Saturation Times in a Two-Phase Continuum Theory for Windblown Sand

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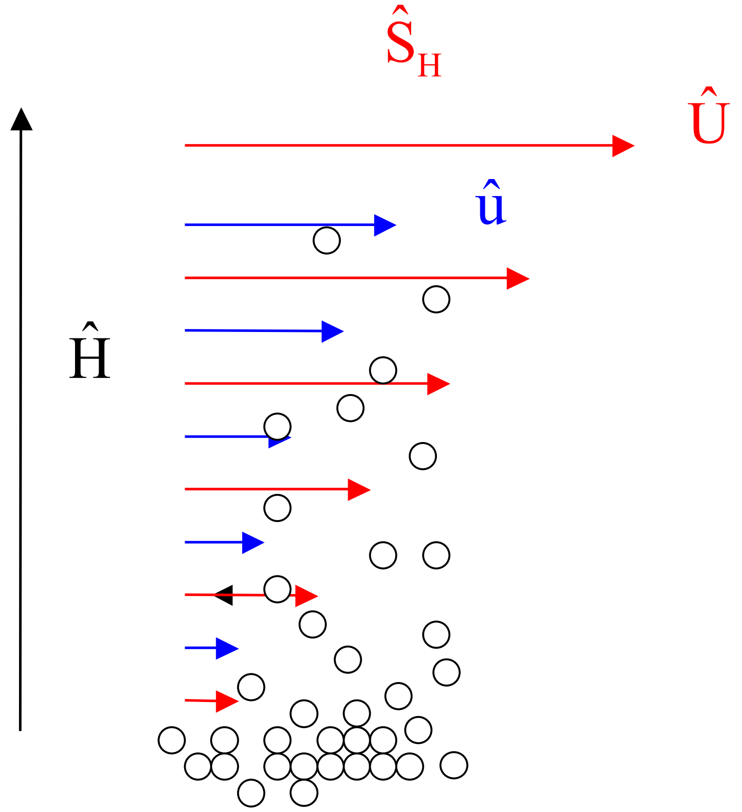
For saltating particles in a uniform turbulent shearing flow, determine the time to restore a steady state after a change in wind speed.

Aeolian Transport

Two-phase, turbulent flow



Friction velocity / Shields parameter

$$\hat{u}^* \equiv \left(\hat{S}_H / \hat{\rho}^f\right)^{1/2} \qquad S^* \equiv \hat{S}_H / \left(\hat{\rho}^s \hat{g} \hat{d}\right)$$

Air drag: $c\hat{D}(\hat{U} - \hat{u})$

$$\hat{D} \equiv \frac{3}{10}\frac{\hat{\rho}^f}{\hat{d}}\left[(\hat{U} - \hat{u})^2 + 3\hat{T}\right]^{1/2} + \frac{18\hat{\mu}^f}{\hat{d}^2}$$

Continuum Theory

Particle horizontal momentum

$$0 = \frac{ds}{dy} + \frac{cD}{\sigma}(U - u)$$

Particle vertical momentum:

$$0 = -\frac{dp}{dy} - c$$

Particle fluctuation energy

$$0 = -\frac{dq}{dy} + su' - \gamma$$

Particle pressure

$$p = cT$$

Particle shear stress

?

Particle energy flux

?

Continuum Theory

Single particle trajectories without vertical drag

$$\text{Upward:}\quad \xi'_y \frac{d\xi'_x}{dy} = D(U-\xi'_x)$$

$$\text{Downward:}\quad \xi'_y \frac{d\xi_x}{dy} = -D(U-\xi_x)$$

Multiply by c, sum, and average

$$\overline{c\xi_y'^2 \frac{d}{dy}\left(\xi'_x+\xi_x\right)} = cD\overline{\xi'_y(\xi'_x-\xi_x)}$$

$$2u \equiv \overline{(\xi'_x+\xi_x)}$$

$$p \equiv c\overline{\xi_y'^2} = cT \qquad 2s \equiv c\overline{\xi'_y(\xi'_x-\xi_x)}$$

