A Two-Phase Continuum Theory for Windblown Sand

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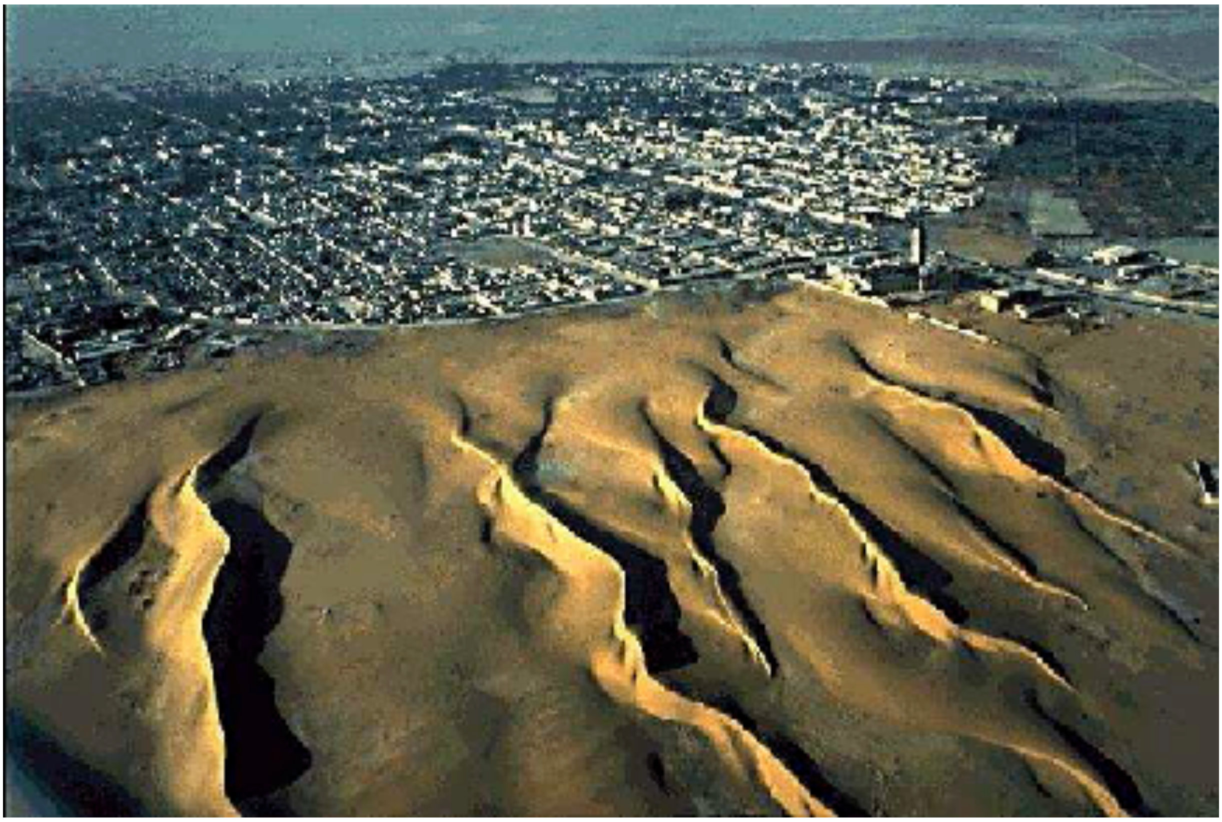
Suspension and transport of sand by saltation and gas velocity fluctuations in a turbulent shearing flow

Desertification

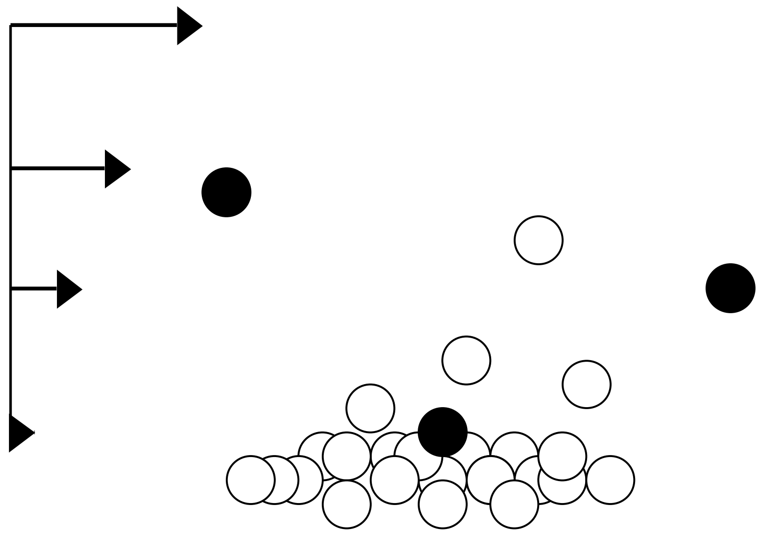
Nouakchott, Mauritania



Dune formation and migration associated with wind-blown sand

Saltation
(sauter: to jump)

A grain lifted from the bed by a strong turbulent eddy is accelerated by the mean turbulent shear flow and, upon colliding with the bed, rebounds and ejects other particles.

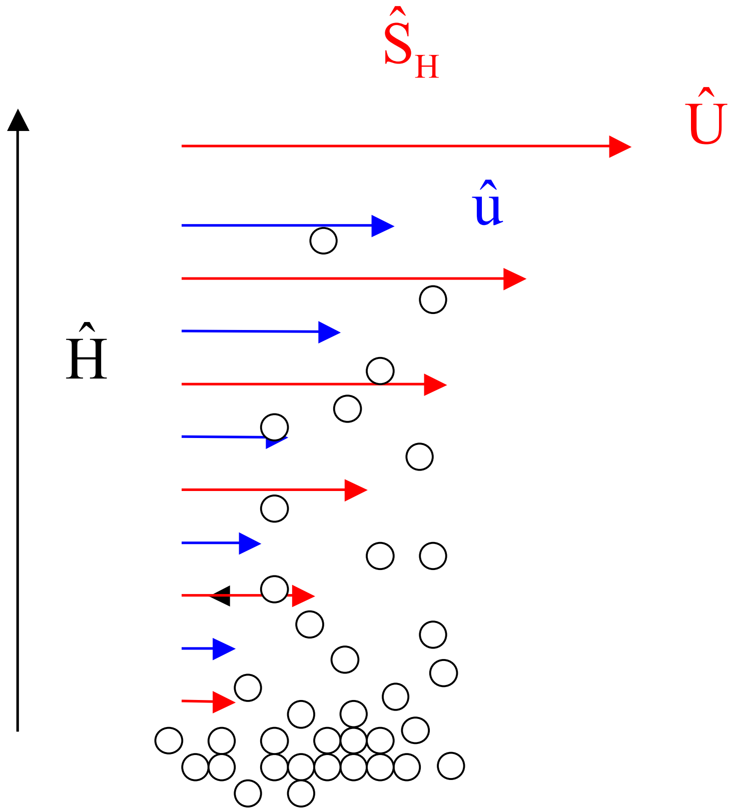


The drag of the particles on the wind eventually limits the number of particles that can participate.

