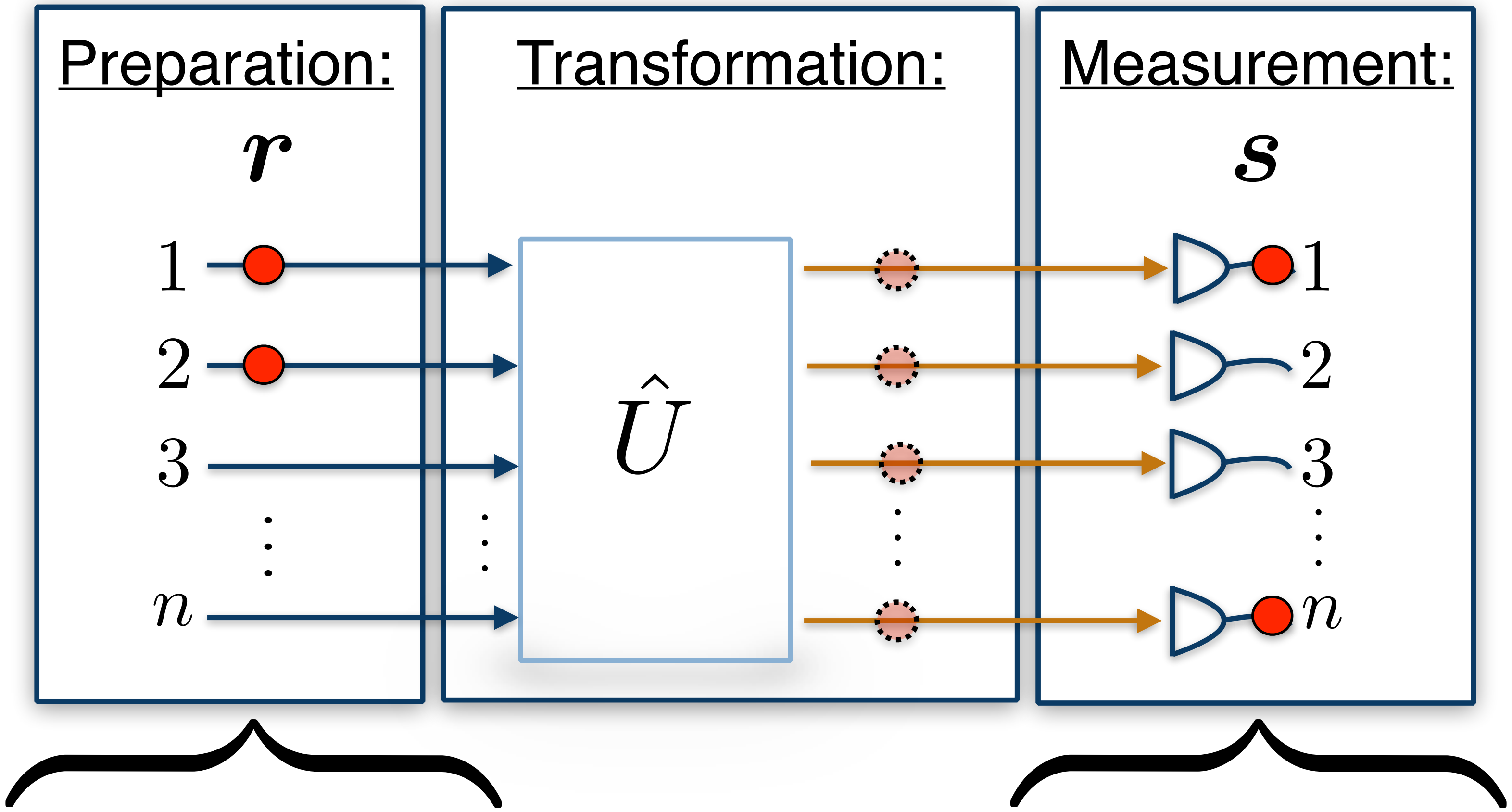
see e.g. [9] Tichy, *J. Phys. B: At. Mol. Opt. Phys.* **47**, 103001 (2014)

Many-Particle Transition



n … number of modes

N … number of particles



mode **occupation** list: $\boldsymbol{r} = (1, 1, 0, \ldots, 0)$ $\boldsymbol{s} = (1, 0, \ldots, 0, 1)$

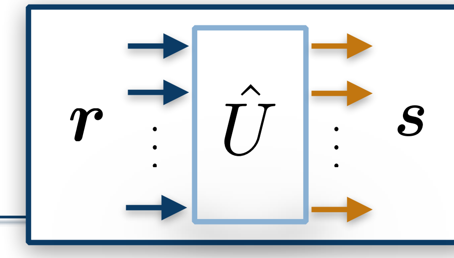
length: $\dim(\boldsymbol{r}) \equiv ||\boldsymbol{r}|| = ||\boldsymbol{s}|| = n$

mode **assignment** list: $\boldsymbol{d}(\boldsymbol{r}) = (1, 2)$ $\boldsymbol{d}(\boldsymbol{s}) = (1, n)$

length: $\dim(\boldsymbol{d}(\boldsymbol{r})) \equiv ||\boldsymbol{d}(\boldsymbol{r})|| = ||\boldsymbol{d}(\boldsymbol{s})|| = N$

see e.g. [9] Tichy, *J. Phys. B: At. Mol. Opt. Phys.* **47**, 103001 (2014)

Transition Probabilities



<u>elements of the unitary:</u> $\boxed{M_{j,k} = \hat{U}_{d_j(\boldsymbol{r}),d_k(\boldsymbol{s})}}$ with $\dim(M) = N \times N$

$N \ldots$ number of particles

correspondence: rows… initial modes

columns… final modes

calculated by:

distinguishable particles:	$P_{\text{dist}}(\boldsymbol{r},\boldsymbol{s},\hat{U}) \propto \text{perm}\left(\|M\|^2\right)$	single transition probabilities
indistinguishable bosons:	$P_{\text{B}}(\boldsymbol{r},\boldsymbol{s},\hat{U}) \propto \|\text{perm}(M)\|^2$	transition amplitudes
indistinguishable fermions:	$P_{\text{F}}(\boldsymbol{r},\boldsymbol{s},\hat{U}) = \|\det(M)\|^2$	

The *<u>permanent</u>* of a matrix is similar to the *<u>determinant</u>* of a matrix but without negative signs for odd permutations

see e.g. [9] Tichy, *J. Phys. B: At. Mol. Opt. Phys.* **47**, 103001 (2014)

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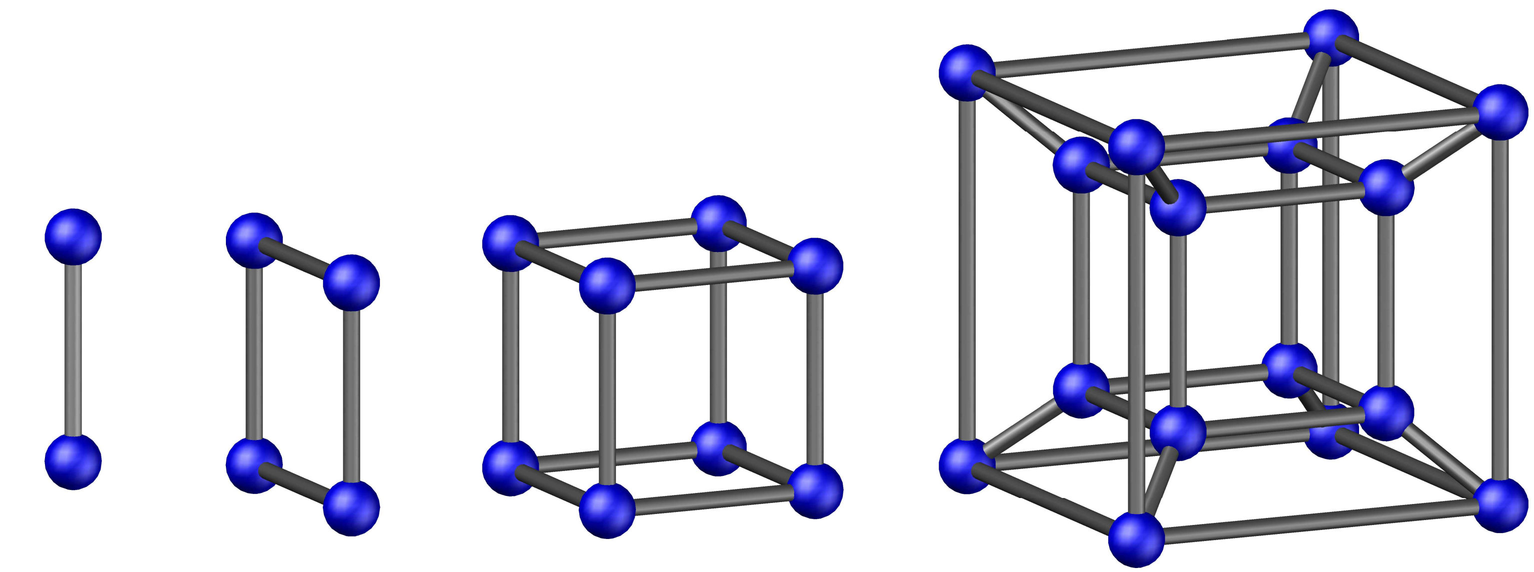
Hypercube Unitary



<u>transition rate:</u> $\mathcal{H}_{i,j} = \kappa$

<u>evolution time:</u> $t = \pi/(4\kappa)$

$$\xRightarrow{\hat{U} = e^{-i\hat{\mathcal{H}}t/\hbar}} \quad \boxed{\hat{U} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}^{\otimes d}}$$

d … dimension

$n = 2^d$ … number of modes

$\dim(\hat{U}) = n \times n$

Hypercube Partitioning

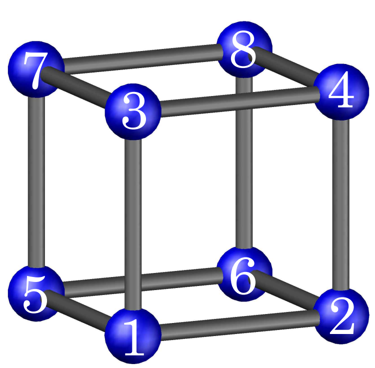
$$\hat{U} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}^{\otimes d}$$

Each tensor power corresponds to a *partitioning value* $p \in \{2, 4, 8, \ldots, 2^d\}$

Partitioning vector : $\boldsymbol{p} = (p_1, p_2, \ldots)\,,\ \ p_i \neq p_j$

Walsh-functions assign $1\ (-1)$ to each mode j : $\mathcal{A}(j, \boldsymbol{p}) = \prod_{m=1}^{||\boldsymbol{p}||} (-1)^{\lfloor \frac{p_m (j-1)}{n} \rfloor}$

Example:
3-dim HC



	Mode number j							
Step-Functions	1	2	3	4	5	6	7	8
$\mathcal{A}(j,2)$	1	1	1	1	-1	-1	-1	-1
$\mathcal{A}(j,4)$	1	1	-1	-1	1	1	-1	-1
$\mathcal{A}(j,8)$	1	-1	1	-1	1	-1	1	-1
$\mathcal{A}(j,(2,4))$	1	1	-1	-1	-1	-1	1	1
$\mathcal{A}(j,(2,8))$	1	-1	1	-1	-1	1	-1	1
$\mathcal{A}(j,(4,8))$	1	-1	-1	1	1	-1	-1	1
$\mathcal{A}(j,(2,4,8))$	1	-1	-1	1	-1	1	1	-1