$$p\frac{du}{dy} = \alpha D s$$

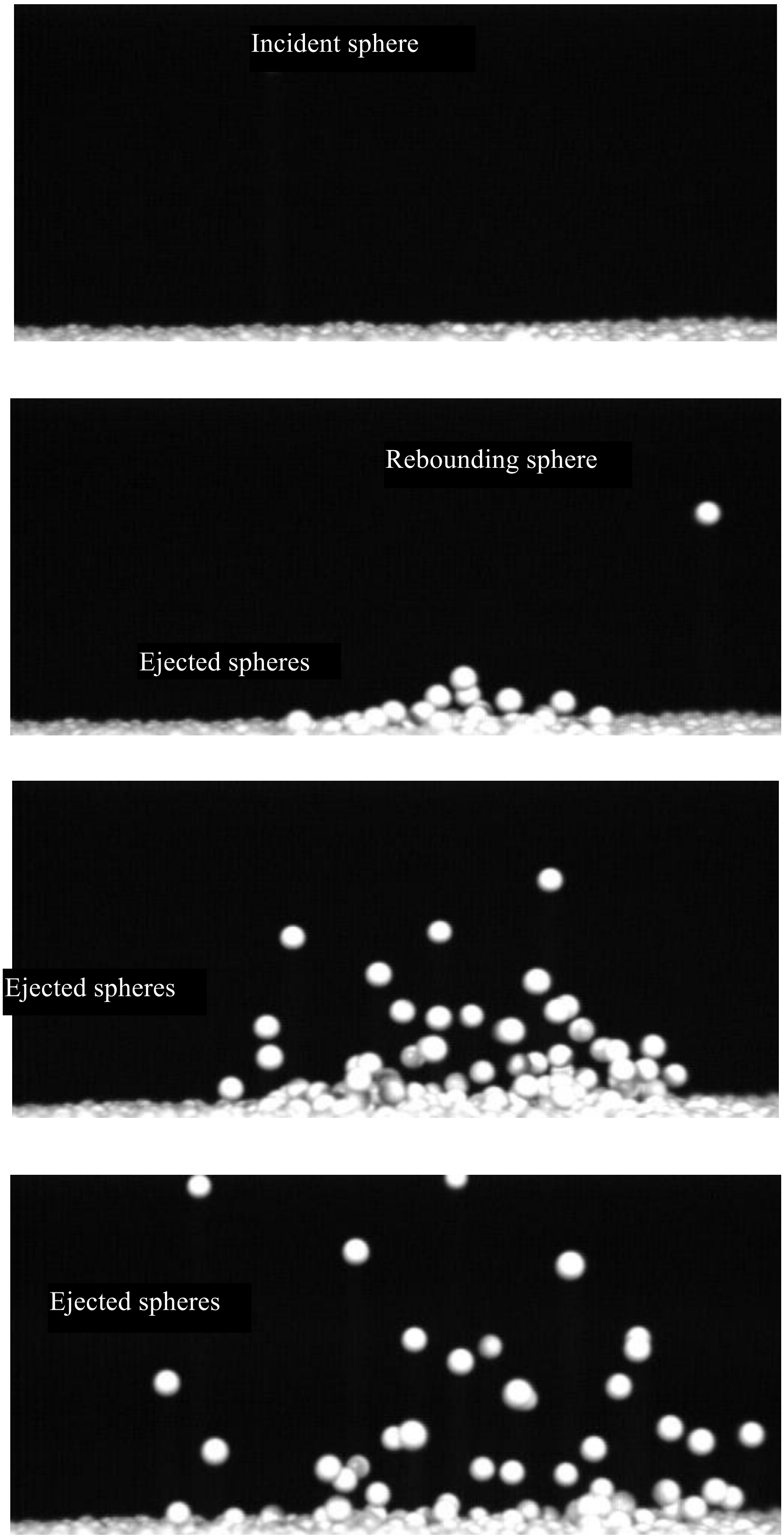
$$2q \equiv c\overline{\xi'_y\left[(\xi'_x-u)^2+\xi_y'^2\right]} + c\overline{\xi_y\left[(\xi_x-u)^2+\xi_y^2\right]}$$

$$\doteq c\left[\overline{\xi'_y(\xi'_x-\xi_x)}\,\overline{(\xi'_x+\xi_x)}/2 - \overline{\xi'_y(\xi'_x-\xi_x)}u\right]$$

$$q = 0$$

Splash

Beladjine, et al. Phys. Rev. E 75, 061305 (2007)



