Treating Bose-Hubbard model by means of classical mechanics
or
Quantum-classical transition in many-body systems

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Quantum chaos

Main conjecture: Energy spectrum of a quantum system with underlying chaotic classical dynamics has the same statistical properties as spectrum of the random matrix (of the same universality class).

Distribution of normalized distances between the nearest levels

$s = (E_{j+1} - E_j)\rho(E_j)$ – normalized distances

$P(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right)$ – chaotic systems

$P(s) = e^{-s}$ – regular (integrable) systems

Integrated distribution $I(s) = \int_0^s P(s')\mathrm{d}s'$

[1] A.R.Kolovsky and A.Buchleitner, *Quantum chaos in Bose-Hubbard model*, Europhys. Lett. **68**, 632 (2004).

[2] A.R.Kolovsky, *Persistent current of atoms in a ring optical lattice*, New J. of Phys. **8**, 197 (2006).

Bose-Hubbard model

$$\widehat{H} = E_0 \sum_{l=1}^{L} \hat{n}_l - \frac{J}{2} \sum_{l=1}^{L} \left(\hat{a}_{l+1}^\dagger \hat{a}_l + h.c. \right) + \frac{U}{2} \sum_{l=1}^{L} \hat{n}_l (\hat{n}_l - 1) \,, \quad \hat{n}_l = \hat{a}_l^\dagger \hat{a}_l$$

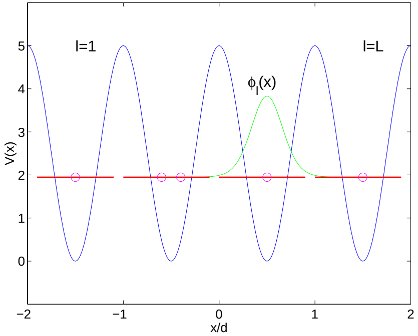
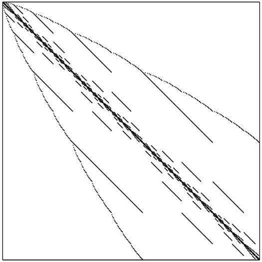


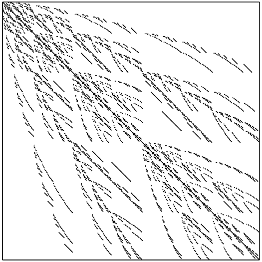
Figure: Cold atoms (open circles) in an optical lattice. Green line is the single-particle Wannier function.

Bose-Hubbard model

$$\widehat{H} = -\frac{J}{2}\sum_{l=1}^{L}\left(\hat{a}^{\dagger}_{l+1}\hat{a}_l + h.c.\right) + \frac{U}{2}\sum_{l=1}^{L}\hat{n}_l(\hat{n}_l - 1)$$

Fock basis: $|\mathbf{n}\rangle = |n_1, n_2, \ldots, n_L\rangle$ where $\sum_l n_l = N$





Canonical transformation: $\hat{b}_k = (1/\sqrt{L})\sum_l \exp(i2\pi kl/L)\hat{a}_l$

$$\widehat{H} = -J\sum_k \cos\left(\frac{2\pi k}{L}\right)\hat{b}^{\dagger}_k\hat{b}_k + \frac{U}{2L}\sum_{k_1,k_2,k_3,k_4}\hat{b}^{\dagger}_{k_1}\hat{b}_{k_2}\hat{b}^{\dagger}_{k_3}\hat{b}_{k_4}\tilde{\delta}(k_1 - k_2 + k_3 - k_4)$$

Transition to chaos

$$\widehat{H} = -\frac{J}{2}\sum_{l=1}^{L}\left(\hat{a}^{\dagger}_{l+1}\hat{a}_l + h.c.\right) + \frac{U}{2}\sum_{l=1}^{L}\hat{n}_l(\hat{n}_l - 1) + \sum_{l=1}^{L}\epsilon_l\hat{n}_l$$

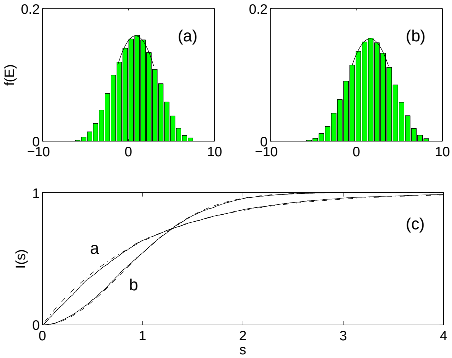


Figure: Density of states (upper panels) and integrated level spacing distribution (lower panel) as compared to the Poisson and Wigner-Dyson distributions. Parameters are $N = 7$, $L = 9$, $J = 1$, $\epsilon = 0.2$, $U = 0.02$ (a) and $U = 0.2$ (b).