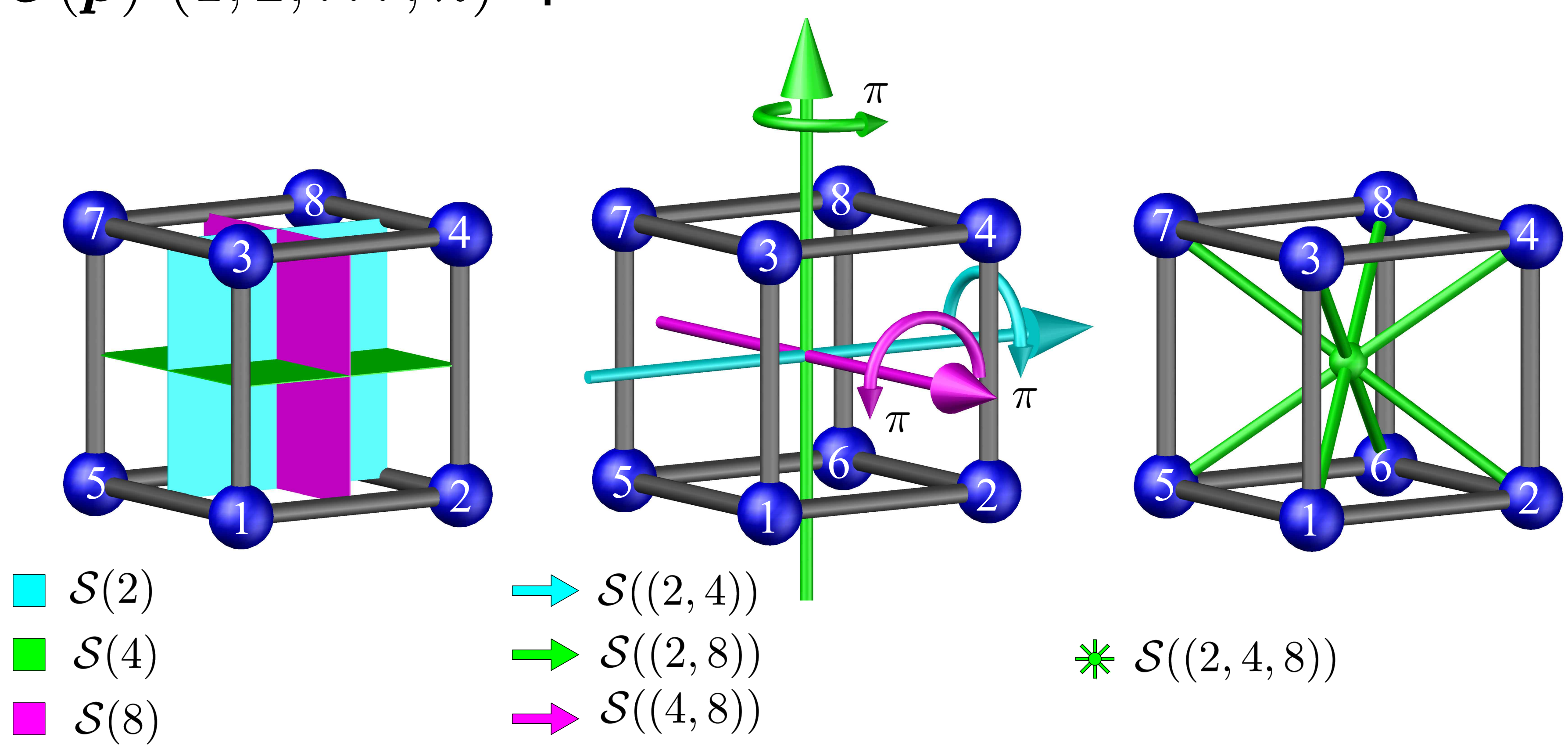
Symmetry Operations

Self-inverse and mutually commuting *__symmetry operations__*:

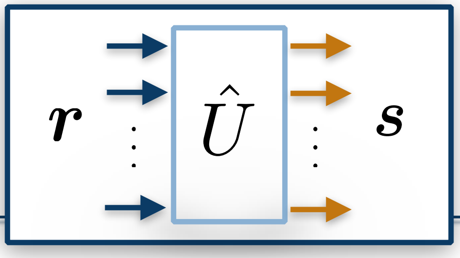
$$\mathcal{S}(\boldsymbol{p}) = \prod_{k=1}^{||\boldsymbol{p}||} \mathbb{1}^{\otimes \log_2(p_k/2)} \otimes \sigma_x \otimes \mathbb{1}^{\otimes \log_2(n/p_k)} \qquad \text{with} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Illustration for $\mathcal{S}(\boldsymbol{p})\ (1, 2, \ldots, n)^\top$:

__Example:__
3-dim HC



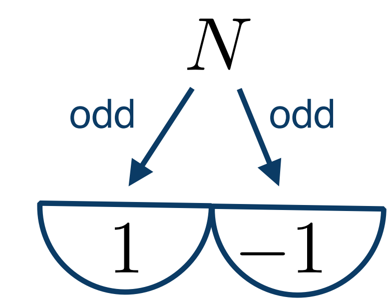
Symmetry Suppression Law



For an initial state $\boldsymbol{r}$ of N particles, which is invariant under the symmetry operation

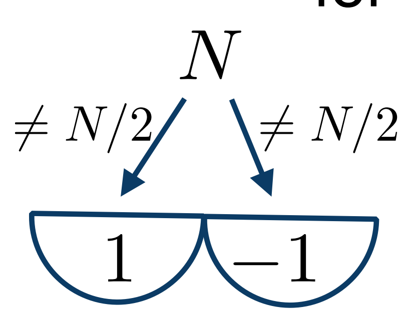
$$\boxed{\mathcal{S}(\boldsymbol{p})\ \boldsymbol{r} = \boldsymbol{r}}$$

Bosons: all final states $\boldsymbol{s}$ with an odd number of particles in modes k for which $\mathcal{A}(k, \boldsymbol{p}) = 1$ are suppressed, i.e.,



$$\prod_{j=1}^{N} \mathcal{A}(d_j(\boldsymbol{s}), \boldsymbol{p}) = -1 \quad \Rightarrow \quad P_{\mathrm{B}}(\boldsymbol{r}, \boldsymbol{s}, \hat{U}) = 0$$

Fermions: all final states $\boldsymbol{s}$ which do not have exactly $N/2$ particles in modes k for which $\mathcal{A}(k, \boldsymbol{p}) = 1$, are suppressed, i.e.,



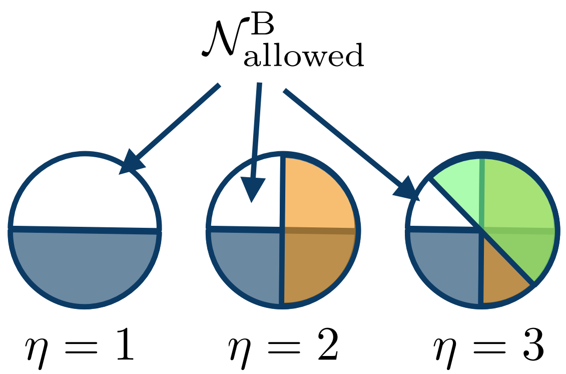
$$\sum_{j=1}^{N} \mathcal{A}(d_j(\boldsymbol{s}), \boldsymbol{p}) \neq 0 \quad \Rightarrow \quad P_{\mathrm{F}}(\boldsymbol{r}, \boldsymbol{s}, \hat{U}) = 0$$

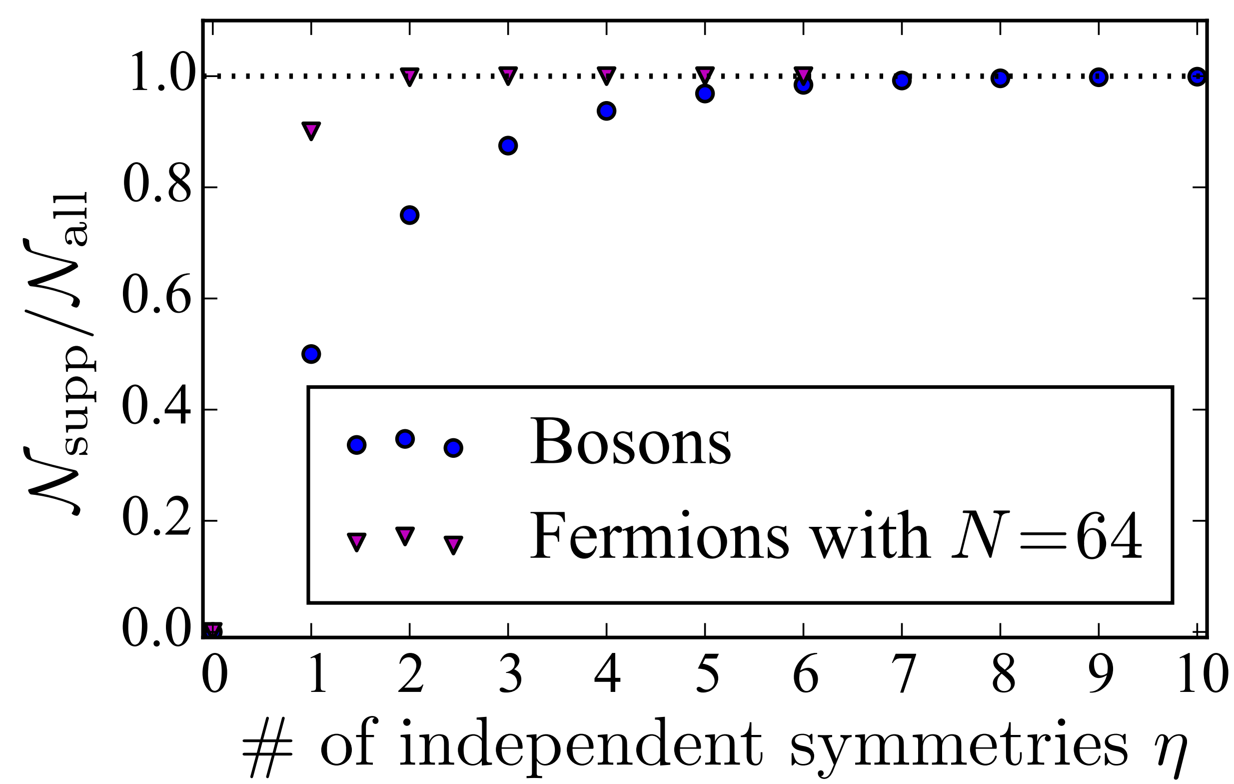
Suppression Ratio

η …number of independent symmetries of the initial state

Bosons: for large n

$$\frac{\mathcal{N}^{\mathrm{B}}_{\mathrm{supp}}}{\mathcal{N}^{\mathrm{B}}_{\mathrm{all}}} \approx 1 - \frac{1}{2^\eta}$$





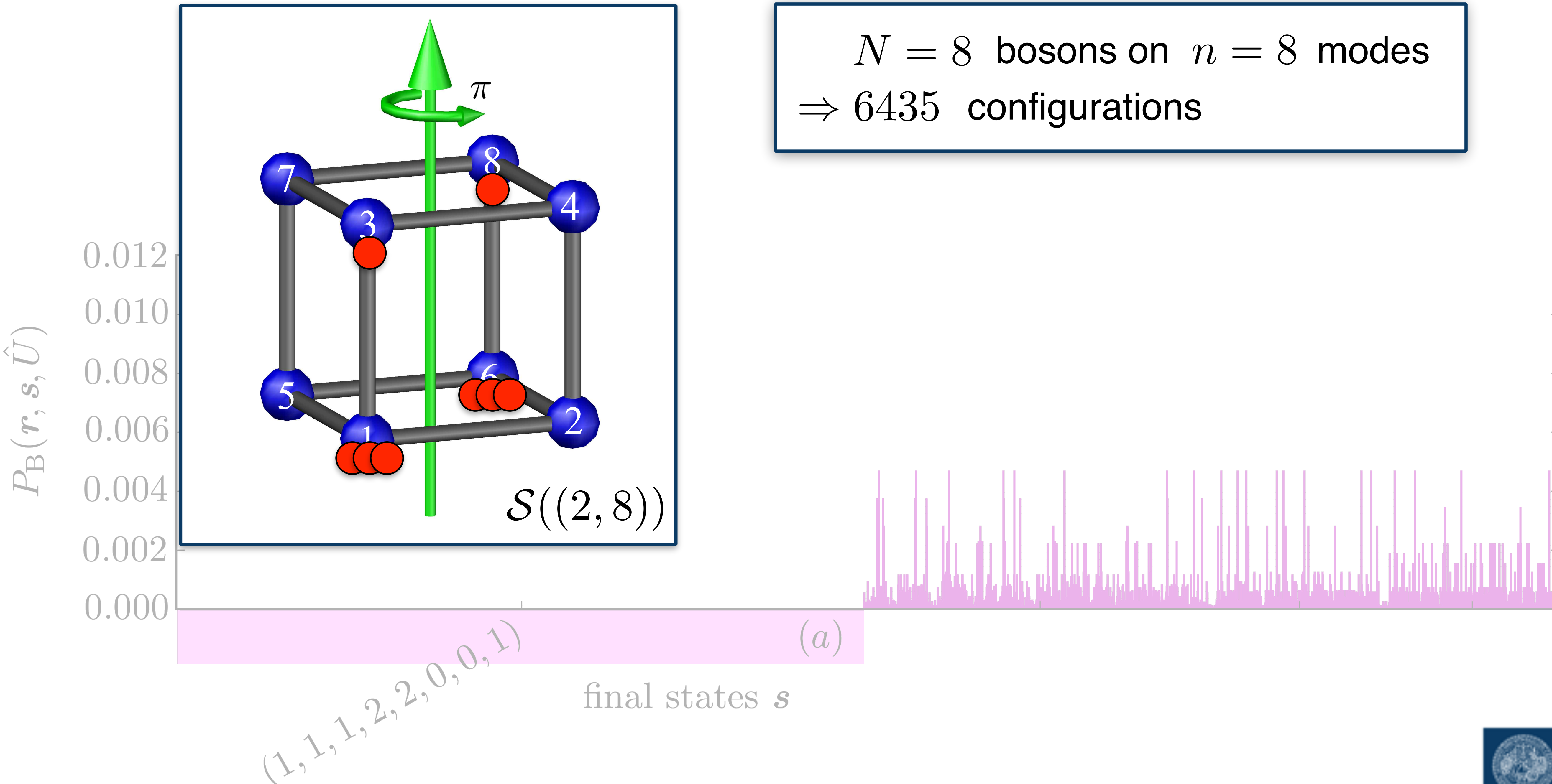
Fermions: for $n \gg N$

$$\frac{\mathcal{N}^{\mathrm{B}}_{\mathrm{supp}}}{\mathcal{N}^{\mathrm{B}}_{\mathrm{all}}} \approx 1 - \frac{N!}{2^{\eta N}\left[\left(\frac{N}{2^\eta}\right)!\right]^{2^\eta}}$$