Momentum of rebounding particles

$$\bar{\xi}' = e(\boldsymbol{\xi})\xi = (0.87 - 0.72\sin\theta)\xi$$

$$\bar{\xi}'_y = \varepsilon(\boldsymbol{\xi})\left|\xi_y\right| = \left(\frac{0.30}{\sin\theta} - 0.15\right)\left|\xi_y\right|$$

Total number N of particles

$$N(\boldsymbol{\xi}) = \begin{cases} 1 + 13(1 - e^2)\left(\dfrac{\xi}{40} - 1\right), & \text{if } \xi > 40 \\ 1, & \text{if } 1 \le \xi \le 40 \\ 0, & \text{if } \xi < 1 \end{cases}$$

Velocity distribution function

$$f(\boldsymbol{\xi}) = \frac{n_0}{2\pi T_0}\exp\left[\frac{-(\xi_x - u_0)^2 - \xi_y^2}{2T_0}\right]$$

$$\left(c_0 = \pi n_0 / 6\right)$$

Mass flux

$$\dot{m} = \int_{\xi_y \le 0} (N-1)\xi_y f(\boldsymbol{\xi}) d\boldsymbol{\xi}$$

$$= \frac{13}{2\pi} \frac{n_0 T_0^2}{u_0 (40-u_0)^2} \left[0.24 + 0.63 \left(\frac{\pi T_0}{20 u_0} \right)^{1/2} \right] e^{\frac{-(40-u_0)^2}{2T_0}}$$

$$- \frac{74\sqrt{2}}{\pi} \frac{n_0}{T_0} e^{\frac{-u_0^2}{2T_0}}$$

Momentum flux

$$\dot{\mathbf{M}} = -\frac{\pi}{6} \int_{\xi_y \le 0} (\bar{\boldsymbol{\xi}}' - \boldsymbol{\xi}) \xi_y f(\boldsymbol{\xi}) d\boldsymbol{\xi}$$

$$\dot{M}_x = c_0 T_0 \left(0.35 + 0.07 \frac{u_0}{T_0^{1/2}} - 0.33 \frac{T_0^{1/2}}{u_0} \right)$$

$$\dot{M}_y = c_0 T_0 \left[0.12 \left(\frac{u_0}{T_0^{1/2}} + \frac{T_0^{1/2}}{u_0} \right) - 0.08 \right] + \frac{c_0 T_0}{2}$$

With $\dot{M}_y = p = c_0 T_0$, $\frac{u_0}{T_0^{1/2}} = 4.6$

Boundary-Value Problem

Steady, uniform flow: $\dot{m} = 0$

$u_0 = 21.7,\ T_0 = 22.3$ and $s_0 \equiv \dot{M}_x = 0.6c_0T_0$

$$\frac{dc}{dy} = -\frac{c}{T_0}$$

$$\frac{du}{dy} = 20D\frac{s}{p} \qquad D = \frac{0.3}{\sigma}\left[(U-u)^2 + 3T_0\right]^{1/2} + \frac{18.3}{\sigma R}$$

$$\frac{ds}{dy} = -cD(U-u) \qquad \sigma = \frac{\rho^s}{\rho^f} \qquad R = \frac{d(gd)^{1/2}}{\mu^f/\rho^f}$$