At higher wind speeds, turbulent suspension, associated with the velocity fluctuations of the wind, becomes increasingly important, and collisions between particles above the bed may play a role in their suspension and transport.

Aeolian Transport

Two-phase, turbulent flow



Friction velocity / Shields parameter

$$\hat{u}^* \equiv \left(\hat{S}_H / \hat{\rho}^f\right)^{1/2} \qquad S^* \equiv \hat{S}_H / \left(\hat{\rho}^s \hat{g} \hat{d}\right)$$

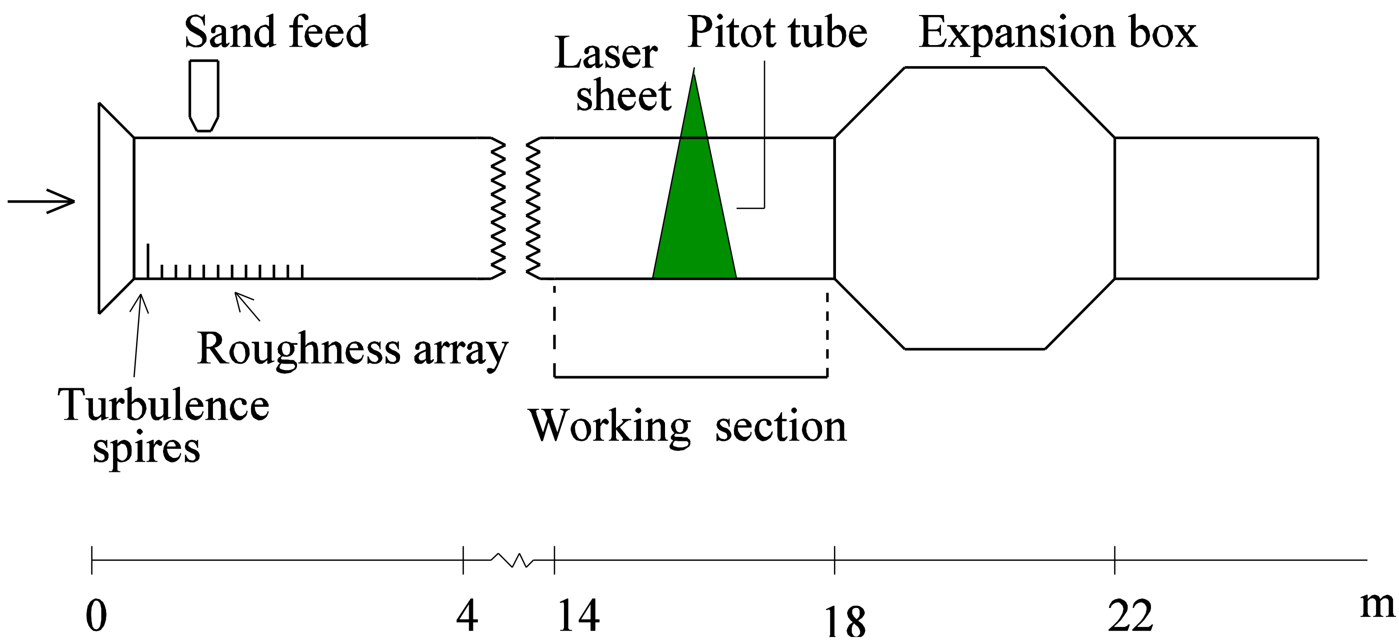
Air drag: $\nu \hat{D}(\hat{U} - \hat{u})$

$$\hat{D} \equiv \frac{3}{10}\frac{\hat{\rho}^f}{\hat{d}}\left[(\hat{U} - \hat{u})^2 + 3\hat{T}\right]^{1/2} + \frac{18\hat{\mu}^f}{\hat{d}^2}$$

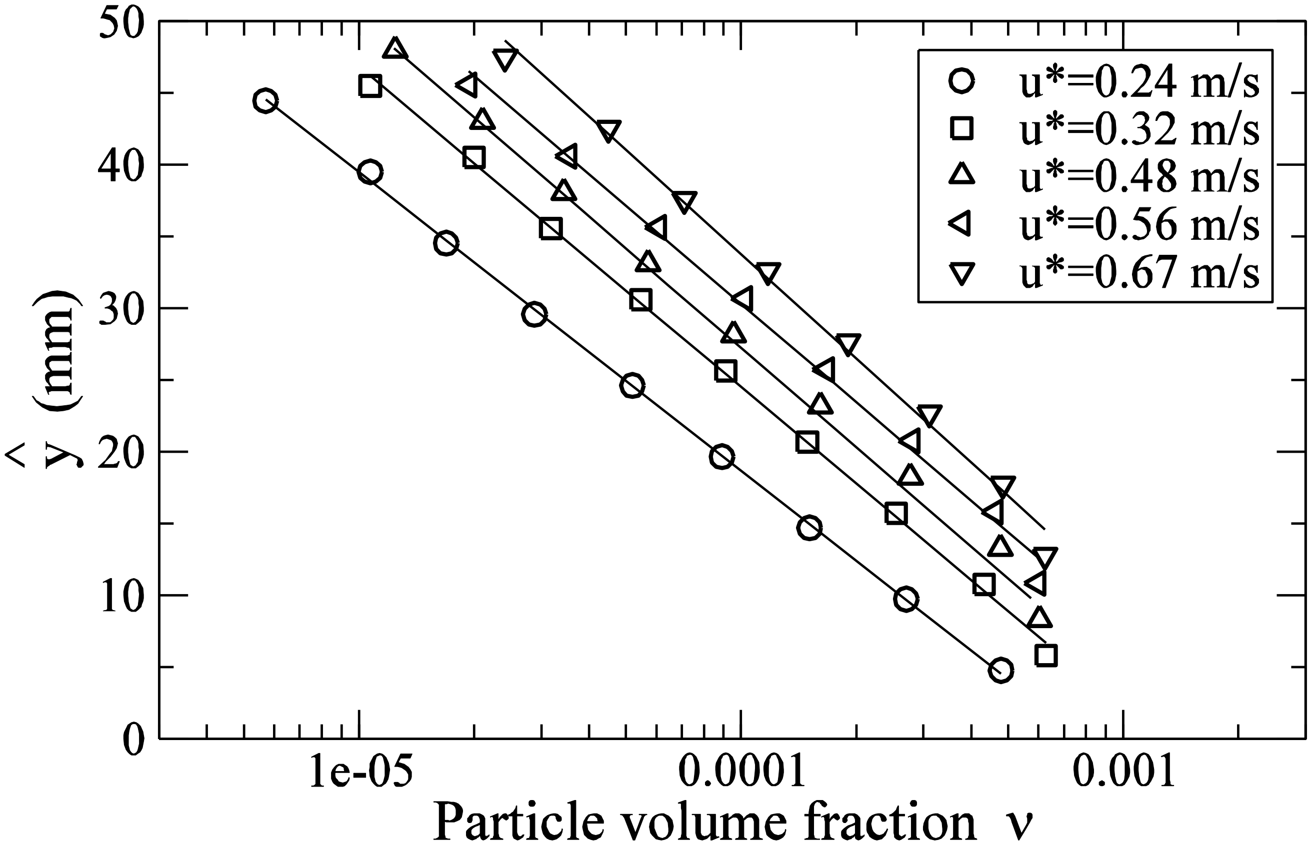
Experiments

Creyssels, Dupont, Valance (Rennes)
Ould El Moctar (Nantes), Rasmussen (Aarhus)