Classical limit

$$H = -\frac{J}{2}\sum_{l=1}^{L}(a_{l+1}^* a_l + c.c.) + \frac{g}{2}\sum_{l=1}^{L}|a_l|^4 \,, \quad g = \frac{UN}{L}$$

Coherent $SU(L)$ states: $|\mathbf{a}\rangle = \frac{1}{\sqrt{N!}}\left(\sum_{l=1}^{L} a_l \hat{a}_l^\dagger\right)^N |vac\rangle$

Equation on the Husimi function $f(\mathbf{a},t) = |\langle \mathbf{a}|\Psi(t)\rangle|^2$

$$\frac{\partial f}{\partial t} = \{H, f\} + O\left(\frac{1}{N}\right)$$

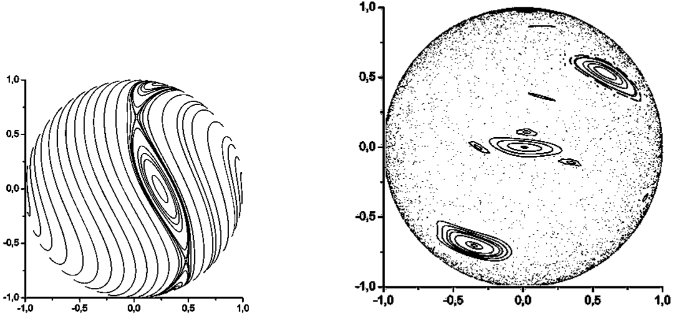
If the initial distribution $f(\mathbf{a}, t=0)$ is a δ-function then the above equation is equivalent to DNLSE equation:

$$i\frac{\partial a_l}{\partial t} = -\frac{J}{2}(a_{l+1} + a_{l-1}) + g|a_l|^2 a_l$$

[3] F. Trimborn, D. Witthaut, and H. J. Korsch, *Exact number conserving phase-space dynamics of the L-site Bose-Hubbard model*, Phys. Rev. A **77**, 043631 (2008)

Phase space and semiclassical quantization

$$H = -\frac{J}{2}\sum_{l=1}^{L}(a_{l+1}^* a_l + c.c.) + \frac{g}{2}\sum_{l=1}^{L}|a_l|^4 \,, \quad \sum_{l=1}^{L}|a_l|^2 = 1$$



$$\rho_{qu}(E) \approx \frac{\mathcal{N}(N)}{N}\rho_{cl}\left(\frac{E}{N}\right) \,, \quad \int \rho_{cl}(E)\mathrm{d}E = 1$$

Density of states

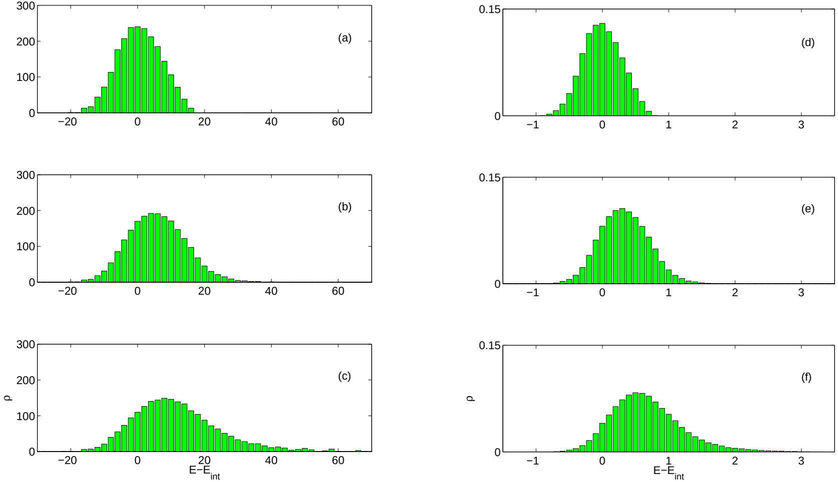


Figure: Density of states of the 5-site BH model for $N = 20$, panels (a-c), as compared to the classical 'density of states', panels (d-f). The energy is measured with respect to the mean interaction energy $E_{int} = gN$.

Low-energy stability islands

Relative volumes of the regular and chaotic components?