Example: N=8 Bosons on the 3-dim. Hypercube

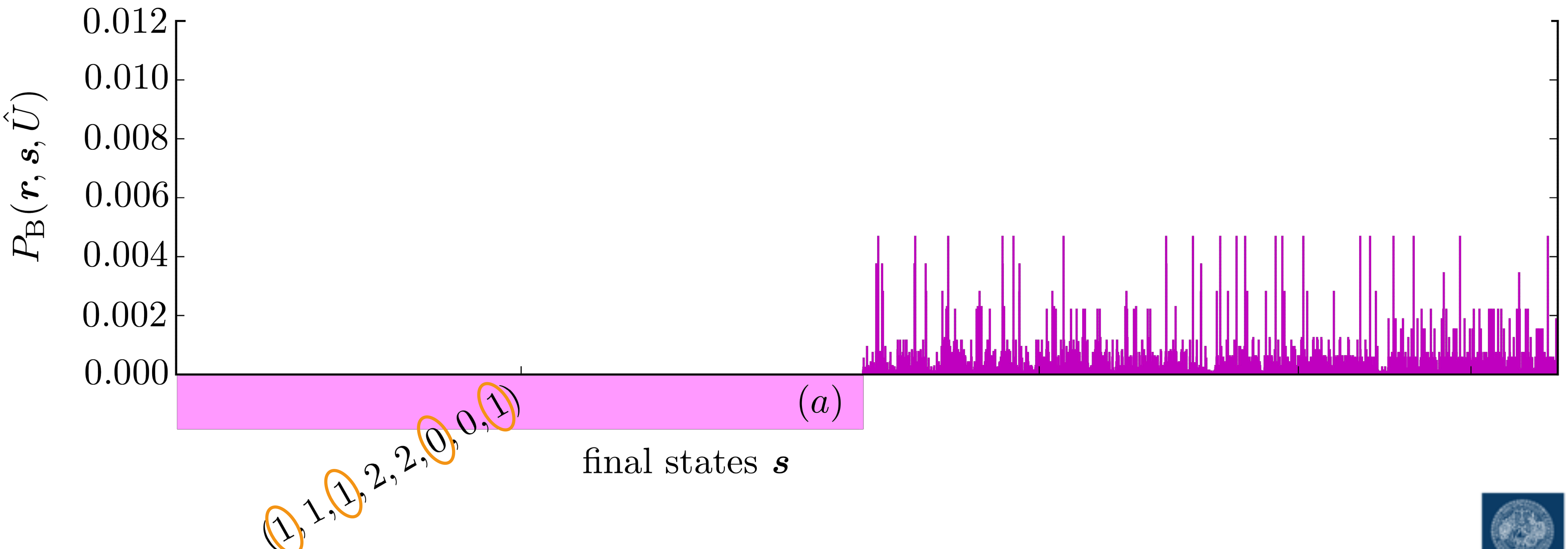
$\boldsymbol{r}_\mathrm{a} = (3,0,1,0,0,3,0,1)$ $\quad \mathcal{S}((2,8))\, \boldsymbol{r}_\mathrm{a} = \boldsymbol{r}_\mathrm{a}$ $\qquad \eta = 1$

$N = 8$ bosons on $n = 8$ modes
$\Rightarrow 6435$ configurations



Example: N=8 Bosons on the 3-dim. Hypercube

$\boldsymbol{r}_{\mathrm{a}} = (3,0,1,0,0,3,0,1)$ $\quad \mathcal{S}((2,8))\ \boldsymbol{r}_{\mathrm{a}} = \boldsymbol{r}_{\mathrm{a}}$ $\quad \eta = 1$

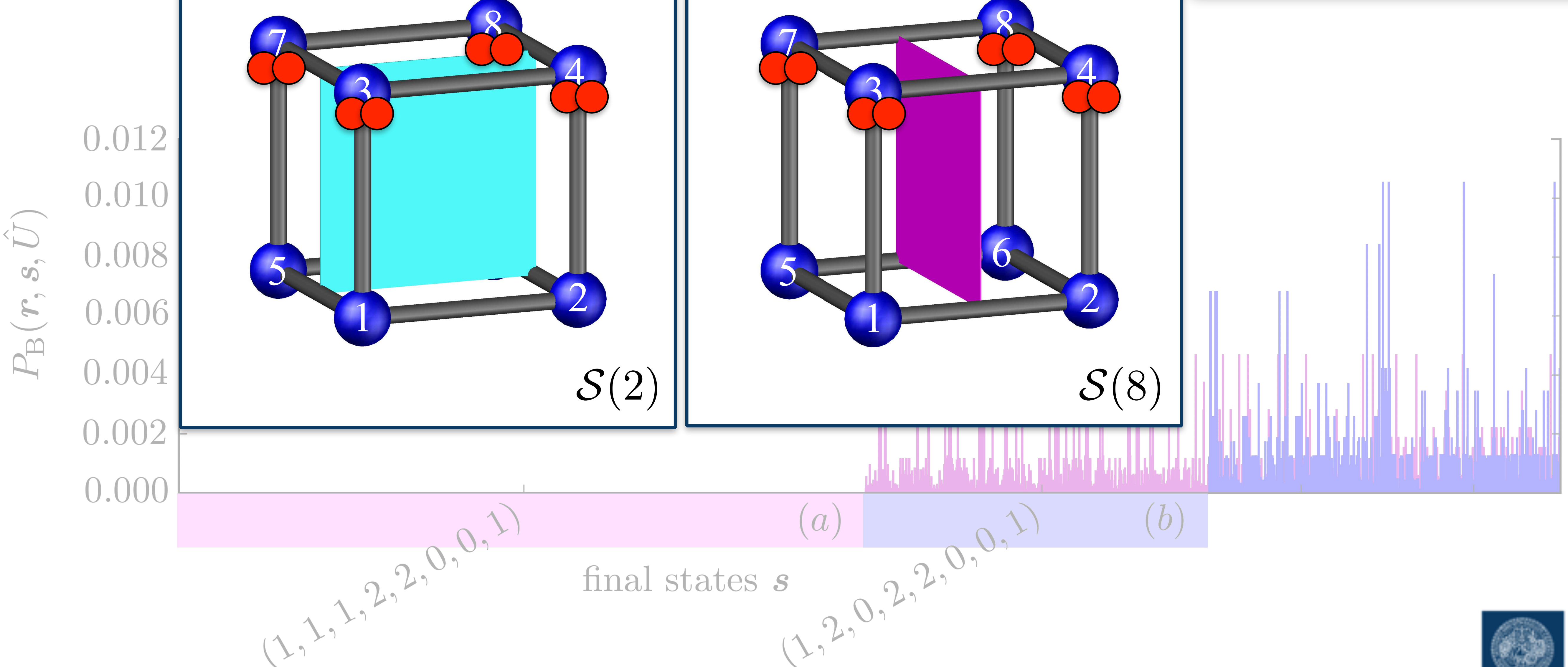


Example: *N*=8 Bosons on the 3-dim. Hypercube

$\boldsymbol{r}_{\mathrm{a}} = (3,0,1,0,0,3,0,1)$ $\quad \mathcal{S}((2,8))\ \boldsymbol{r}_{\mathrm{a}} = \boldsymbol{r}_{\mathrm{a}}$ $\quad \eta = 1$

$\boldsymbol{r}_{\mathrm{b}} = (0,0,2,2,0,0,2,2)$ $\quad \mathcal{S}(2)\ \boldsymbol{r}_{\mathrm{b}} = \boldsymbol{r}_{\mathrm{b}} \quad \mathcal{S}(8)\ \boldsymbol{r}_{\mathrm{b}} = \boldsymbol{r}_{\mathrm{b}}$ $\quad \eta = 2$

$\Rightarrow \mathcal{S}((2,8))\ \boldsymbol{r}_{\mathrm{b}} = \boldsymbol{r}_{\mathrm{b}}$



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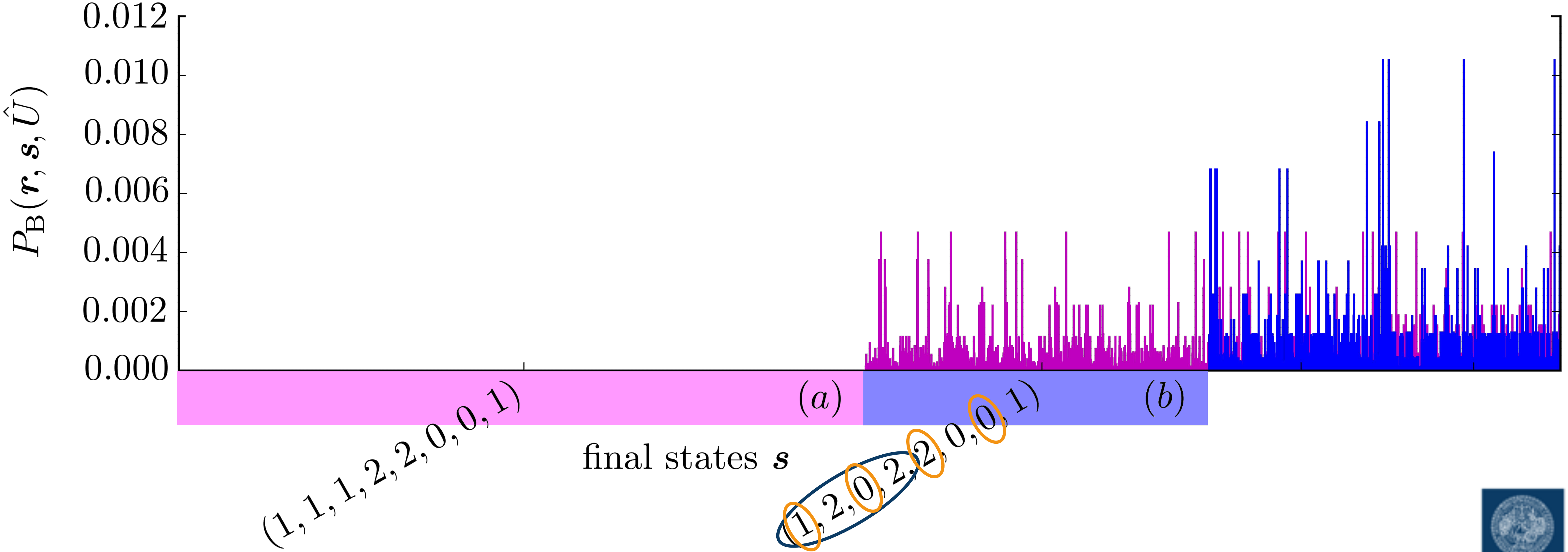