Wind-Sand Tunnel, Aarhus, Denmark



Volume Fraction



$$\nu = \nu_0 \exp\left(-\frac{\hat{y}}{\hat{\ell}}\right) \qquad \hat{\ell} \approx 40\hat{d}$$

Continuum Theory

Particle horizontal momentum

$$0 = \frac{ds}{dy} + \frac{\nu D}{\sigma}(U - u)$$

Particle vertical momentum:

$$0 = -\frac{dp}{dy} - \nu$$

Particle fluctuation energy

$$0 = -\frac{dq}{dy} + su' - \gamma$$

Particle pressure

$$p = \nu T$$

Particle shear stress

?

Particle energy flux

?

Continuum Theory

Single particle trajectories without vertical drag

$$\text{Upward:}\quad \xi'_y \frac{d\xi'_x}{dy} = D(U-\xi'_x)$$

$$\text{Downward:}\quad \xi'_y \frac{d\xi_x}{dy} = -D(U-\xi_x)$$

Multiply by ν, sum, and average

$$\overline{\nu\xi_y'^2 \frac{d}{dy}}(\xi'_x+\xi_x) = \nu\overline{D\xi'_y(\xi'_x-\xi_x)}$$

$$2u \equiv \overline{(\xi'_x+\xi_x)}$$

$$p \equiv \nu\overline{\xi_y'^2} = \nu T \quad 2s \equiv \nu\overline{\xi'_y(\xi'_x-\xi_x)}$$

$$p\frac{du}{dy} = \alpha D s$$

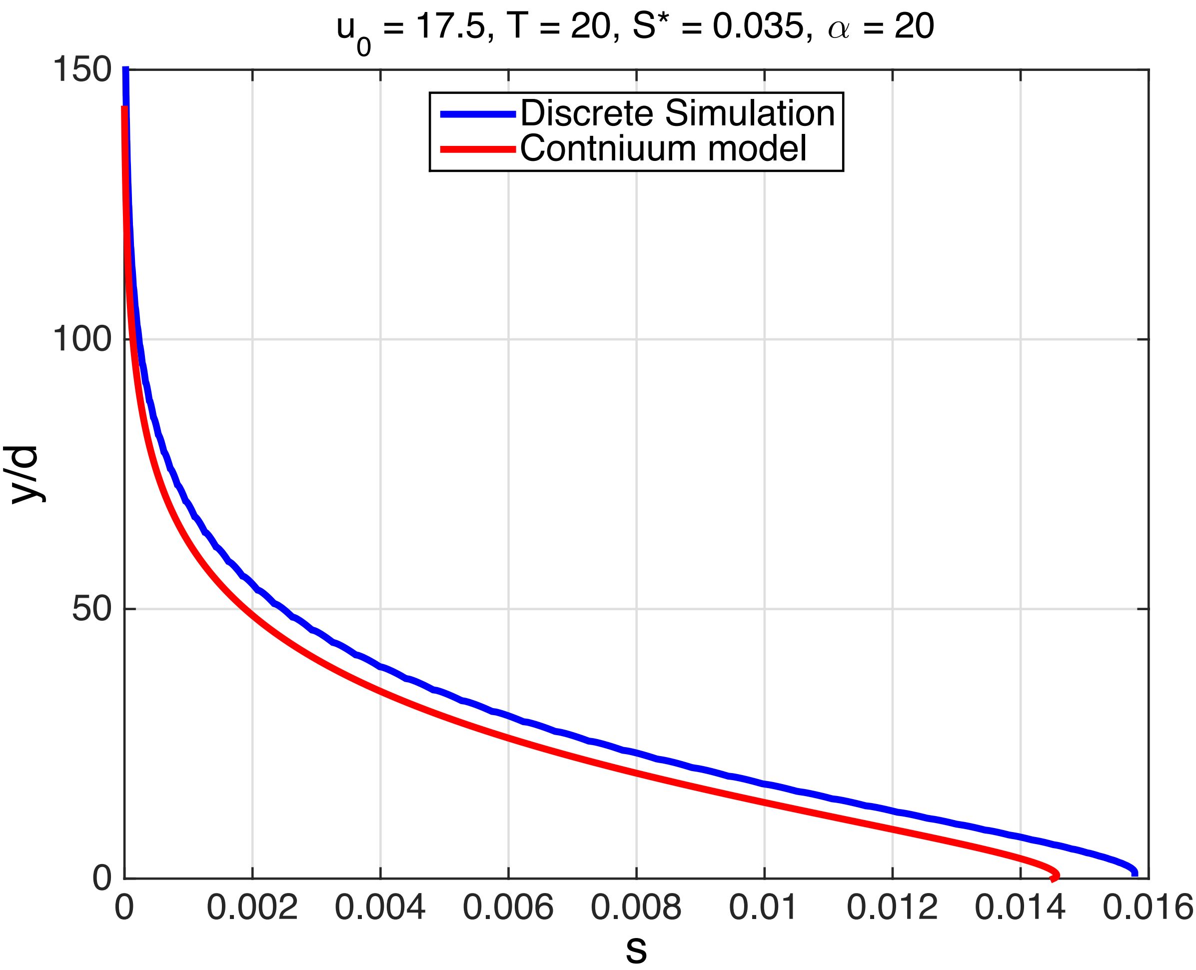
$$2q \equiv \nu\overline{\xi'_y\left[(\xi'_x-u)^2+\xi_y'^2\right]} + \nu\overline{\xi_y\left[(\xi_x-u)^2+\xi_y^2\right]}$$

$$B\nu\left[\overline{\xi'_y(\xi'_x-\xi_x)}\,\overline{(\xi'_x+\xi_x)}/2 - \overline{\xi'_y(\xi'_x-\xi_x)}u\right]$$