$v_{reg} = v_{reg}(E) \to 1$ if $E \to E_{min} \approx -J$

Effective Hamiltonians:

$H_{eff} = (\delta_k + g)I + g\sqrt{I^2 - 4M^2}\cos(2\theta)\ , \quad |M| \le I/2\ , \delta_k = J[1 - \cos(2\pi k/L)]$

Quantizing these effective Hamiltonians we have

$$E^{(k)} = E_0^{(k)} + \hbar\Omega^{(k)}(n + 1/2)\ , \quad \Omega^{(k)} \sim \sqrt{g}k$$

which is nothing else as the Bogoliubov spectrum for low-energy excitations of a BEC.

[4] A.R.Kolovsky, *Semiclassical quantization of the Bogoliubov spectrum*, Phys. Rev. Lett. **99**, 020401 (2007); Phys. Rev. E **76**, 026207 (2007)

Bogoliubov spectrum

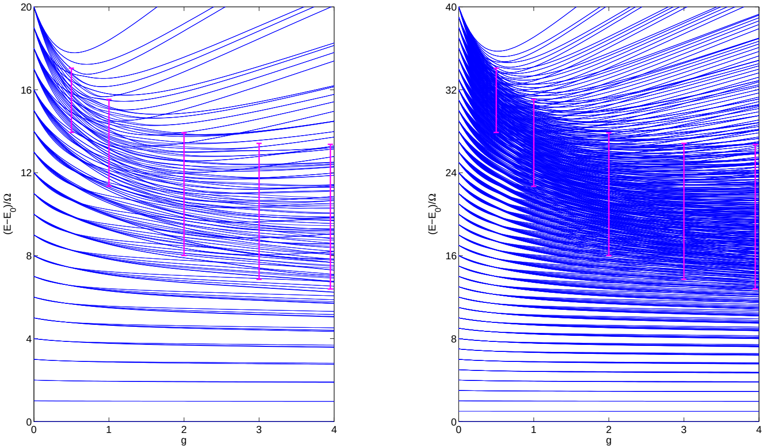
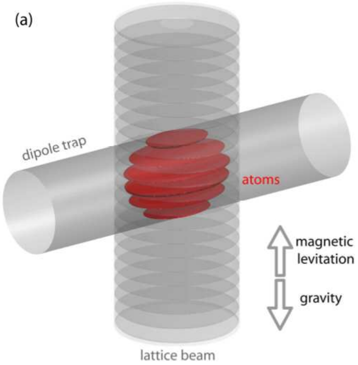


Figure: Energy spectrum of the 3-site BH model for $N = 20$ (left) and $N = 40$ (right) as the function of macroscopic interaction constant $g = UN/L$. The energy is measured with respect to the ground energy and is scaled according to the Bogoliubov frequency $\Omega(g)$.

Bloch oscillations

$$\widehat{H} = -\frac{J}{2}\sum_l \left(\hat{a}^\dagger_{l+1}\hat{a}_l + h.c.\right) + \frac{U}{2}\sum_l \hat{n}_l(\hat{n}_l - 1) + dF\sum_l l\hat{n}_l$$



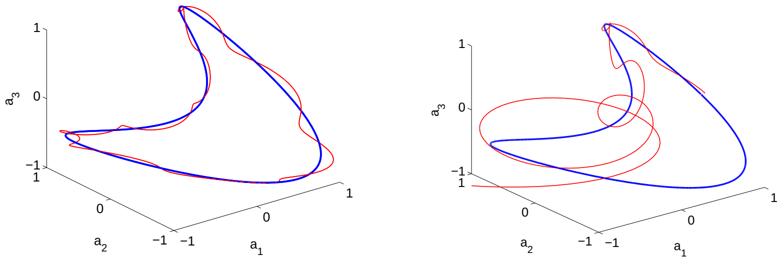
$$\widehat{H}(t) = -\frac{J}{2}\sum_l \left(\hat{a}^\dagger_{l+1}\hat{a}_l e^{-i\omega_B t} + h.c.\right) + \frac{U}{2}\sum_l \hat{n}_l(\hat{n}_l - 1)\ , \quad \omega_B = \frac{dF}{\hbar}$$

Semiclassical approach

$$i\dot{a}_l = -\frac{J}{2}\left(a_{l+1}e^{i\omega_B t} + a_{l-1}e^{-i\omega_B t}\right) + g|a_l|^2 a_l$$

Solution for the uniform initial condition

$$a_l(t) = \frac{1}{\sqrt{L}}\exp\left(i\frac{J}{F}\sin(\omega_B t) - igt\right) \ , \quad p(t) = p_0 \sin(\omega_B t)$$