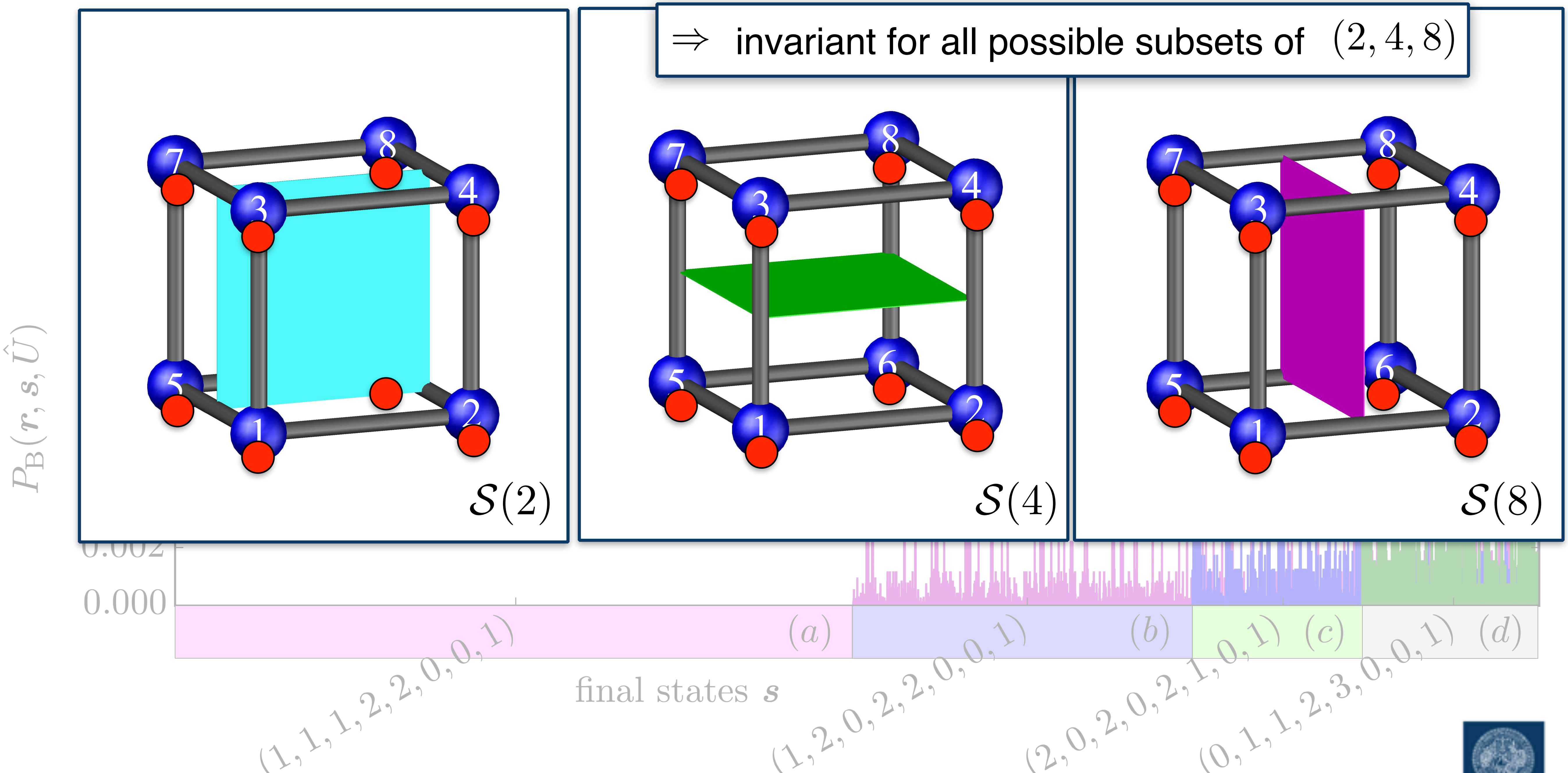
Example: N=8 Bosons on the 3-dim. Hypercube

$\boldsymbol{r}_\mathrm{a} = (3,0,1,0,0,3,0,1)$ $\quad \mathcal{S}((2,8))\ \boldsymbol{r}_\mathrm{a} = \boldsymbol{r}_\mathrm{a}$ $\quad \eta = 1$

$\boldsymbol{r}_\mathrm{b} = (0,0,2,2,0,0,2,2)$ $\quad \mathcal{S}(2)\ \boldsymbol{r}_\mathrm{b} = \boldsymbol{r}_\mathrm{b}$ $\quad \mathcal{S}(8)\ \boldsymbol{r}_\mathrm{b} = \boldsymbol{r}_\mathrm{b}$ $\quad \eta = 2$

$\boldsymbol{r}_\mathrm{c} = (1,1,1,1,1,1,1,1)$ $\quad \mathcal{S}(2)\ \boldsymbol{r}_\mathrm{c} = \boldsymbol{r}_\mathrm{c}$ $\quad \mathcal{S}(4)\ \boldsymbol{r}_\mathrm{c} = \boldsymbol{r}_\mathrm{c}$ $\quad \mathcal{S}(8)\ \boldsymbol{r}_\mathrm{c} = \boldsymbol{r}_\mathrm{c}$ $\quad \eta = 3$

$\Rightarrow$ invariant for all possible subsets of $(2,4,8)$

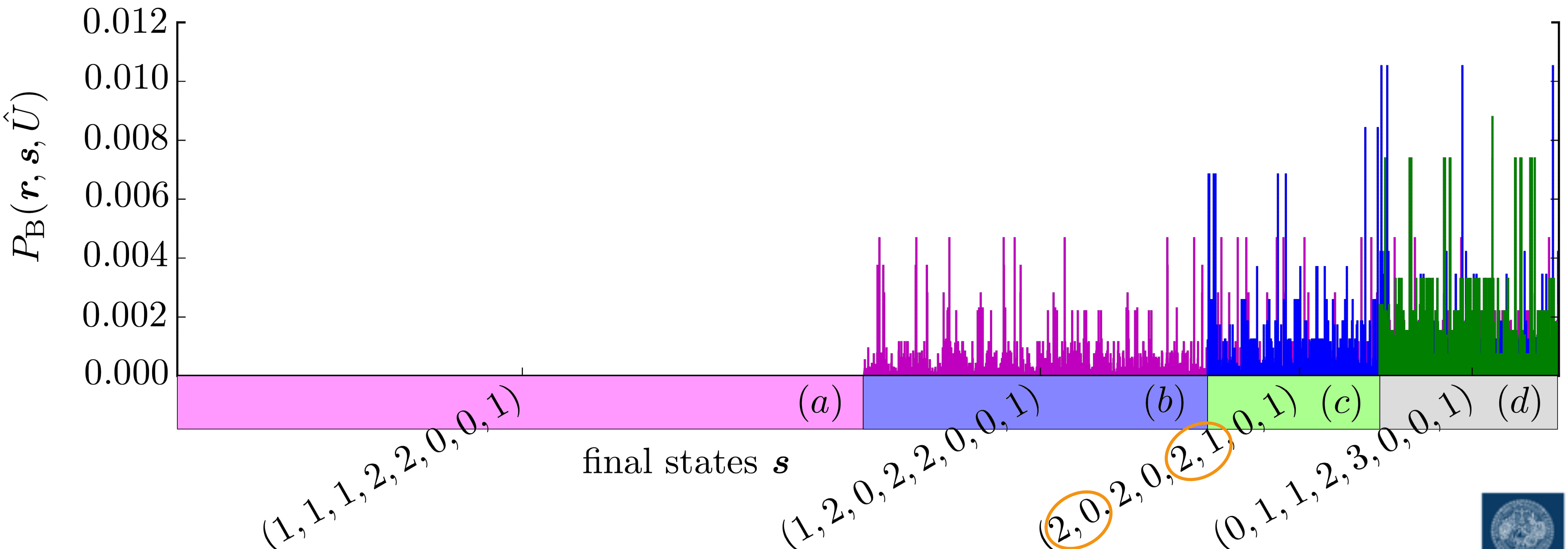


Example: *N*=8 Bosons on the 3-dim. Hypercube

$\boldsymbol{r}_{\mathrm{a}} = (3,0,1,0,0,3,0,1)$ $\mathcal{S}((2,8))\ \boldsymbol{r}_{\mathrm{a}} = \boldsymbol{r}_{\mathrm{a}}$ $\eta = 1$

$\boldsymbol{r}_{\mathrm{b}} = (0,0,2,2,0,0,2,2)$ $\mathcal{S}(2)\ \boldsymbol{r}_{\mathrm{b}} = \boldsymbol{r}_{\mathrm{b}}$ $\mathcal{S}(8)\ \boldsymbol{r}_{\mathrm{b}} = \boldsymbol{r}_{\mathrm{b}}$ $\eta = 2$

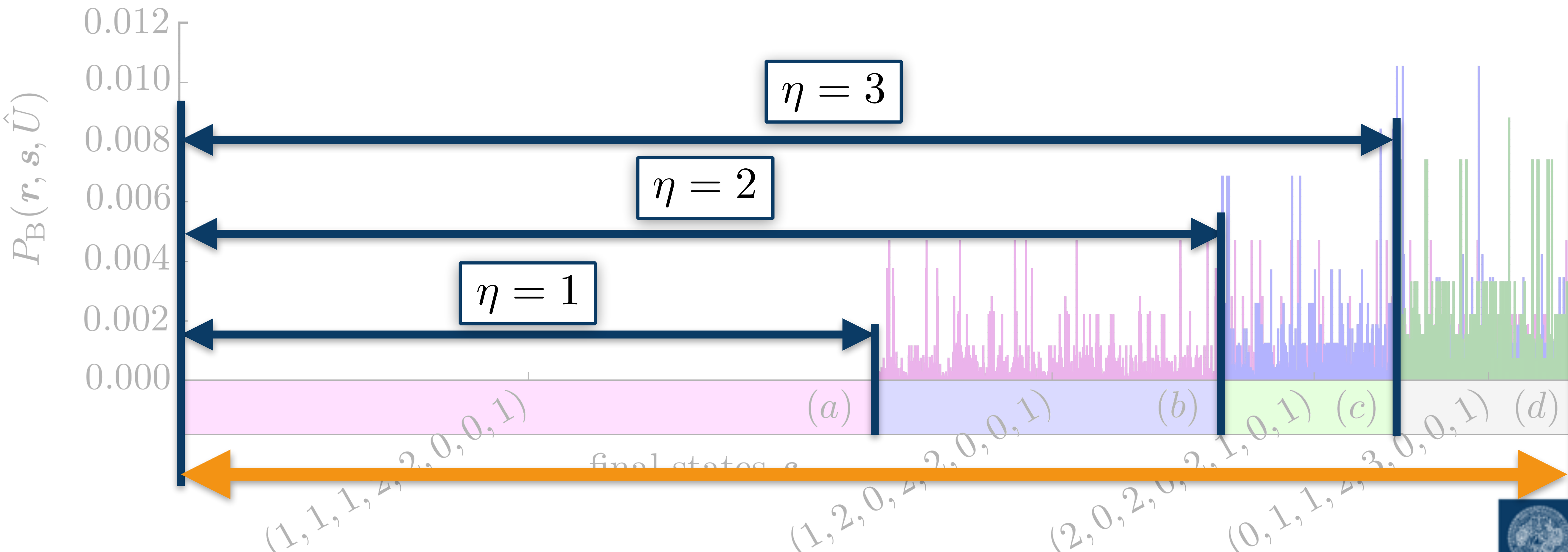
$\boldsymbol{r}_{\mathrm{c}} = (1,1,1,1,1,1,1,1)$ $\mathcal{S}(2)\ \boldsymbol{r}_{\mathrm{c}} = \boldsymbol{r}_{\mathrm{c}}$ $\mathcal{S}(4)\ \boldsymbol{r}_{\mathrm{c}} = \boldsymbol{r}_{\mathrm{c}}$ $\mathcal{S}(8)\ \boldsymbol{r}_{\mathrm{c}} = \boldsymbol{r}_{\mathrm{c}}$ $\eta = 3$



Example: N=8 Bosons on the 3-dim. Hypercube

$\boldsymbol{r}_\mathrm{a} = (3,0,1,0,0,3,0,1)$
$\boldsymbol{r}_\mathrm{b} = (0,0,2,2,0,0,2,2)$
$\boldsymbol{r}_\mathrm{c} = (1,1,1,1,1,1,1,1)$

suppression ratio: $$\boxed{\frac{\mathcal{N}^\mathrm{B}_\mathrm{supp}}{\mathcal{N}^\mathrm{B}_\mathrm{all}} \approx 1 - \frac{1}{2^\eta}}$$