$$q = 0$$

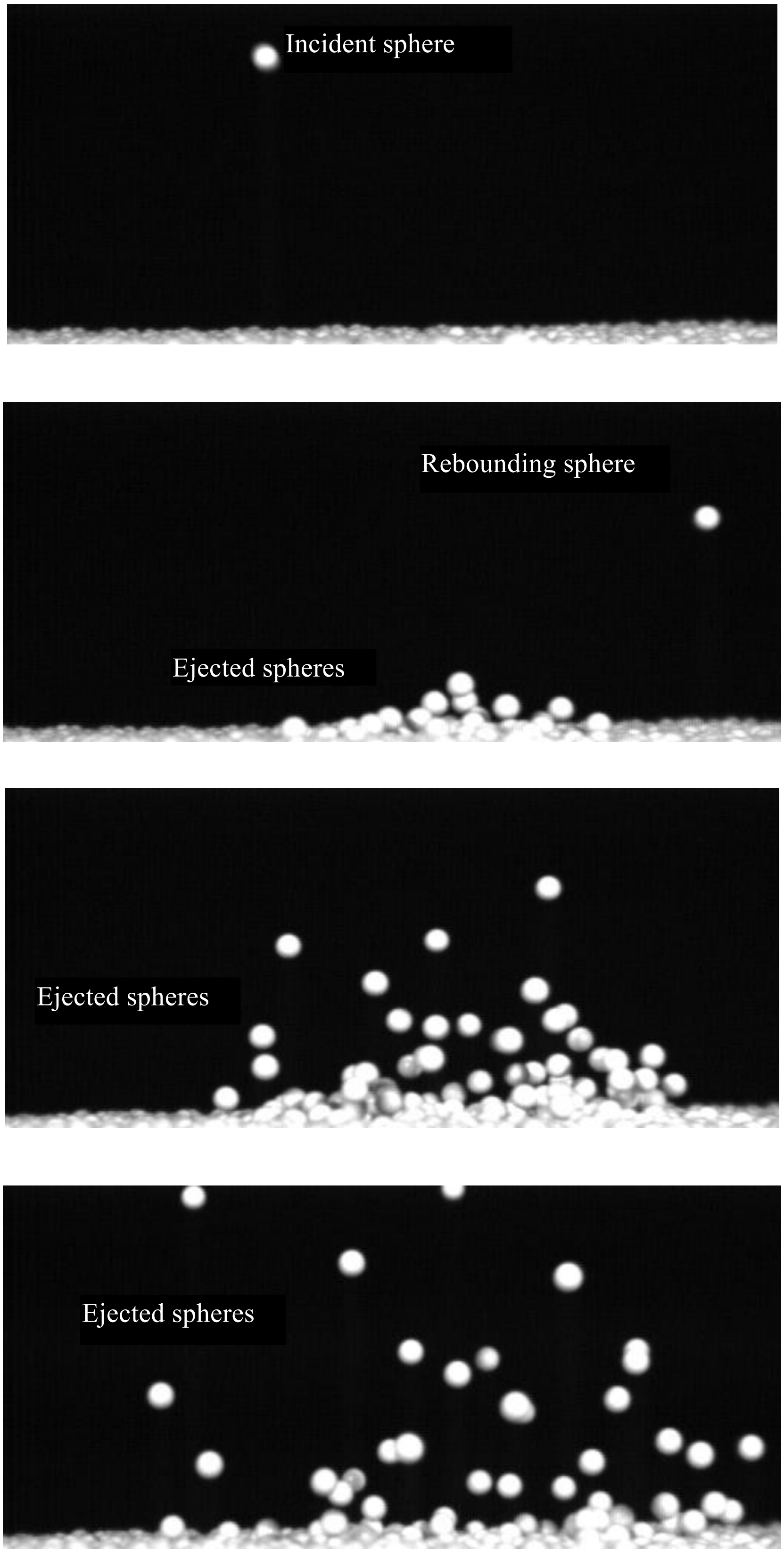
Particle Shear Stress

Continuum versus discrete simulation



Splash

Beladjine, et al. Phys. Rev. E 75, 061305 (2007)



Momentum of rebounding particles

$$\bar{\xi}' = e(\boldsymbol{\xi})\xi = (0.87 - 0.72\sin\theta)\xi$$

$$\bar{\xi}'_y = \varepsilon(\boldsymbol{\xi})\left|\xi_y\right| = \left(\frac{0.30}{\sin\theta} - 0.15\right)\left|\xi_y\right|$$

Total number N of particles

$$N(\boldsymbol{\xi}) = \begin{cases} 1 + 13(1 - e^2)\left(\dfrac{\xi}{40} - 1\right), & \text{if } \xi > 40 \\ 1, & \text{if } 1 \le \xi \le 40 \\ 0, & \text{if } \xi < 1 \end{cases}$$

Velocity distribution function

$$f(\boldsymbol{\xi}) = \frac{n_0}{2\pi T_0} \exp\left[\frac{-(\xi_x - u_0)^2 - \xi_y^2}{2T_0}\right]$$

$$(\nu_0 = \pi n_0 / 6)$$

Mass flux

$$\dot{m} = \int_{\xi_y \le 0} (N-1)\xi_y f(\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

$$= \frac{13}{2\pi}\frac{n_0 T_0^2}{u_0(40-u_0)^2}\left[0.24 + 0.63\left(\frac{\pi T_0}{20 u_0}\right)^{1/2}\right] e^{\frac{-(40-u_0)^2}{2T_0}}$$

$$-\frac{74\sqrt{2}}{\pi}\frac{n_0}{T_0} e^{\frac{-u_0^2}{2T_0}}$$

Momentum flux

$$\dot{\mathbf{M}} = -\frac{\pi}{6}\int_{\xi_y \le 0} (\bar{\boldsymbol{\xi}}' - \boldsymbol{\xi})\xi_y f(\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

$$\dot{M}_x = \nu_0 T_0 \left(0.35 + 0.07\frac{u_0}{T_0^{1/2}} - 0.33\frac{T_0^{1/2}}{u_0}\right)$$

$$\dot{M}_y = \nu_0 T_0 \left[0.12\left(\frac{u_0}{T_0^{1/2}} + \frac{T_0^{1/2}}{u_0}\right) - 0.08\right] + \frac{\nu_0 T_0}{2}$$

With $\dot{M}_y = p = \nu_0 T_0, \ \frac{u_0}{T_0^{1/2}} = 4.6$

Boundary-Value Problem

Steady, fully-developed flow: $\dot{m} = 0$

$u_0 = 20, \ T_0 = 20$ and $s_0 \equiv \dot{M}_x = 0.6 \nu_0 T_0$

$$\frac{d\nu}{dy} = -\frac{\nu}{T_0}$$

$$\frac{du}{dy} = 20D\frac{s}{p}$$

$$D = \frac{0.3}{\sigma}\left[(U-u)^2 + 3T_0\right]^{1/2} + \frac{18.3}{\sigma R}$$

$$\frac{ds}{dy} = -\nu D(U-u) \qquad \sigma = \frac{\rho^s}{\rho^f} \qquad R = \frac{d(gd)^{1/2}}{\mu^f/\rho^f}$$

$$\frac{dU}{dy} = \frac{(S^*-s)\sigma}{\left[(S^*-s)\sigma\right]^{1/2}\kappa\left(y+y_0\right)}$$

$y = 0: \; u_0 = 20, \; T_0 = 20, \; s = 0.6 v_0 T_0, \; U = 0$

$y = 210: \; s = 0$ Parameter: v_0

Include suspension by the turbulent velocity fluctuations

Particle vertical momentum

$$0 = -\frac{d\hat{p}}{d\hat{y}} - \hat{\rho}^s v' \hat{g} + \hat{D} \overline{v' \Delta \hat{V}}$$

Turbulent suspension force

$$\overline{v' \Delta \hat{V}} = -\frac{\hat{\mu}^T}{\hat{\rho}^f} \frac{d\overline{v}}{d\hat{y}}$$