Stability analysis

$$F_{cr} \approx \begin{cases} 3g \,, & F < 2J \\ \sqrt{10gJ} \,, & F > 2J \end{cases} .$$

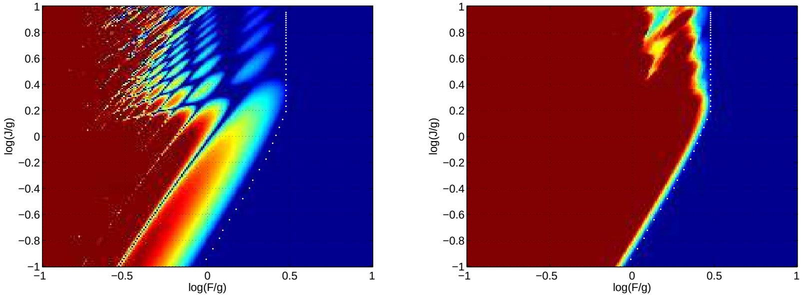


Figure: Increment of the dynamical instability (sum of the positive Lyapunov exponents) for $L = 3$ (left) and $L = 15$ (right).

[6] A.R.Kolovsky, e-print: cond-mat/0412195 (2004).

[7] Yi Zheng, M. Kostrun, and J. Javanainen, Phys. Rev. Lett. **93** 230401 (2004).

Strong *vs.* weak field regime

$$p(t) = \begin{cases} \exp(-\gamma t)\sin(\omega_B t) & , \quad F \ll F_{cr} \\ \exp\left(-2\bar{n}[1-\cos(Ut/\hbar)]\right)\sin(\omega_B t) & , \quad F \gg F_{cr} \end{cases}$$

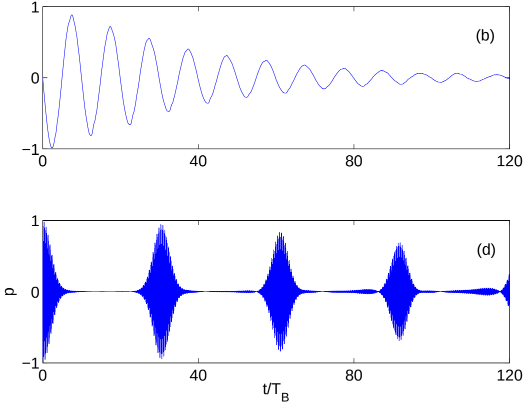


Figure: Numerical simulations of Bloch oscillations of interacting atoms for $F < F_{cr}$ (upper panel) and $F > F_{cr}$ (lower panel).

Strong field – quasiperiodic Bloch oscillations

$$p(t) = \exp\left(-2\bar{n}[1 - \cos(Ut/\hbar)]\right)\sin(\omega_B t)$$

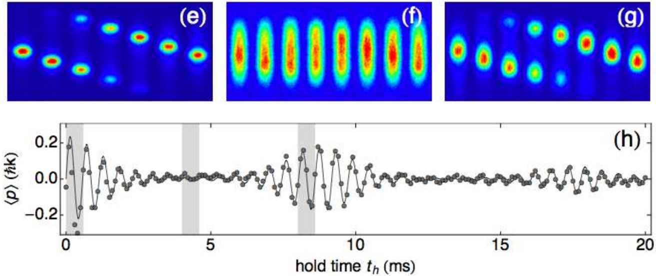


Figure: Dynamics of the mean momentum for $F > F_{cr}$ according to Ref. [11].

[9] A.R.Kolovsky, *New Bloch period for interacting cold atoms in 1D optical lattices*, Phys. Rev. Lett. **90**, 213002 (2003).

[11] F.Meinert, M.J.Mark, E.Kirilov, K.Lauber, P.Weinmann, M.Gröbner, and H.-C. Nägerl, *Interaction-induced quantum phase revivals and evidence for the transition to the quantum chaotic regime in 1D atomic Bloch oscillations*, (2013).

Weak field – decaying Bloch oscillations

$$p(t) = \exp(-\gamma t)\sin(\omega_B t)\ , \quad \gamma \sim \bar{n}^2 U^2$$

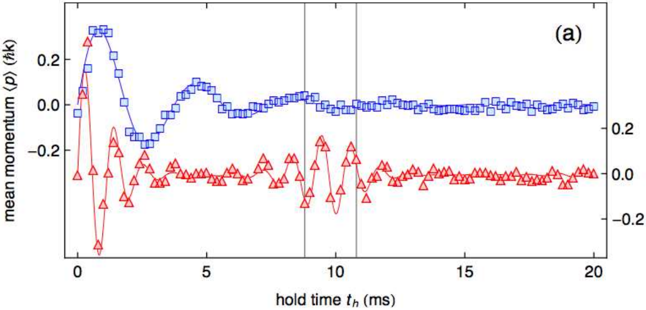


Figure: Dynamics of the mean momentum for $F < F_{cr}$ according to Ref. [11].