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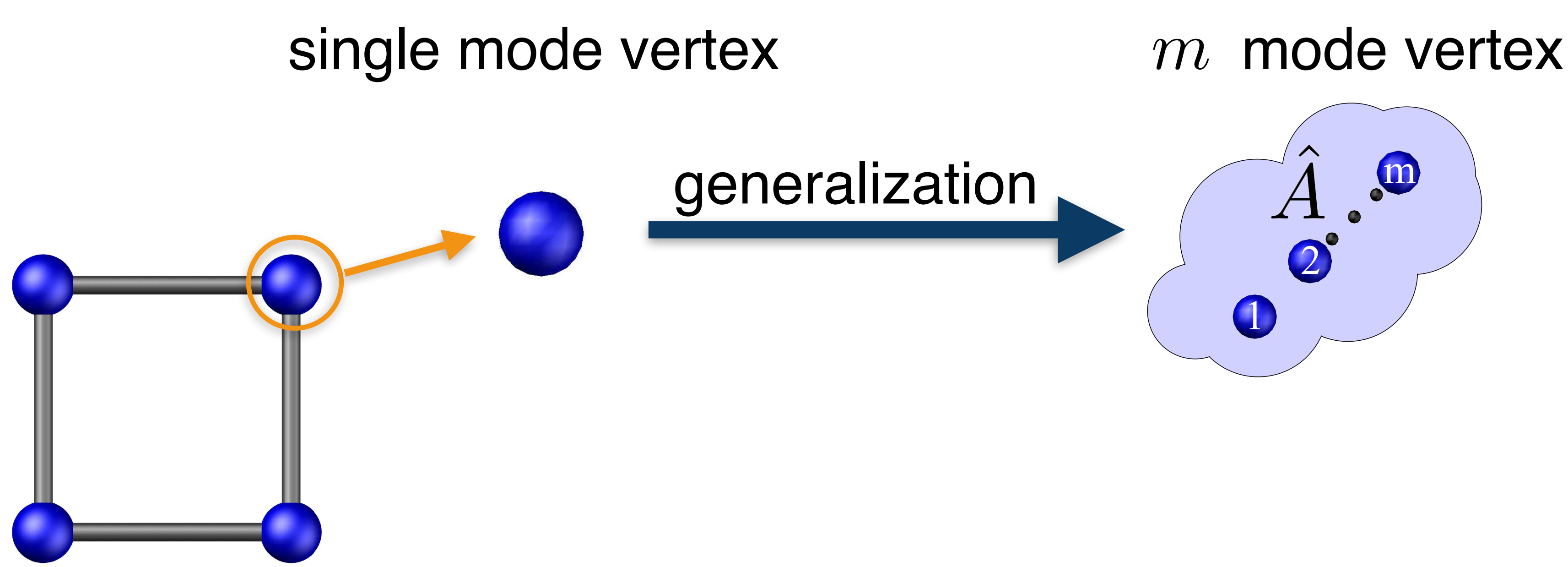
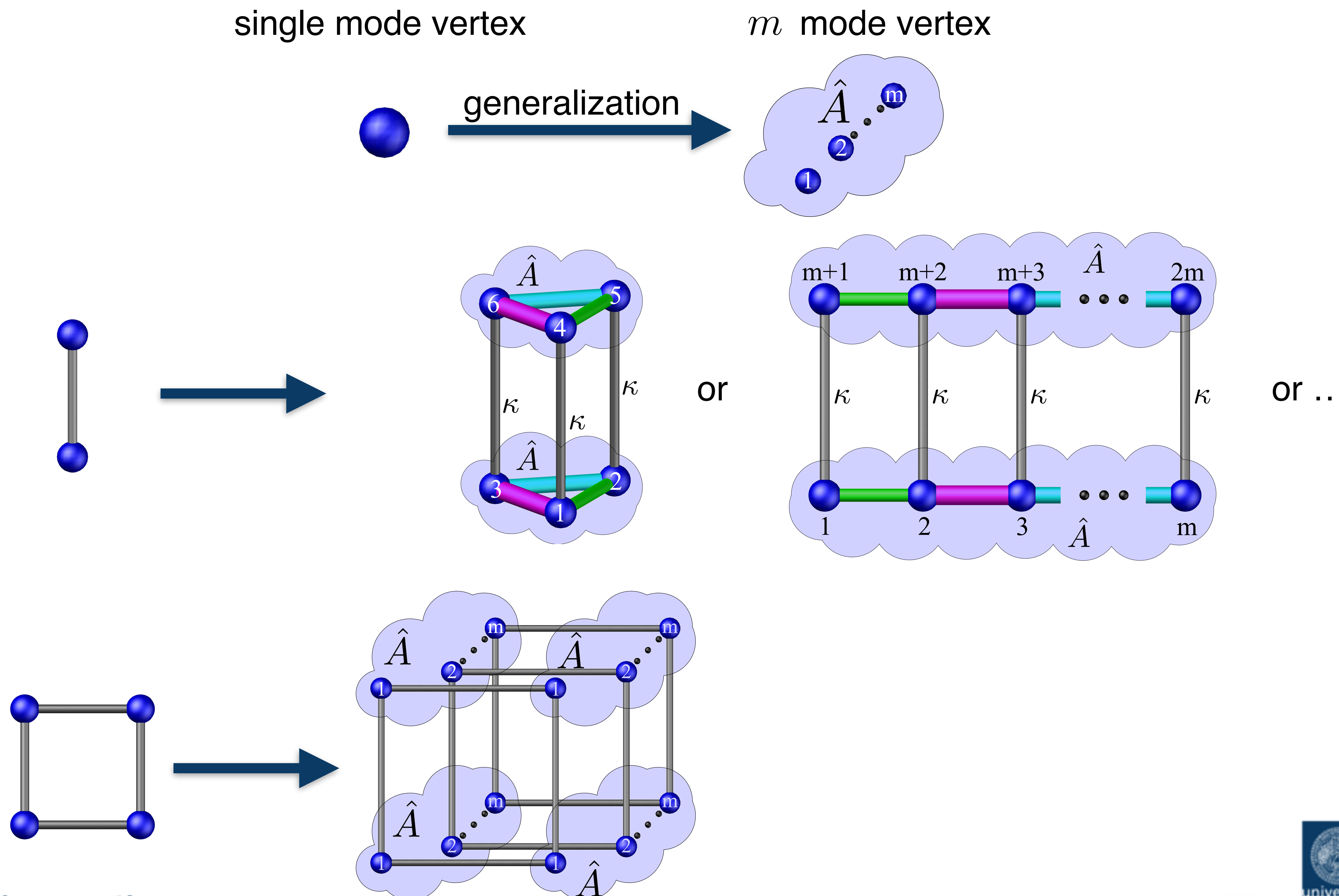
Generalization of the Vertices

- Nodes are allowed to have diverse internal degrees of freedom
- Each vertex is described by the same but arbitrary $m \times m$ subunitary $\hat{A}$:
- Each SUBvertex is equally coupled to d identical counterparts in the HC ordering

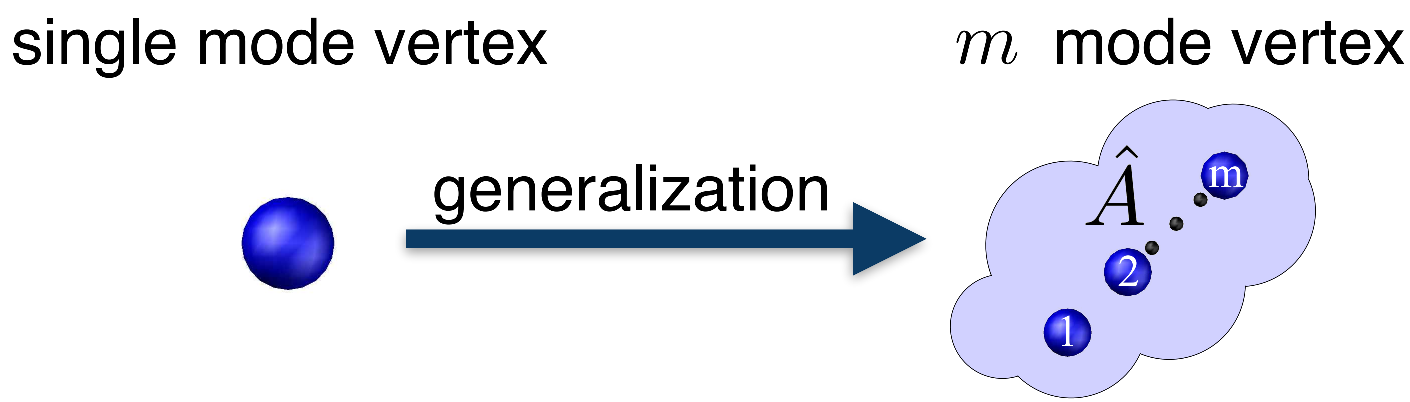


$$\hat{U} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}^{\otimes d} \longrightarrow \boxed{\hat{U} = \frac{1}{\sqrt{2^d}} \hat{A} \otimes \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}^{\otimes d}}$$

Generalization: Examples



Generalization: Suppression Law



$$\hat{U} = \frac{1}{\sqrt{2^d}} \hat{A} \otimes \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}^{\otimes d}$$

The same suppression laws hold

$$\boxed{\mathcal{S}(\boldsymbol{p})\ \boldsymbol{r} = \boldsymbol{r}}$$

Bosons:

$$\prod_{j=1}^{N} \mathcal{A}(d_j(\boldsymbol{s}), \boldsymbol{p}) = -1 \qquad \Rightarrow \qquad P_{\mathrm{B}}(\boldsymbol{r}, \boldsymbol{s}, \hat{U}) = 0$$

Fermions:

$$\sum_{j=1}^{N} \mathcal{A}(d_j(\boldsymbol{s}), \boldsymbol{p}) \neq 0 \qquad \Rightarrow \qquad P_{\mathrm{F}}(\boldsymbol{r}, \boldsymbol{s}, \hat{U}) = 0$$

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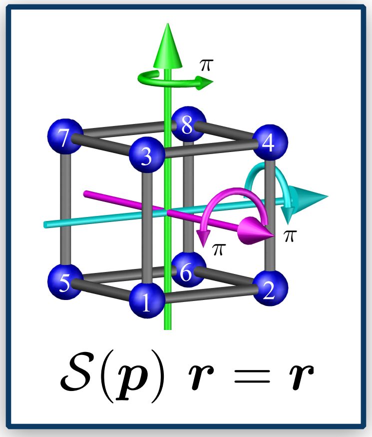
Summary

- Suppression laws for many-body QT on HC graphs
  - symmetry based
  - analytic

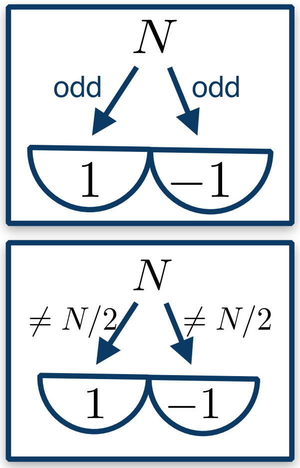


- Each symmetry of the initial state groups all modes into two partitions of equal size

- The final occupation of these partitions determines the suppression

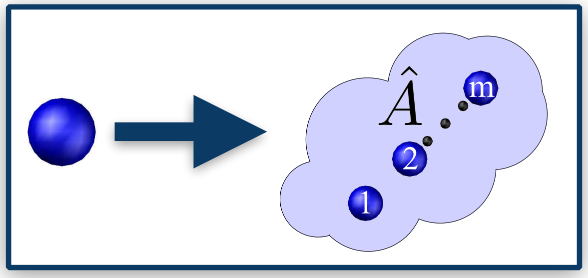
  Bosons: Suppression depends on the parity of the occupation

  Fermions: Suppression for all imbalanced occupation



- Generalization of the suppression law:

  HCs with arbitrary identical subgraphs on all vertices



Outlook

- Supp. law could suit for the certification of many-particle indistinguishability [17,18]
- Realizations in
  - atomic lattices [19,20]
  - optomechanical systems [21]
  - optical systems [22]

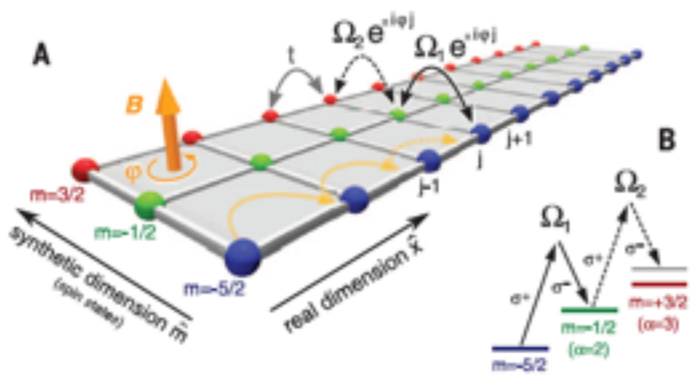


Figure reference: [19]

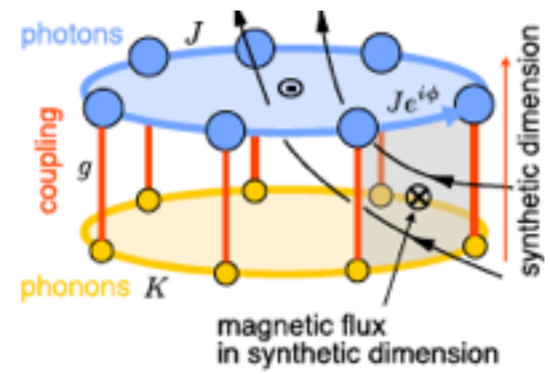


Figure reference: [21]