Turbulent viscosity

$$\hat{\mu}^T = \hat{\rho}^f \frac{0.09}{0.165} \kappa \left(\hat{y} + \hat{y}_0 \right) \hat{k}^{1/2}$$

Velocity fluctuations

$$\hat{k}^{1/2} = \frac{(0.09)^{1/2}}{0.165}\left[\frac{(\hat{S}_H - \hat{s})}{\hat{\rho}^f}\right]^{1/2}$$

Boundary-Value Problem

Steady, fully-developed flow: $\dot{m} = 0$

$$u_0 = 20,\ T_0 = 20 \text{ and } s_0 \equiv \dot{M}_x = 0.6 v_0 T_0$$

$$\frac{dv}{dy} = -\frac{v}{T_0 + D\mu^T}$$

$$\mu^T \equiv \frac{0.09}{0.165}\frac{1}{\sigma}k^{1/2}0.41\left(y + y_0\right)$$

$$k^{1/2} \equiv \frac{(0.09)^{1/2}}{0.165}\left[(S^* - s)\sigma\right]^{1/2}$$

$$\frac{du}{dy} = 20D\frac{s}{p} \qquad D = \frac{0.3}{\sigma}\left[(U-u)^2 + 3T_0\right]^{1/2} + \frac{18.3}{\sigma R}$$

$$\frac{ds}{dy} = -vD(U-u) \qquad \sigma = \frac{\rho^s}{\rho^f} \qquad R = \frac{d(gd)^{1/2}}{\mu^f/\rho^f}$$

$$\frac{dU}{dy} = \frac{(S^* - s)\sigma}{\mu^T}$$

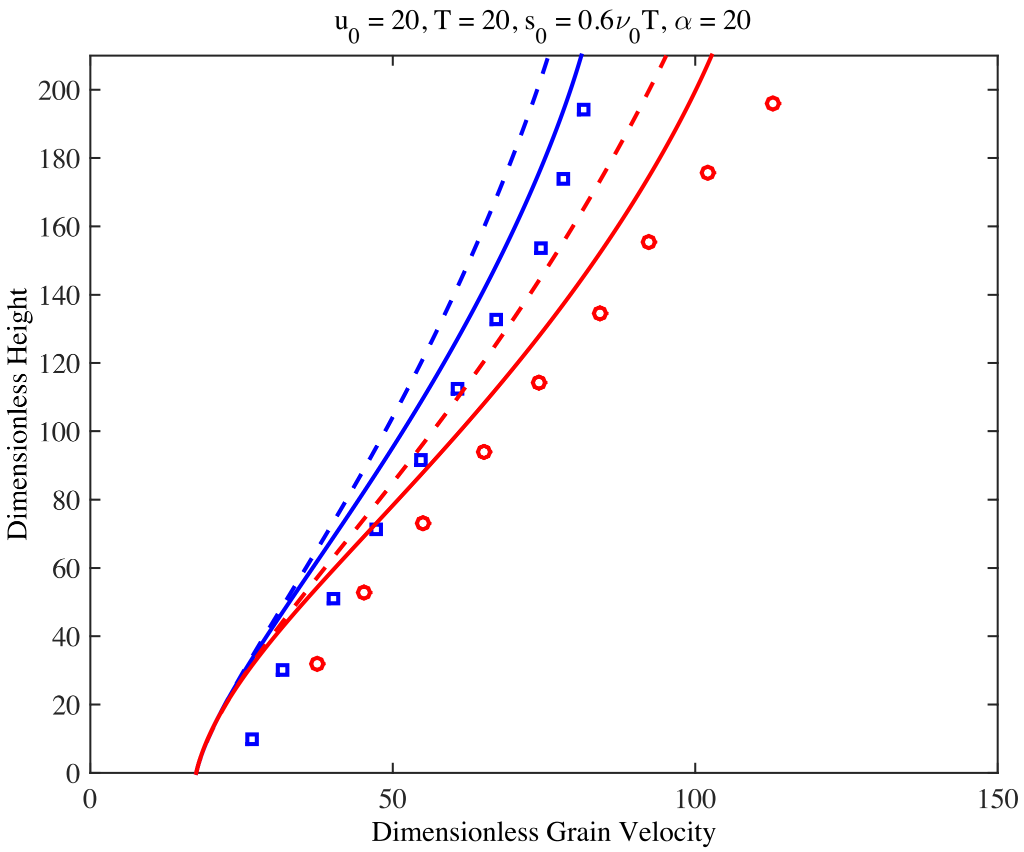
$$y = 0: \ u_0 = 20, \ \ T_0 = 20, \ \ s = 0.6\nu_0 T_0, \ \ U = 0$$