[10] A.Buchleitner and A.R.Kolovsky, *Interaction-induced decoherence of atomic Bloch oscillations*, Phys. Rev. Lett. **91**, 253002 (2003).

[11] F.Meinert, M.J.Mark, E.Kirilov, K.Lauber, P.Weinmann, M.Gröbner, and H.-C. Nägerl, *Interaction-induced quantum phase revivals and evidence for the transition to the quantum chaotic regime in 1D atomic Bloch oscillations*, (2013).

Quantum ensemble

$$\frac{\partial f}{\partial t} = \{H, f\} + O\left(\frac{1}{N}\right) \ , \quad f(\mathbf{a}, t=0) = |\langle \mathbf{a}|BEC\rangle|^2$$

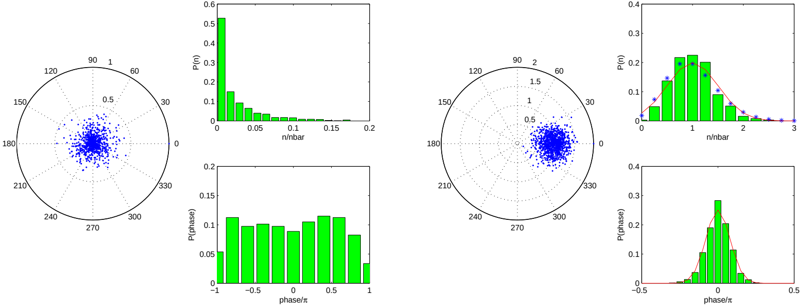


Figure: Ensemble of classical trajectories in the Bloch (amplitudes $b_{k\neq 0}$) and Wannier (amplitudes a_l) representations. Parameters are $L = 5$ and $N = 20$.

[8] A.R.K., H.J.Korsch, and E.M.Graefe, *Bloch oscillations of Bose-Einstein condensates: Quantum counterpart of dynamical instability*, Phys. Rev. A **80**, 023617 (2009).

Classical *vs.* quantum dynamics: internal decoherence

$$\frac{\partial f}{\partial t} = \{H, f\} + O\left(\frac{1}{N}\right) , \quad f(\mathbf{a}, t=0) = |\langle \mathbf{a}|BEC\rangle|^2$$

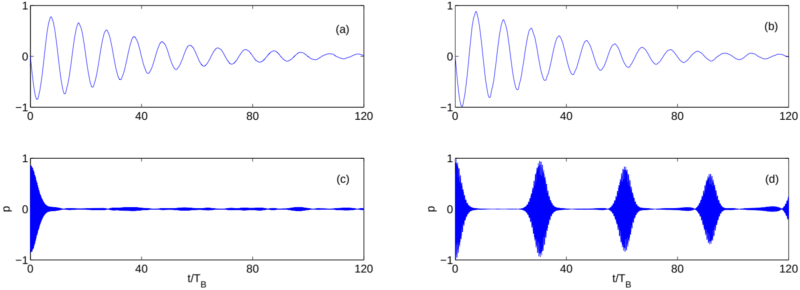


Figure: Dynamics of the mean momentum for $dF/J = 0.1$ and $dF/J = 10$, calculated by using the classical (left) and quantum (right) approaches.

Conclusions

- We addressed the energy spectrum of the Bose-Hubbard model by using "semiclassical" approach based on classical Hamiltonian.
- In particular, we obtained the Bogoliubov spectrum for low-energy excitations of a BEC by quantizing classical tori where $\hbar_{eff} = 1/N$.
- We addressed dynamics of the Bose-Hubbard system induced by a static field (Bloch oscillations).
- Using classical analysis we predicted two qualitatively different regimes of Bloch oscillations which have been observed in the laboratory experiment.
- Remarkably, regime of decaying Bloch oscillations is perfectly reproduced by pure classical dynamics. Here we have a loop: underlying classical chaos $\rightarrow$ quantum chaos $\rightarrow$ internal decoherence $\rightarrow$ classical dynamics.

[12] A.R.Kolovsky, *Bose-Hubbard Hamiltonian: Quantum Chaos approach* , Int. J. of Modern Physics B **30** (2016), 1630009 (2016) [arXiv:1507.0341 (2015)]