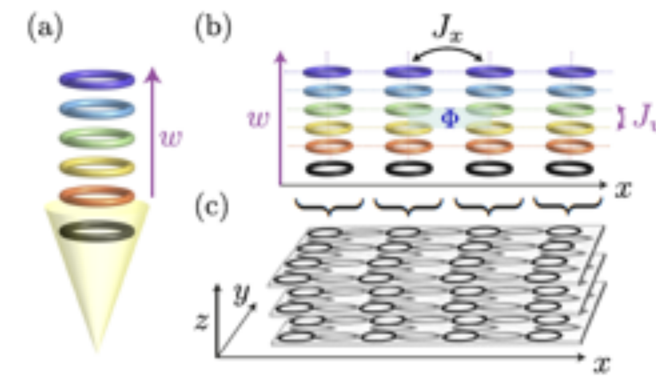


Figure reference: [22]

- If only fewer dimensions are available, make use of
  - long-range connections [23,24]
  - internal degrees of freedom [25]

Figure reference: [24]

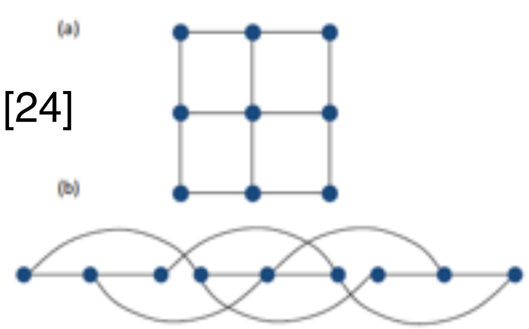
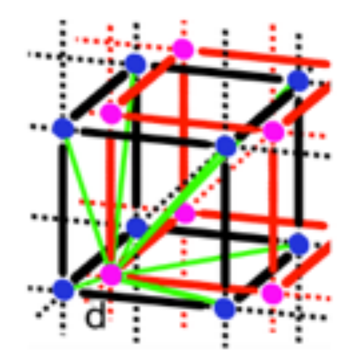


Figure reference: [25]



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[21] Schmidt et al., *Optica*, **2**, 635-641 (2015)

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[25] Boada et al., *Phys. Rev. Lett.*, **108**, 133001 (2012)

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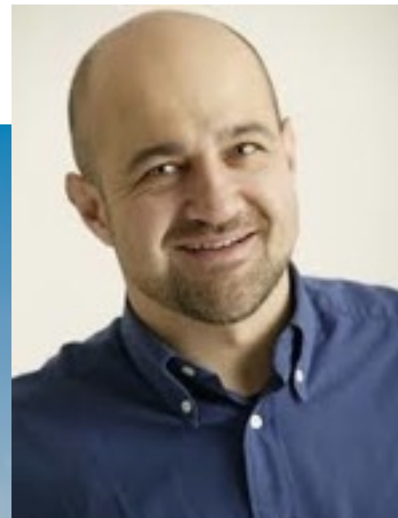

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Thanks for your attention

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[21] Schmidt et al., *Optica,* **2**, 635-641 (2015)

[22] Ozawa et al., arXiv:1510.03910 (2015)

[23] Daqing et al., *Nat. Phys.*, **7**, 481-484 (2011)

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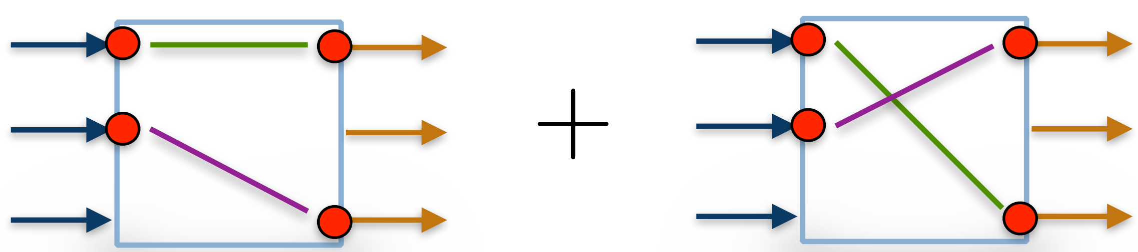
[25] Boada et al., *Phys. Rev. Lett.*, **108**, 133001 (2012)

Appendix A: Transition Probabilities - Example

Distinguishable Particles:



$$p_{1,1} \cdot p_{2,3} + p_{1,3} \cdot p_{2,1}$$

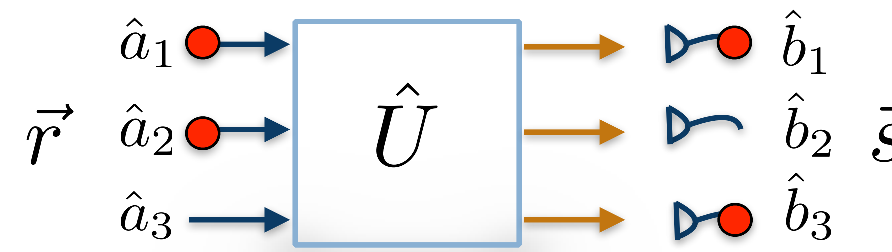
$$\vec{d}(\vec{r}) = (1,2) \qquad \vec{\sigma}_1 = (1,3) \qquad \vec{\sigma}_2 = (3,1)$$

$p_{i,j} = |\hat{U}_{i,j}|^2$

$\vec{\sigma}_i \in S_{\vec{d}(\vec{s})}$

$S_{\vec{d}(\vec{s})}$ all permutations of $\vec{d}(\vec{s})$

$$P_{\text{dist}}(\vec{r}, \vec{s}, \hat{U}) = \sum_{\vec{\sigma} \in S_{\vec{d}(\vec{s})}} \prod_{j=1}^{N} \left| \hat{U}_{d_j(\vec{r}), \sigma_j} \right|^2$$



$$\vec{r} = (1,1,0) \Rightarrow \vec{s} = (1,0,1)$$

$$\vec{d}(\vec{r}) = (1,2) \qquad \vec{d}(\vec{s}) = (1,3)$$

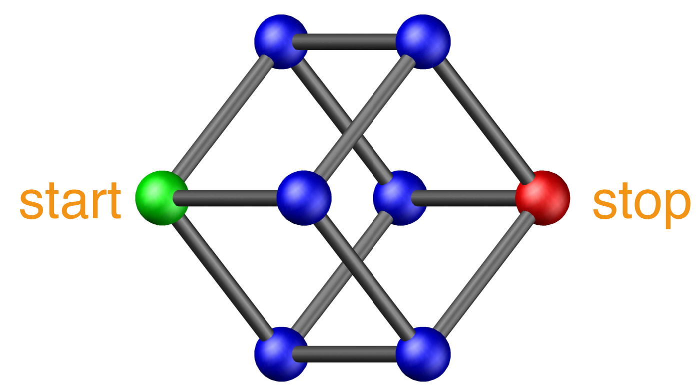
Bosons:

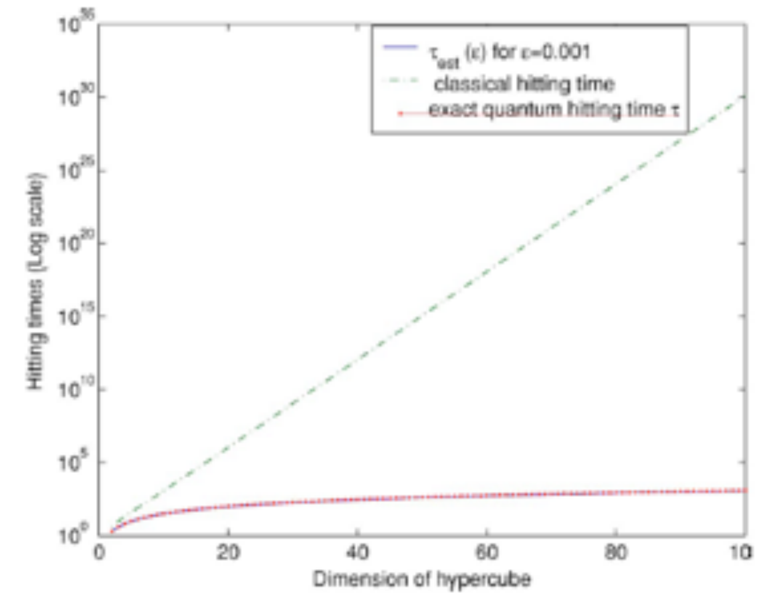
$$P_{\mathcal{B}}(\vec{r}, \vec{s}, \hat{U}) \propto \left| \sum_{\vec{\sigma} \in S_{\vec{d}(\vec{s})}} \prod_{j=1}^{N} \hat{U}_{d_j(\vec{r}), \sigma_j} \right|^2$$

Appendix B - Robust Exp. Speed-Up

Quantum walks on hypercube graphs:

- exponential speed-up of hitting times [A1-A5]





Reference: [A3]

- robust exponential speed-ups [A1-A5]

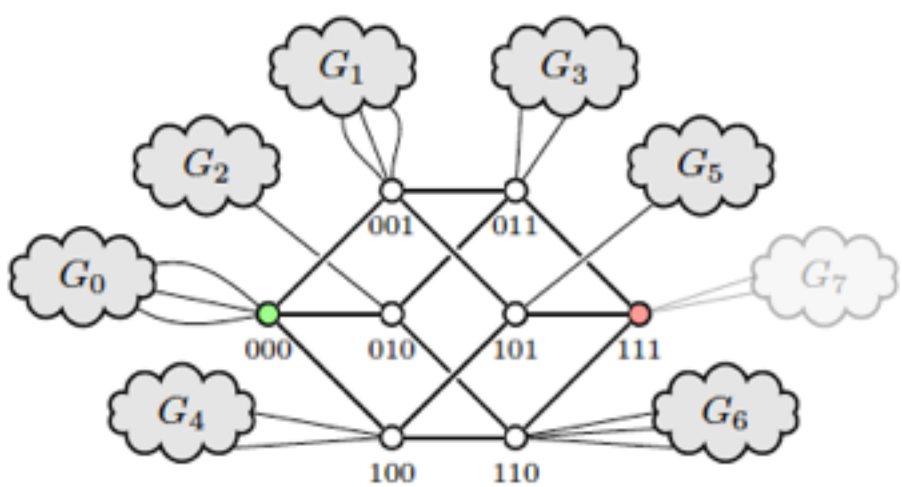


Figure reference: [A4]

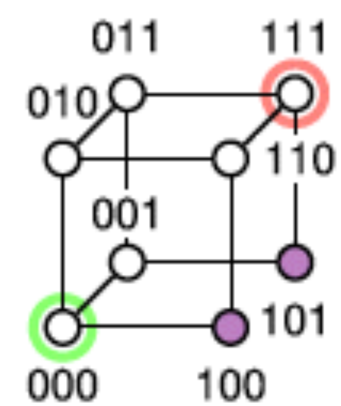


Figure reference: [A5]

[A1] Kempe, *Probab. Theory Relat. Fields,* **133**, 215-235 (2005)
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[A4] Makmal et al., *Phys. Rev. A,* **90**, 022314 (2014)
[A5] Makmal et al., *Phys. Rev. A,* **93**, 022322 (2016)

Appendix C - Walsh Functions

$$\hat{U} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}^{\otimes d}$$

Each tensor power corresponds to a *partitioning value* $p \in \{2, 4, 8, \ldots, 2^d\}$

<u>*Rademacher functions:*</u> assign $1\ (-1)$ to each mode j

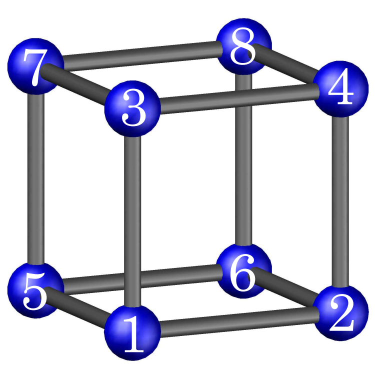
$$x(j,p) = (-1)^{\lfloor \frac{p(j-1)}{n} \rfloor}$$

<u>*Walsh functions:*</u>

$$\boxed{\mathcal{A}(j,\boldsymbol{p}) = \prod_{m=1}^{||\boldsymbol{p}||} x(j,p_m)}$$

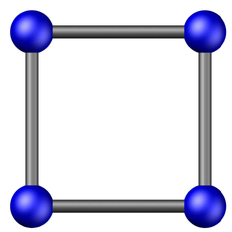
where partitioning vector: $\boldsymbol{p} = (p_1, p_2, \ldots)$, $p_i \neq p_j$