$y = 210: \ s = 0$ Parameter: ν_0

Influence of Suspension

S*=0.035 and 0.098

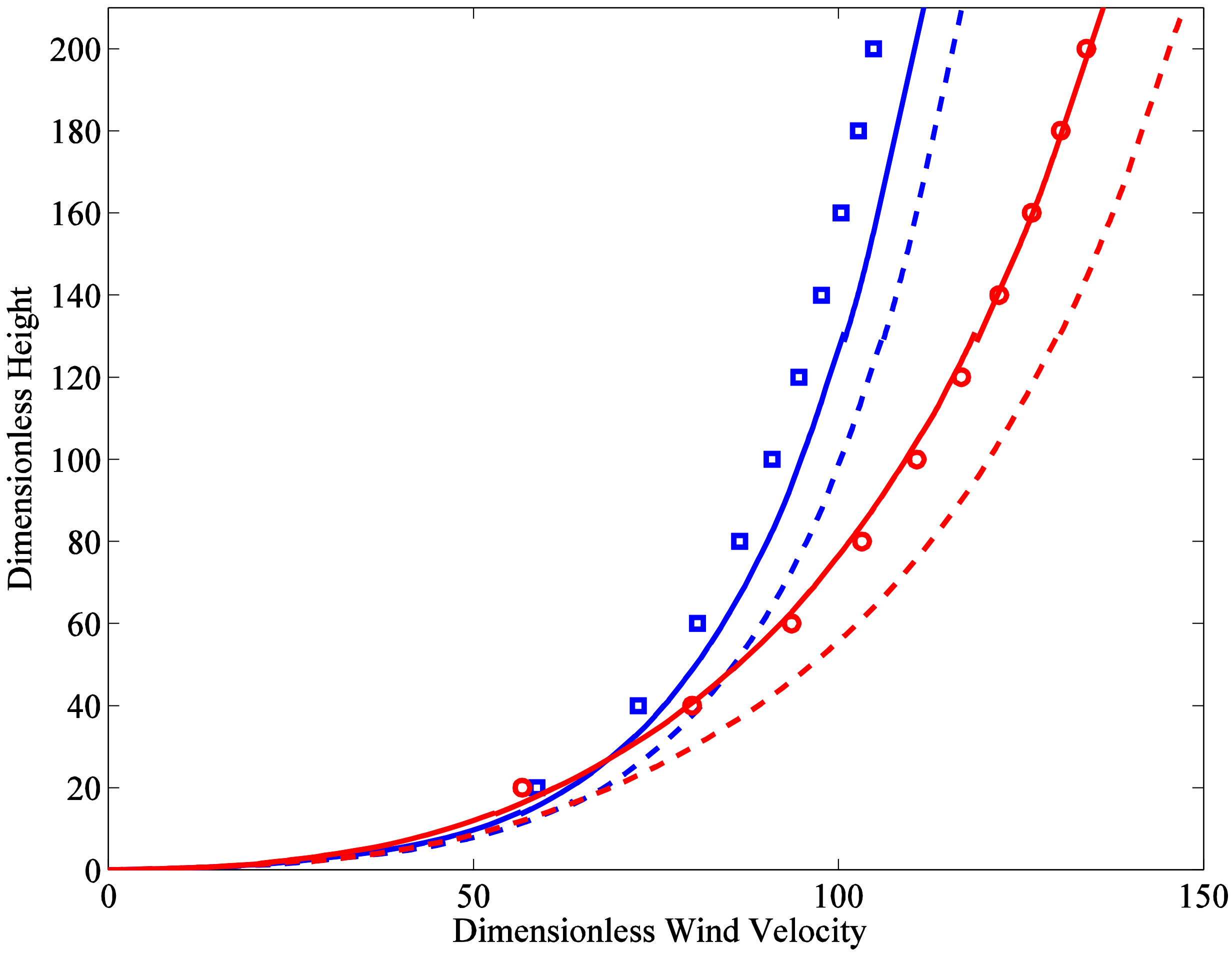
Grain Velocities



Suspension moves the grain velocities in the right direction.

Influence of Suspension

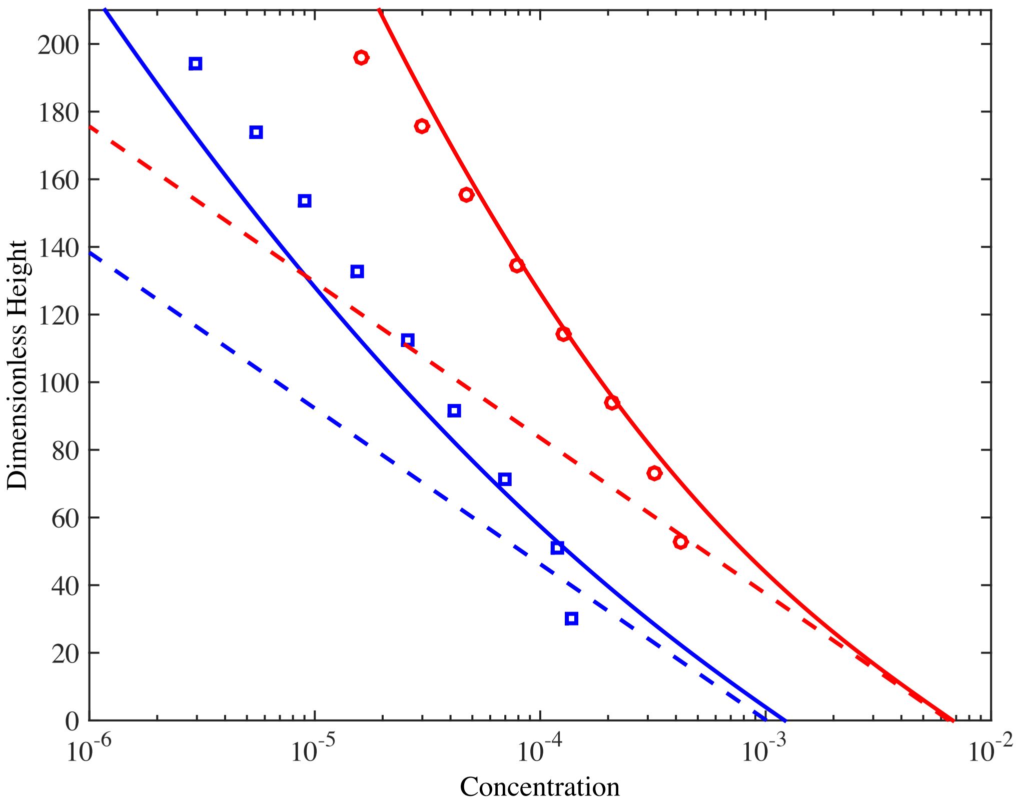
Air Velocities



Suspension moves the air velocities in the right directions.

Influence of Suspension

Particle Concentration



Above 50 particle diameters, suspension increases the dimensionless decay length.

Uniform, Unsteady

(saturation time)

$$c\frac{\partial u}{\partial t} = -cv\frac{\partial u}{\partial y} + \frac{\partial s}{\partial y} + c\frac{D}{\sigma}(U-u) \qquad s = \frac{\sigma p}{\alpha D}\frac{\partial u}{\partial y}$$

$$c\frac{\partial v}{\partial t} = -cv\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}(-p+b) - c\frac{D}{\sigma}v - c \quad b = \frac{\sigma p}{D}\frac{\partial v}{\partial y}$$

$$\frac{\partial c}{\partial t} = -c\frac{\partial v}{\partial y} - v\frac{\partial c}{\partial y} + \varepsilon\frac{\partial}{\partial y}\left(\frac{\partial c}{\partial y} - \frac{c}{T}\right) \qquad p = cT$$

$$\frac{\partial U}{\partial t} = \sigma\frac{\partial S}{\partial y} - cD(U-u) \qquad S = \frac{1}{\sigma}\left[\kappa\left(y+y_0\right)\frac{\partial U}{\partial y}\right]^2$$

$$\sigma = \frac{\rho^s}{\rho^f},\ D = 0.3\left[\left(U-u\right)^2 + v^2\right]^{1/2} + \frac{18.3}{Re},\ Re = \frac{d(gd)^{1/2}}{\mu/\rho^f}$$

$$y=0:\ \ s = 0.6cT,\ \ U = 0,\ \ v = \beta\left(u-u_0\right),\ \ \frac{\partial c}{\partial y} = -\frac{c}{T}$$

$$T = \left(u/4.6\right)^2 \quad \beta = 1.35\times 10^{-4}$$

$$y=H:\ \ s=0,\ \ S = S_0^* + S^{*\prime},\ \ b=0,\ \ cu = 0.001$$

$$t=0:\ \text{steady solution } c^0,\ u^0,\ v^0 = 0,\ U^0.$$

Linearize about the steady solution: $S^{*\prime} / S_0^* << 1$

$$c^0 \frac{\partial u'}{\partial t} = -c^0 v' \frac{\partial u^0}{\partial y} + \frac{\partial s'}{\partial y} + c' \frac{D^0}{\sigma}(U^0 - u^0)$$

$$+c^0 \frac{D'}{\sigma}(U^0 - u^0) + c^0 \frac{D^0}{\sigma}(U' - u')$$

$$s' = \frac{T}{\alpha D^0}\left(-D' \frac{c^0}{D^0} \frac{\partial u^0}{\partial y} + c' \frac{\partial u^0}{\partial y} + c^0 \frac{\partial u'}{\partial y} \right)$$

$$D^0 = 0.3\left|U^0 - u^0\right| + \frac{18.3}{R} \quad D' = 0.3\left|U' - u'\right|$$

$$c^0 \frac{\partial v'}{\partial t} = \frac{\partial}{\partial y}\left(-p' + b'\right) - c^0 \frac{D^0}{\sigma} v' - c' \quad b' = \frac{\sigma c^0 T}{D^0} \frac{\partial v'}{\partial y}$$

$$\frac{\partial c'}{\partial t} = -c^0 \frac{\partial v'}{\partial y} - v' \frac{\partial c^0}{\partial y} + \varepsilon\left(\frac{\partial c'}{\partial y} - \frac{c'}{T} \right) \qquad p' = c'T$$

$$\frac{\partial U'}{\partial t} = \sigma \frac{\partial S'}{\partial y} - c' D^0 (U^0 - u^0) - c^0 D' (U^0 - u^0)$$

$$-c^0 D^0 (U' - u')$$

$$S' = \frac{2\left[\kappa\left(y + y_0\right)\right]^2}{\sigma} \frac{\partial U^0}{\partial y} \frac{\partial U'}{\partial y}$$