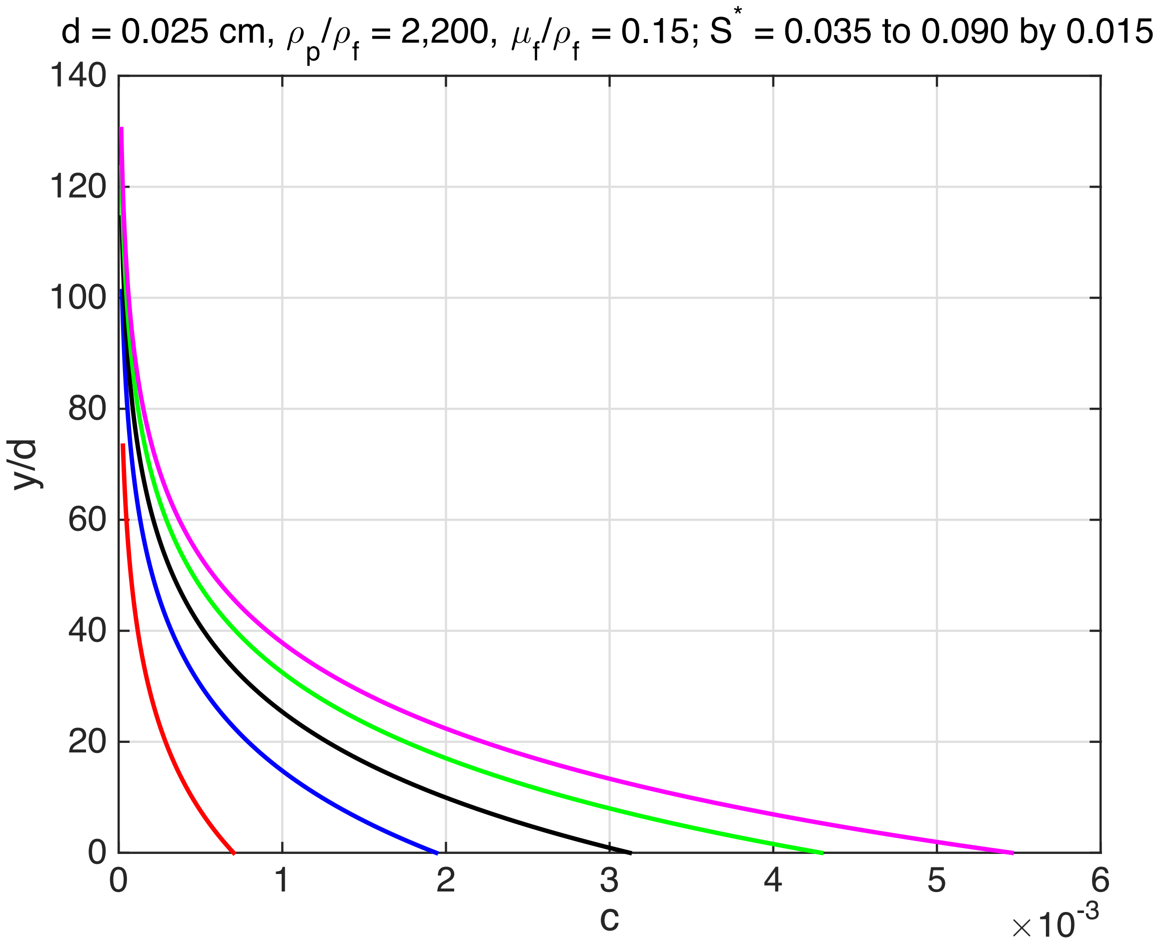
$$\frac{dU}{dy} = \frac{\left[(S^*-s)\sigma\right]^{1/2}}{\kappa\left(y+y_0\right)}$$