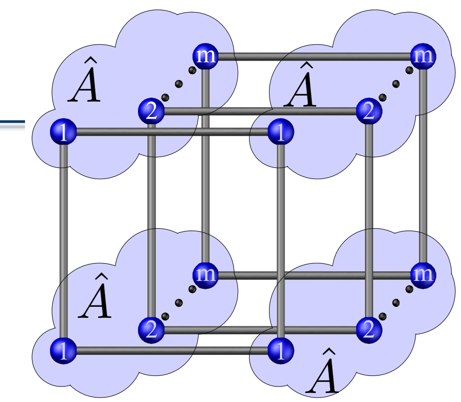
<u>Example:</u>
3-dim HC



Step-Functions	Mode number j 1	2	3	4	5	6	7	8
$x(j,2)$	1	1	1	1	-1	-1	-1	-1
$x(j,4)$	1	1	-1	-1	1	1	-1	-1
$x(j,8)$	1	-1	1	-1	1	-1	1	-1
$\mathcal{A}(j,(2,4)) = x(j,2)\ x(j,4)$	1	1	-1	-1	-1	-1	1	1
$\mathcal{A}(j,(2,8)) = x(j,2)\ x(j,8)$	1	-1	1	-1	-1	1	-1	1
$\mathcal{A}(j,(4,8)) = x(j,4)\ x(j,8)$	1	-1	-1	1	1	-1	-1	1
$\mathcal{A}(j,(2,4,8)) = x(j,2)\ x(j,4)\ x(j,8)$	1	-1	-1	1	-1	1	1	-1

Appendix D - Formal Generalization





$$\hat{U} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}^{\otimes d}$$

$$n = 2^d$$

$$\longrightarrow$$

$$\boxed{\hat{U} = \frac{1}{\sqrt{2^d}} \hat{A} \otimes \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}^{\otimes d}}$$

$$n = m\ 2^d$$

Modifications in order to apply the symmetry suppression laws:

$$\mathcal{S}(p) = \mathbb{1}^{\otimes \log_2(p/2)} \otimes \sigma_x \otimes \mathbb{1}^{\otimes \log_2(n/p)}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathcal{S}(p) = \mathbb{1}^{\otimes \log_2(p/2)} \otimes \Sigma_x \otimes \mathbb{1}^{\otimes \log_2(2^d/p)}$$

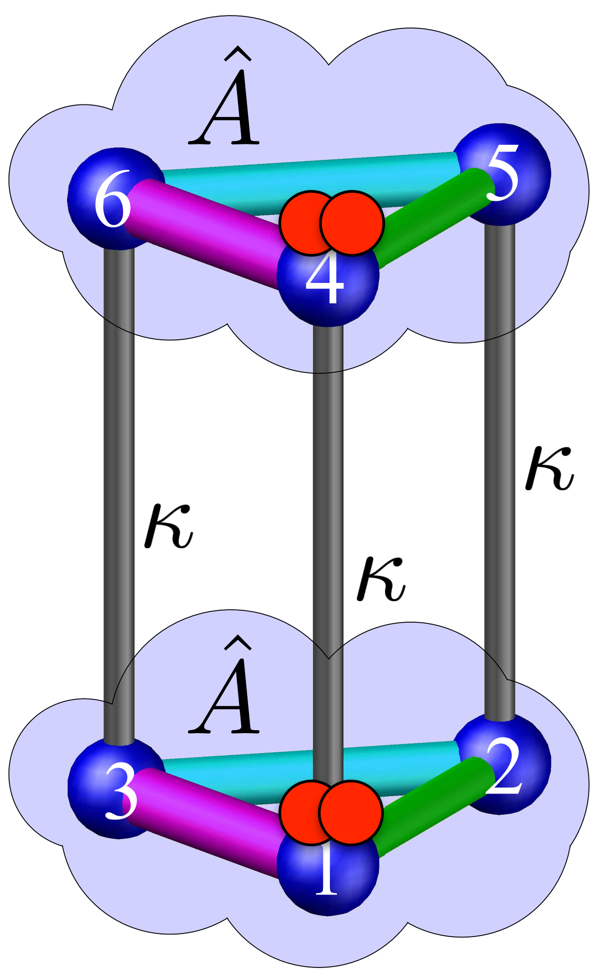
$$\Sigma_x = \begin{pmatrix} \hat{0}_{m\times m} & \mathbb{1}_{m\times m} \\ \mathbb{1}_{m\times m} & \hat{0}_{m\times m} \end{pmatrix}$$

Appendix E - Example: Bosonic Interference

$N = 4$ bosons, $n = 6$ modes, $d = 1$ dimension

$p = 2$ is the only partitioning value

$\Rightarrow$ partitioning: $\mathcal{A}(j,2) = \begin{cases} 1 & \text{for } \; j \in \{1,2,3\} \\ -1 & \text{for } \; j \in \{4,5,6\} \end{cases}$



$\boldsymbol{r}_{\mathrm{a}} = (\underline{2}, 0, 0, \underline{2}, 0, 0)$

$\mathcal{S}(2)\ \boldsymbol{r}_{\mathrm{a}} = \boldsymbol{r}_{\mathrm{a}}$

$\Rightarrow$ suppression law holds

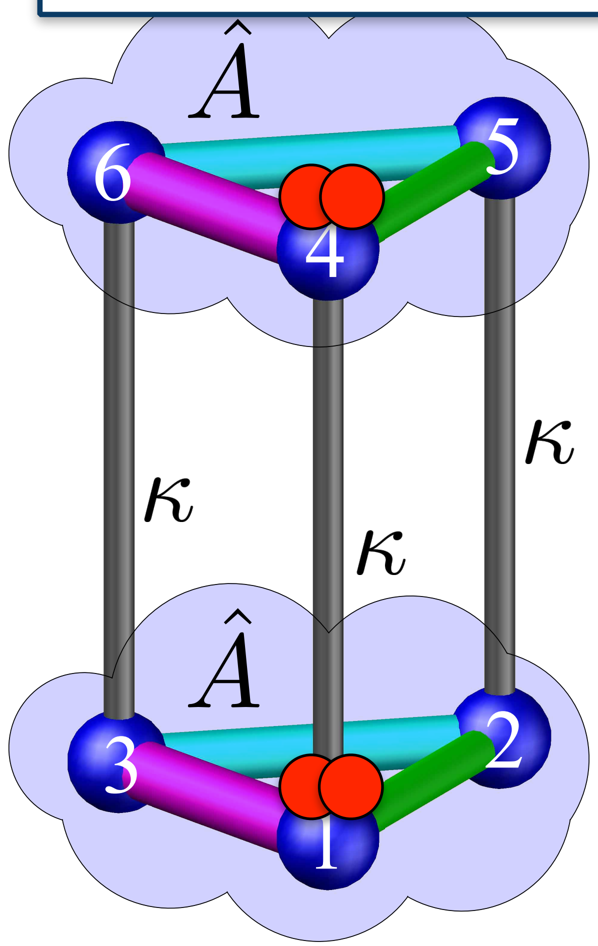
Suppressed final states: Odd particle number on any substructure

	$P_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{a}}, \boldsymbol{s}, \hat{U})$
$\boldsymbol{s}_1 = (3, 0, 0, 0, 1, 0)$	0
$\boldsymbol{s}_2 = (1, 1, 1, 0, 1, 0)$	0

Appendix E - Example: Bosonic Interference

Condition for suppression

- is **independent** on the **final** particle occupation within the substructure



$$\boldsymbol{r}_{\mathrm{a}} = (2, 0, 0, 2, 0, 0)$$

$$\mathcal{S}(2)\ \boldsymbol{r}_{\mathrm{a}} = \boldsymbol{r}_{\mathrm{a}}$$

$\Rightarrow$ suppression law holds

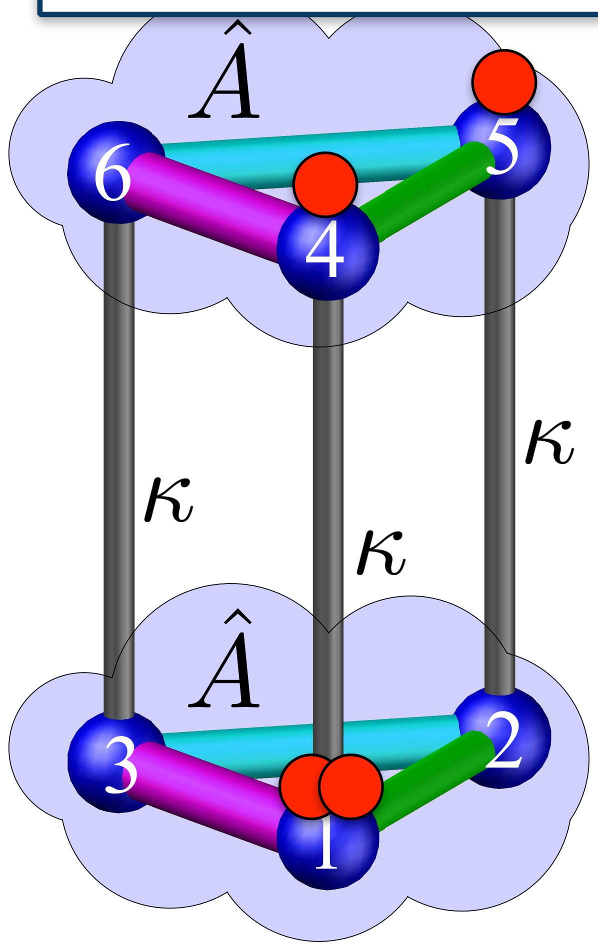
Suppressed final states: Odd particle number on any substructure

	$P_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{a}}, \boldsymbol{s}, \hat{U})$
$\boldsymbol{s}_1 = (3, 0, 0, 0, 1, 0)$	0
$\boldsymbol{s}_2 = (1, 1, 1, 0, 1, 0)$	0

Appendix E - Example: Bosonic Interference

Condition for suppression

- is **independent** on the **final** particle occupation within the substructure



$\boldsymbol{r}_{\mathrm{a}} = (2,0,0,2,0,0)$

$\mathcal{S}(2)\ \boldsymbol{r}_{\mathrm{a}} = \boldsymbol{r}_{\mathrm{a}}$

$\Rightarrow$ suppression law holds

$\boldsymbol{r}_{\mathrm{b}} = (\underline{2},0,0,\underline{1},\underline{1},0)$

$\mathcal{S}(2)\ \boldsymbol{r}_{\mathrm{b}} \neq \boldsymbol{r}_{\mathrm{b}}$

$\Rightarrow$ suppression law **NOT VALID**

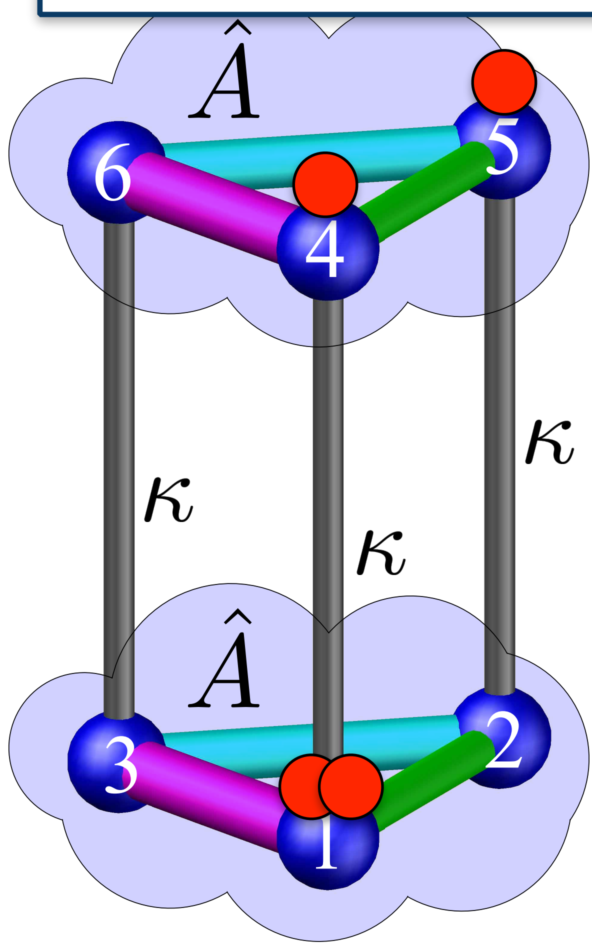
Suppressed final states: Odd particle number on any substructure

	$P_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{a}}, \boldsymbol{s}, \hat{U})$	$P_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{b}}, \boldsymbol{s}, \hat{U})$
$\boldsymbol{s}_1 = (3,0,0,0,1,0)$	0	$\neq 0$
$\boldsymbol{s}_2 = (1,1,1,0,1,0)$	0	$\neq 0$

Appendix E - Example: Bosonic Interference

Condition for suppression

- is **independent** on the **final** particle occupation within the substructure
- **depends** on the **initial** particle occupation within the substructure