Take $S_0^* = 0.05$.

Boundary conditions

$$y = 0:\ s' = 0.6c'T,\ U' = 0,\ v' = \beta u',\ \frac{\partial c'}{\partial y} = -\frac{c'}{T}$$

$$y = H:\ s' = 0,\ S^{*\prime} = 0.005,\ b' = 0,\ c^0 u' = 0.001$$

Initial conditions

$$t = 0:\ \ u' \equiv 0,\ \ v' \equiv 0,\ \ U' \equiv 0,\ \ c' \equiv 0,\ \ S^* = S^{*\prime}$$

Dimensional parameters
cgs

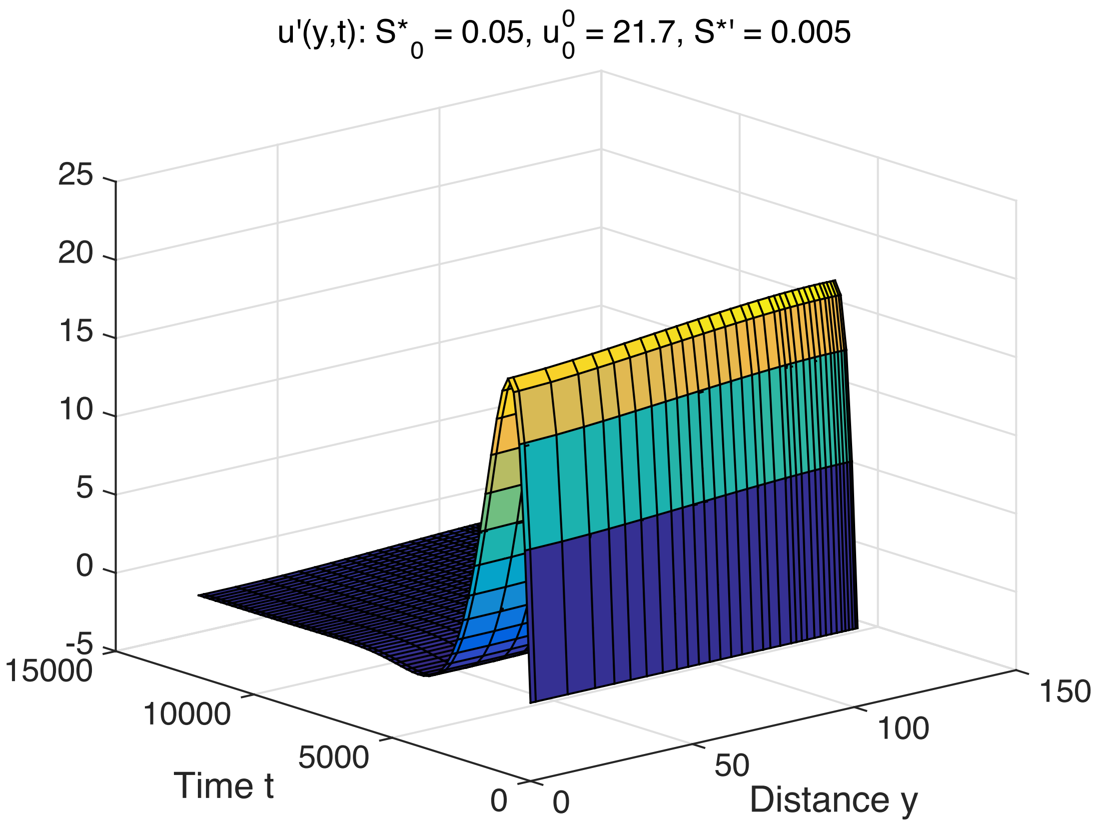
$$d = 0.025,\quad g = 980,\quad \mu_f / \rho_f = 0.15$$