$y = 0:\ u_0 = 21.7,\ T_0 = 22.3,\ s = 0.6c_0T_0,\ U = 0$

$y = H:\ s = 0 \quad cu = 0.001$ Parameters: c_0, H

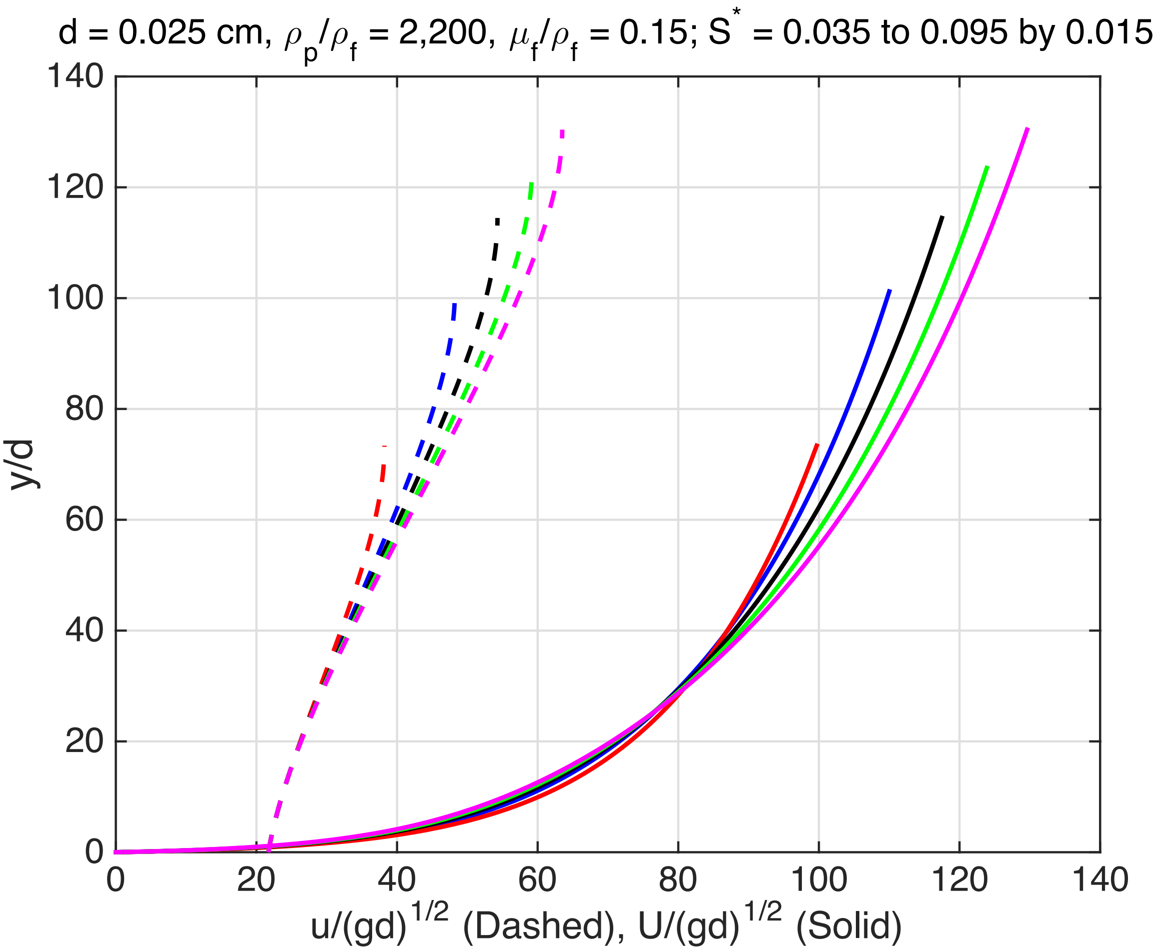
Predicted Profiles

Concentration



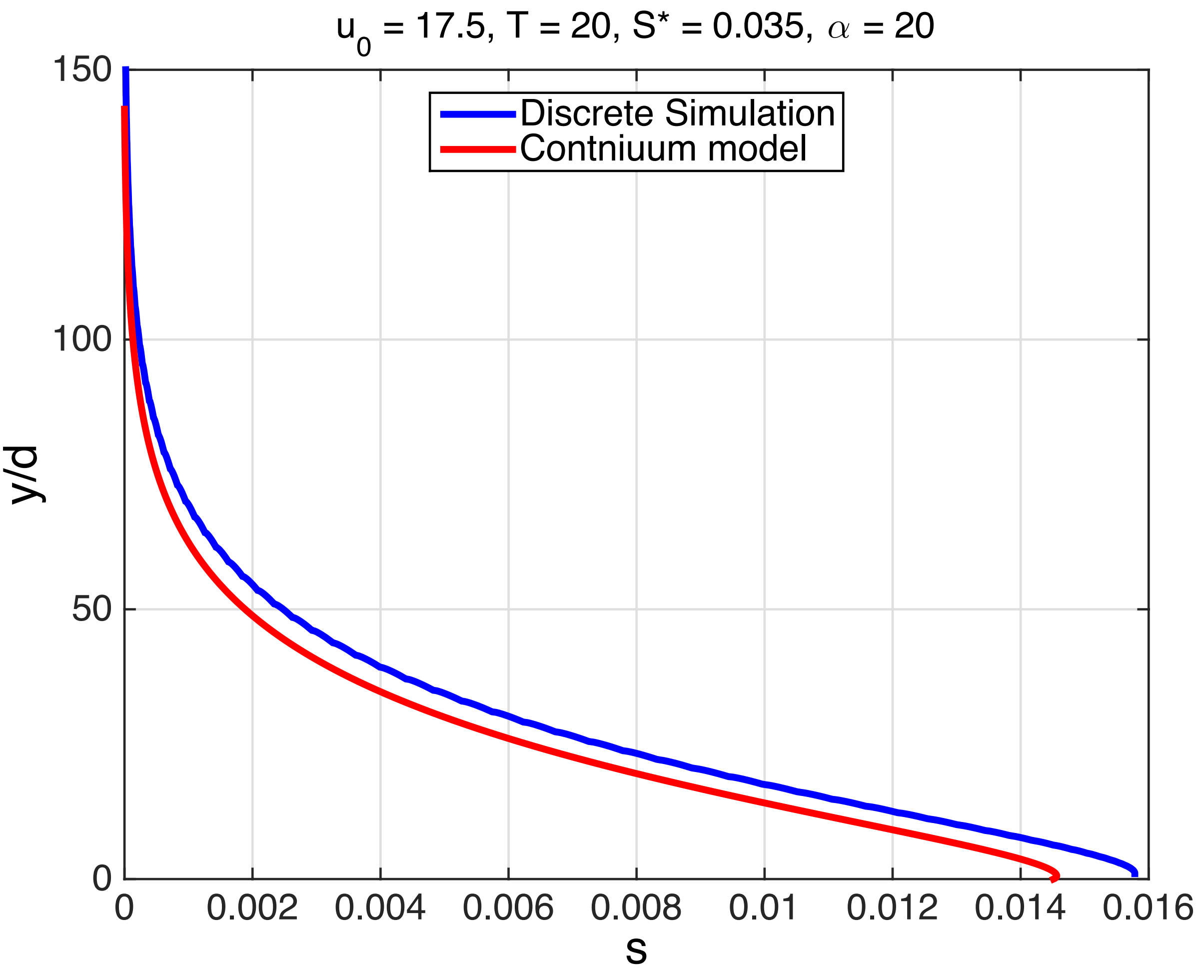
Predicted Profiles

Velocities



Particle Shear Stress

Continuum versus discrete simulation



Uniform, Unsteady

Phrase an initial-boundary-value problem to describe the adjustment of an initially steady flow to an abrupt change in wind speed.

Pressure

$$T_0 = \frac{u_0^2}{(4.6)^2}$$

Shear stress

$$s_0 = 0.6 c_0 T_0$$

Mass flux

$$c_0 v_0 = \frac{39}{\pi^2} \frac{c_0 T_0^2}{u_0 (40 - u_0)^2} \left[0.24 + 0.63 \left(\frac{\pi T_0}{20 u_0} \right)^{1/2} \right] e^{\frac{-(40-u_0)^2}{2T_0}}$$

$$- \frac{444\sqrt{2}}{\pi^2} \frac{c_0}{T_0} e^{\frac{-u_0^2}{2T_0}}$$

Expand about $u_0^0 = 21.7$, at which $v_0 = 0$.

Then

$$v_0 \doteq \beta_0 \left(u_0 - u_0^0 \right), \qquad \beta_0 = 1.35 \times 10^{-4}$$