$\boldsymbol{r}_{\mathrm{a}} = (2,0,0,2,0,0)$

$\mathcal{S}(2)\ \boldsymbol{r}_{\mathrm{a}} = \boldsymbol{r}_{\mathrm{a}}$

$\Rightarrow$ suppression law holds

$\boldsymbol{r}_{\mathrm{b}} = (\underline{2},0,0,\underline{1},\underline{1},0)$

$\mathcal{S}(2)\ \boldsymbol{r}_{\mathrm{b}} \neq \boldsymbol{r}_{\mathrm{b}}$

$\Rightarrow$ suppression law **NOT VALID**

$\boxed{\text{Suppressed final states: Odd particle number on any substructure}}$

	$P_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{a}}, \boldsymbol{s}, \hat{U})$	$P_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{b}}, \boldsymbol{s}, \hat{U})$
$\boldsymbol{s}_1 = (3,0,0,0,1,0)$	0	$\neq 0$
$\boldsymbol{s}_2 = (1,1,1,0,1,0)$	0	$\neq 0$

Appendix F - Realizations of higher dim. Graphs

Realizations in

- atomic lattices [A6,A7]

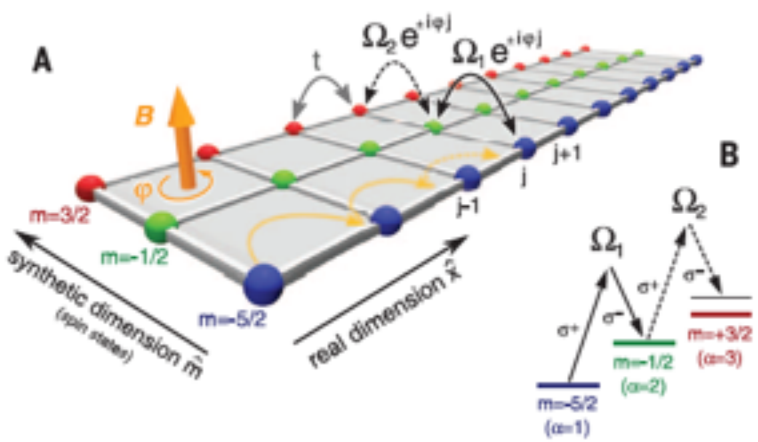


Figure reference: [A6]

- optomechanical systems [A8]

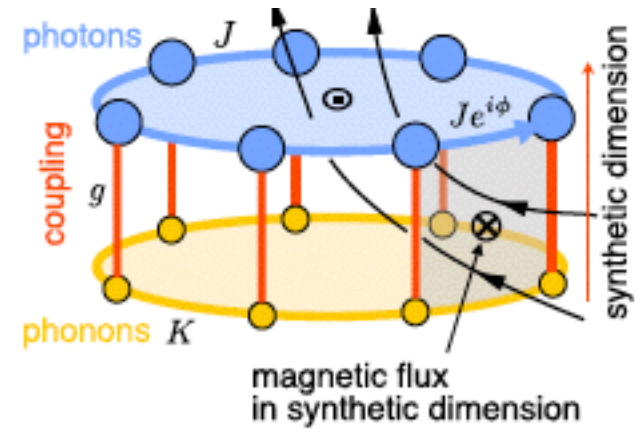


Figure reference: [A8]

- optical systems [A9]

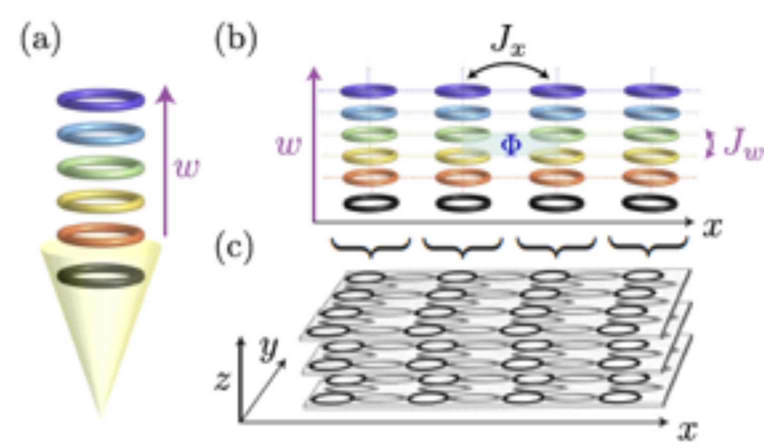


Figure reference: [A9]

[A6] Mancini et al., *Science,* **349**, 1510-1513 (2015)

[A7] Stuhl et al., *Science,* **349**, 1514-1518 (2015)

[A8] Schmidt et al., *Optica,* **2**, 635-641 (2015)

[A9] Ozawa et al., arXiv:1510.03910 (2015)

Appendix G - Independent Symmetries

- Symmetry operators $\mathcal{S}(\boldsymbol{p})$ are self-inverse and mutually commute
- Define: $$\Gamma = \{\boldsymbol{p} | \mathcal{S}(\boldsymbol{p})\boldsymbol{r} = \boldsymbol{r}\}$$

  and all possible sets $$\Lambda_k \subseteq \Gamma \;:\; \forall \boldsymbol{p} \in \Gamma \;\exists\; T \subseteq \Lambda_k \;:\; \prod_{\boldsymbol{p}_j \in T} \mathcal{S}(\boldsymbol{p}_j) = \mathcal{S}(\boldsymbol{p})$$
- Then, the number of independent symmetries of $\boldsymbol{r}$ is given by

$$\eta = \min\{|\Lambda_1|, |\Lambda_2|, \dots\}$$

- Example: $\boldsymbol{r}$ is invariant under $\mathcal{S}(2)$ $\mathcal{S}(8)$ and $\mathcal{S}(2,8)$

$$\Lambda_1 = \{2, 8, (2,8)\} \quad \Lambda_3 = \{2, (2,8)\}$$
$$\Lambda_2 = \{2, 8\} \quad \Lambda_4 = \{8, (2,8)\}$$
$$\Rightarrow \quad \eta = 2$$

Appendix H - Uncertainties

- Imperfect unitary [A10]: $\tilde{U}_{j,k} = U_{j,k}(1 + \delta_{j,k})$ $\quad\delta_{j,k} \in \mathbb{C}$

  mean deviation $||\delta|| = \langle|\delta_{j,k}|\rangle_{j,k}$ $\quad\delta_{j,k} \ll 1$

$$P_{\mathrm{B,F}}(\boldsymbol{r}, \boldsymbol{s}, \tilde{U}) \approx N||\delta||^2 P_{\mathrm{dist}}(\boldsymbol{r}, \boldsymbol{s}, U)$$

- Partial distinguishable particles:
  - via selected single-particle basis [A10, A11]

    ON basis $\{|\tilde{\phi}_1\rangle, \ldots, |\tilde{\phi}_N\rangle\}$ via Gram-Schmidt o.n.

    imperfect single-particle state

$$|\phi_j\rangle = \sum_{k=1}^{j} c_{j,k}|\tilde{\phi}_k\rangle$$

$$\mathcal{N}_{\mathrm{forbidden}} \lesssim \mathcal{N}_{\mathrm{supp}} \left(1 - \prod_{k=2}^{N} |c_{k,1}|^2\right)$$

  - Tensor-Permanent Approach [A11]

    upper bound is derived for $|P_{\mathrm{B}}(\boldsymbol{r}, \boldsymbol{s}, U) - P_{\mathrm{part}}(\boldsymbol{r}, \boldsymbol{s}, U)|$

[A10] Tichy et al., *Phys. Rev. Let.,* **113**, 020502 (2014)

[A11] Tichy, *Phys. Rev. A,* **91**, 022316 (2015)

Appendix G - Symmetry Operations

Self-inverse and mutually commuting *symmetry operations:*

$$\mathcal{S}(p) = \mathbb{1}^{\otimes \log_2(p/2)} \otimes \sigma_x \otimes \mathbb{1}^{\otimes \log_2(n/p)}$$

$$\dim\left(\mathcal{S}(p)\right) = n \times n$$

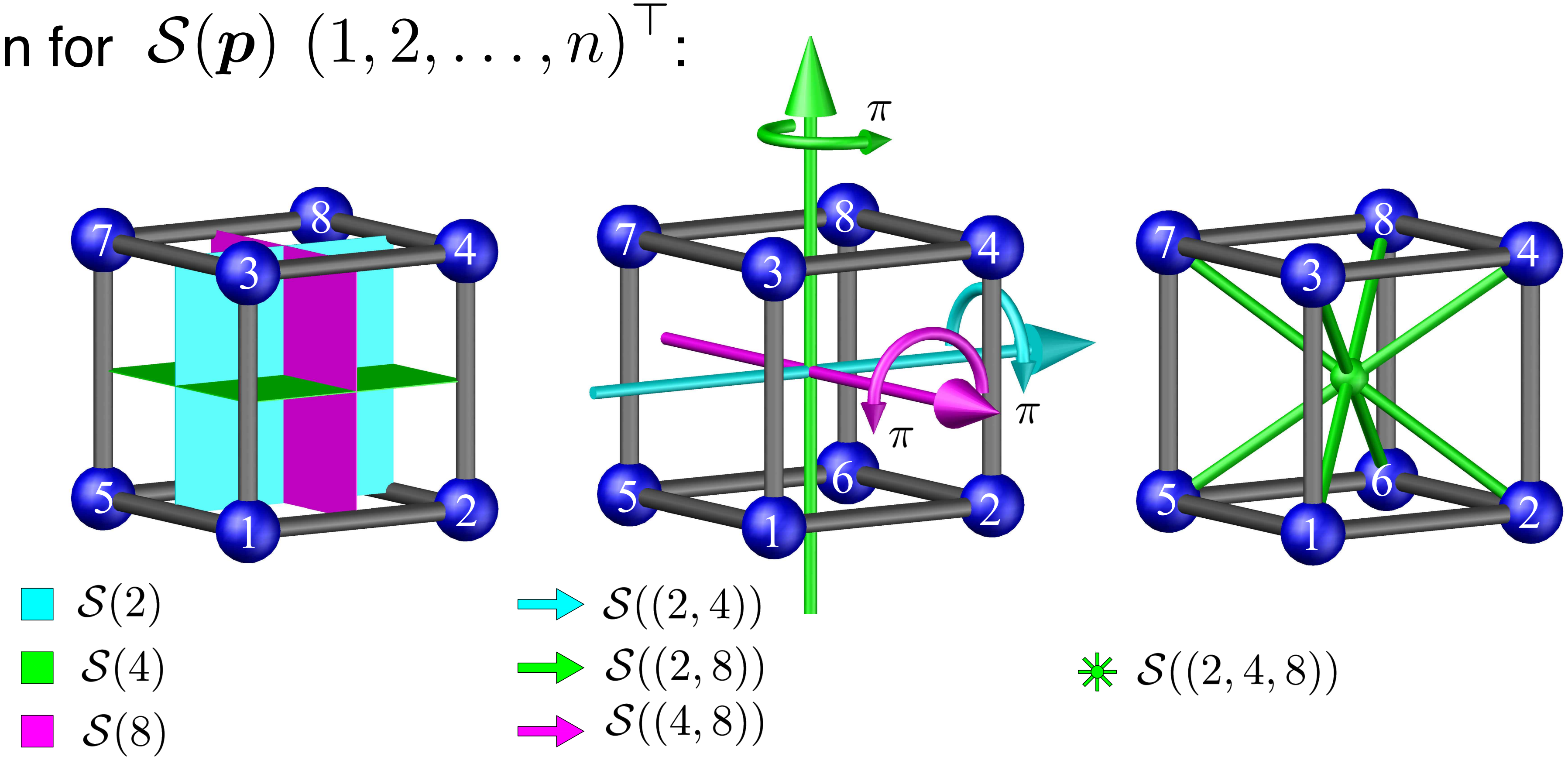
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Consecutive action of symmetry operations:

$$\boxed{\prod_{k=1}^{||\boldsymbol{p}||} \mathcal{S}(p_k) = \mathcal{S}(\boldsymbol{p})}$$

Illustration for $\mathcal{S}(\boldsymbol{p})\ (1, 2, \ldots, n)^\top$:

<u>Example:</u>
3-dim HC



Appendix H - Suppression Ratio for Fermions

η …number of independent symmetries of the initial state

Bosons: for large n

$$\boxed{\frac{\mathcal{N}^{\mathrm{B}}_{\mathrm{supp}}}{\mathcal{N}^{\mathrm{B}}_{\mathrm{all}}} \approx 1 - \frac{1}{2^\eta}}$$

Fermions: for $n \gg N$

$$\boxed{\frac{\mathcal{N}^{\mathrm{B}}_{\mathrm{supp}}}{\mathcal{N}^{\mathrm{B}}_{\mathrm{all}}} \approx 1 - \frac{N!}{2^{\eta N}\left[\left(\frac{N}{2^\eta}\right)!\right]^{2^\eta}}}$$