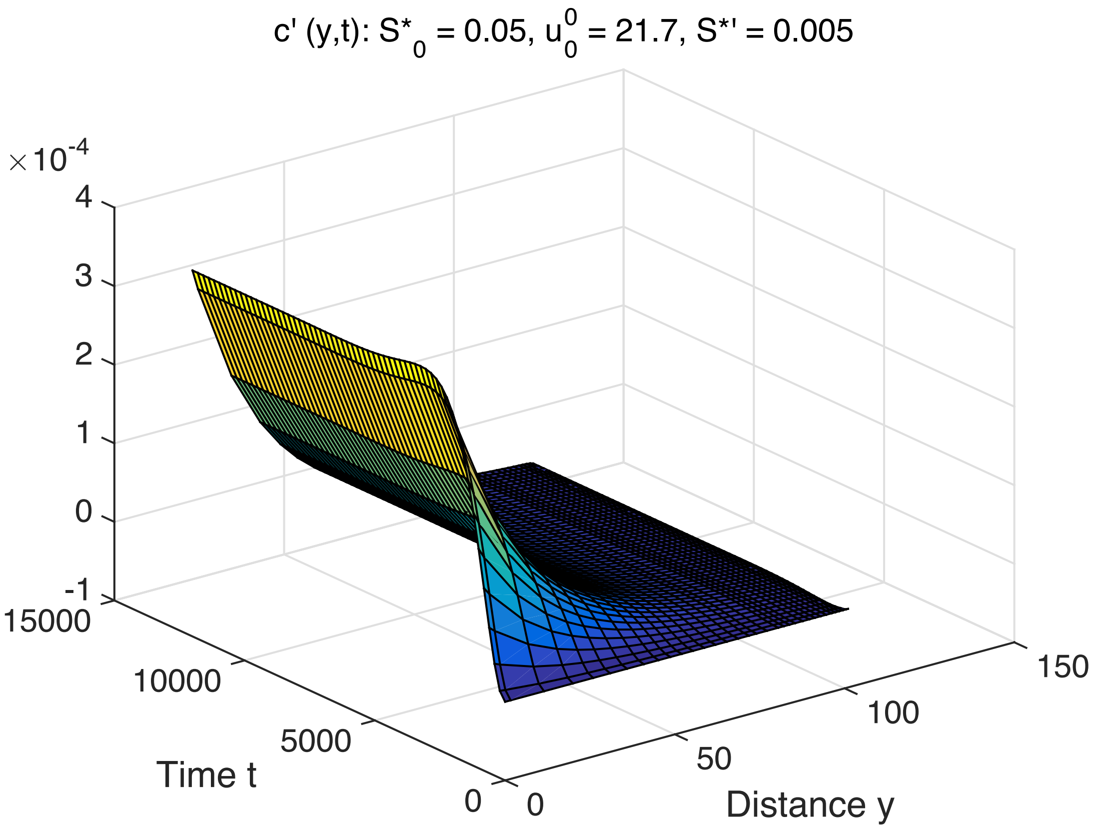
Dimensionless parameters

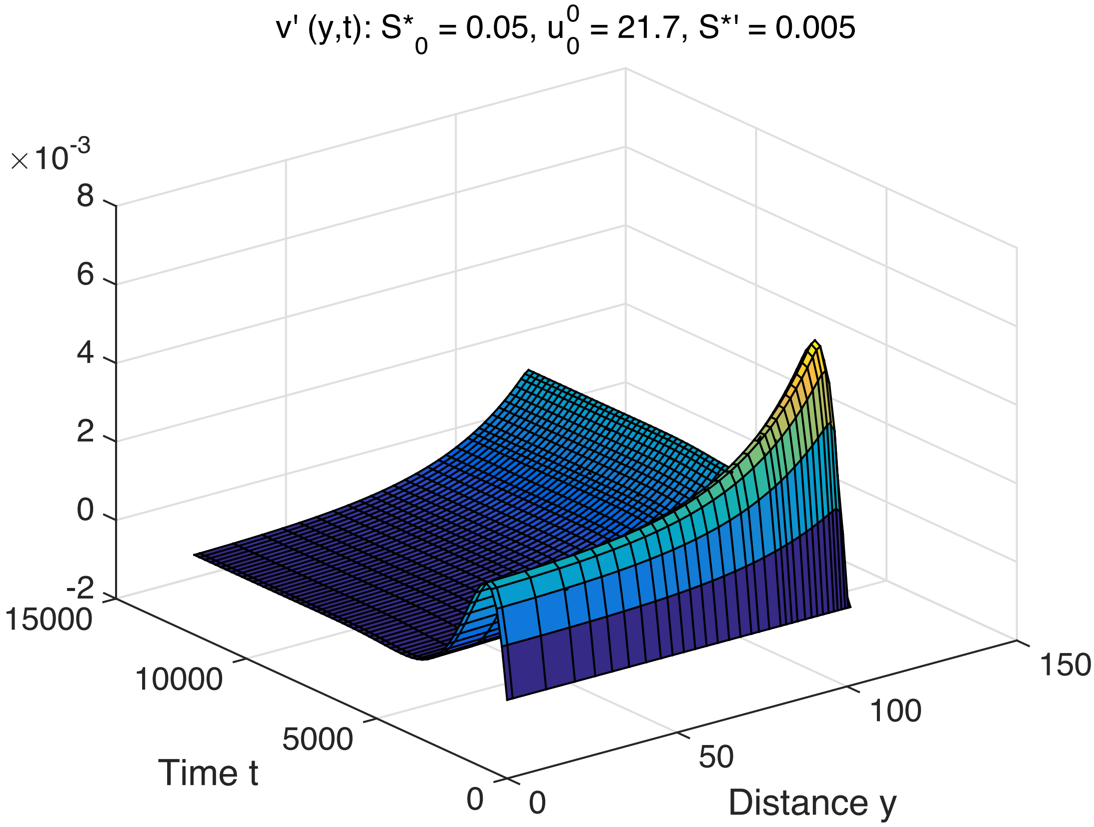
$$u_0^0 = 21.7,\ T^0 = 22,\ \sigma = 2200,\ \alpha = 20$$

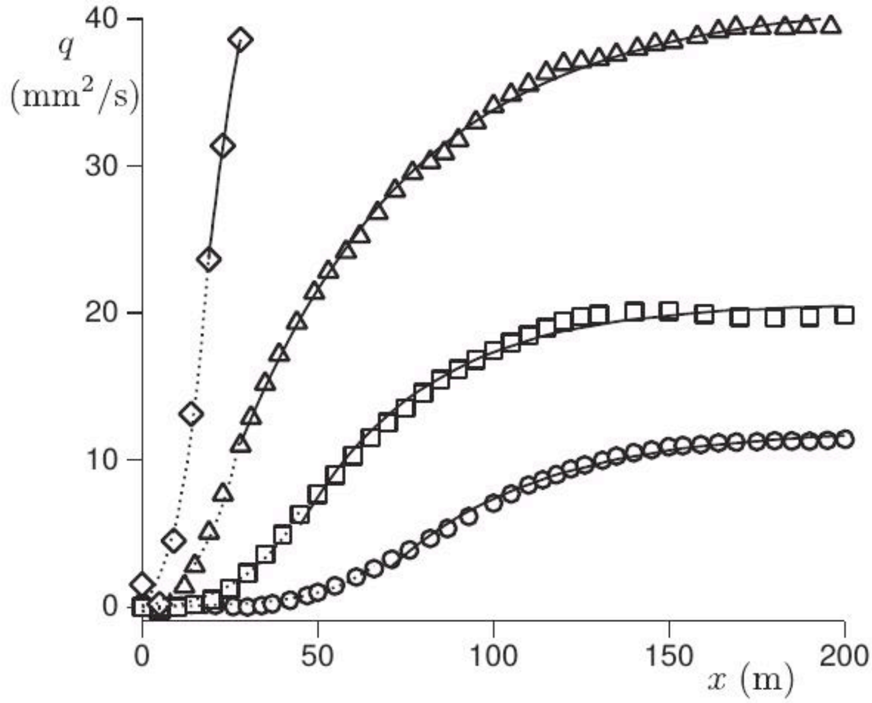
$$\kappa = 0.41,\ \varepsilon = 0.01,\ \beta = 1.35 \times 10^{-4}$$

Solve the system using Matlab “pdepe”.


u'(y,t): $S^*_0 = 0.05$, $u^0_0 = 21.7$, $S^{*\prime} = 0.005$
25
20
15
10
5
0
-5
15000
10000
5000
0
Time t
0
50
100
150
Distance y


c' (y,t): $S^*_0 = 0.05$, $u^0_0 = 21.7$, $S^{*\prime} = 0.005$
$\times 10^{-4}$
4
3
2
1
0
-1
15000
10000
5000
0
Time t
0
50
100
150
Distance y





B. Andreotti, P. Claudin, O. Pouliquen, Geomorphology 123, 343–348 (2010)

What have we done?

We've formulated a two-phase continuum theory for saltation that also accommodates turbulent and collisional suspension.

The theory permits boundary-value problems to be phrased and solved for steady, uniform flows and initial-boundary-value problems to be phrased and solved both for unsteady, uniform flows and steady, developing flows.

The solutions reproduce the distributions and feature of flows seen in laboratory flows, including the characteristic times for the adjustment of the different mechanisms of suspension to changes in conditions.

At least one of these adjustments occurs in a non-monotone way.