Uniform, Unsteady
(saturation time)

$$c\frac{\partial u}{\partial t} = -cv\frac{\partial u}{\partial y} + \frac{\partial s}{\partial y} + c\frac{D}{\sigma}(U-u) \qquad s = \frac{\sigma p}{\alpha D}\frac{\partial u}{\partial y}$$

$$c\frac{\partial v}{\partial t} = -cv\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}(-p+b) - c\frac{D}{\sigma}v - c \quad b = \frac{\sigma p}{D}\frac{\partial v}{\partial y}$$

$$\frac{\partial c}{\partial t} = -c\frac{\partial v}{\partial y} - v\frac{\partial c}{\partial y} + \varepsilon\frac{\partial}{\partial y}\left(\frac{\partial c}{\partial y} - \frac{c}{T}\right) \qquad p = cT$$

$$\frac{\partial U}{\partial t} = \sigma\frac{\partial S}{\partial y} - cD(U-u) \qquad S = \frac{1}{\sigma}\left[\kappa\left(y+y_0\right)\frac{\partial U}{\partial y}\right]^2$$

$$\sigma = \frac{\rho^s}{\rho^f},\ D = 0.3\left[\left(U-u\right)^2 + v^2\right]^{1/2} + \frac{18.3}{\mathrm{Re}},\ \mathrm{Re} = \frac{d(gd)^{1/2}}{\mu/\rho^f}$$

$$y = 0:\ \ s = 0.6cT,\ \ U = 0,\ \ v = \beta\left(u - u^0\right),\ \ \frac{\partial c}{\partial y} = -\frac{c}{T}$$

$$T = \left(u/4.6\right)^2 \quad \beta = 1.35\times 10^{-4}$$

$$y = H:\ \ s = 0,\ \ S = S^{*0} + S^{*\prime},\ \ b = 0,\ \ \frac{\partial c}{\partial y} = -\frac{c}{T}$$

$$t = 0:\ \text{steady solution } c^0(y),\ u^0(y),\ v^0 \equiv 0,\ U^0(y).$$

Linearize about the steady solution: $S^{*\prime} / S_0^* << 1$

$$c^0 \frac{\partial u'}{\partial t} = -c^0 v' \frac{\partial u^0}{\partial y} + \frac{\partial s'}{\partial y} + c' \frac{D^0}{\sigma}(U^0 - u^0)$$
$$+c^0 \frac{D'}{\sigma}(U^0 - u^0) + c^0 \frac{D^0}{\sigma}(U' - u')$$

$$s' = \frac{T_0^0}{\alpha D^0}\left(-D' \frac{c^0}{D^0}\frac{\partial u^0}{\partial y} + c' \frac{\partial u^0}{\partial y} + c^0 \frac{\partial u'}{\partial y} + \frac{2c^0}{u_0^0} u_0'\right)$$

$$D^0 = 0.3\left|U^0 - u^0\right| + \frac{18.3}{R} \quad D' = 0.3\left|U' - u'\right|$$

$$c^0 \frac{\partial v'}{\partial t} = \frac{\partial}{\partial y}\left(-p' + b'\right) - c^0 \frac{D^0}{\sigma} v' - c' \quad p' = T_0^0\left(c' + \frac{2c^0}{u_0^0} u_0'\right)$$

$$\frac{\partial c'}{\partial t} = -c^0 \frac{\partial v'}{\partial y} - v' \frac{\partial c^0}{\partial y} + \varepsilon\left(\frac{\partial c'}{\partial y} - \frac{c'}{T_0^0}\right) \quad b' = \frac{\sigma c^0 T_0^0}{D^0}\frac{\partial v'}{\partial y}$$

$$\frac{\partial U'}{\partial t} = \sigma \frac{\partial S'}{\partial y} - c' D^0 (U^0 - u^0) - c^0 D'(U^0 - u^0)$$
$$-c^0 D^0 (U' - u')$$

$$S' = \frac{2\left[\kappa\left(y + y_0\right)\right]^2}{\sigma}\frac{\partial U^0}{\partial y}\frac{\partial U'}{\partial y}$$

Take $S_0^* = 0.05$.

Boundary conditions

$$y = 0:\ s' = 0.6c'T,\ U' = 0,\ v' = \beta u',\ \frac{\partial c'}{\partial y} = -\frac{c'}{T}$$

$$y = H:\ s' = 0,\ S^{*\prime} = 0.005,\ b' = 0,\ \frac{\partial c'}{\partial y} = -\frac{c'}{T}$$

Initial conditions

$$t = 0:\quad u' \equiv 0,\quad v' \equiv 0,\quad U' \equiv 0,\quad c' \equiv 0,\quad S^* = S^{*\prime}$$

Dimensional parameters
cgs

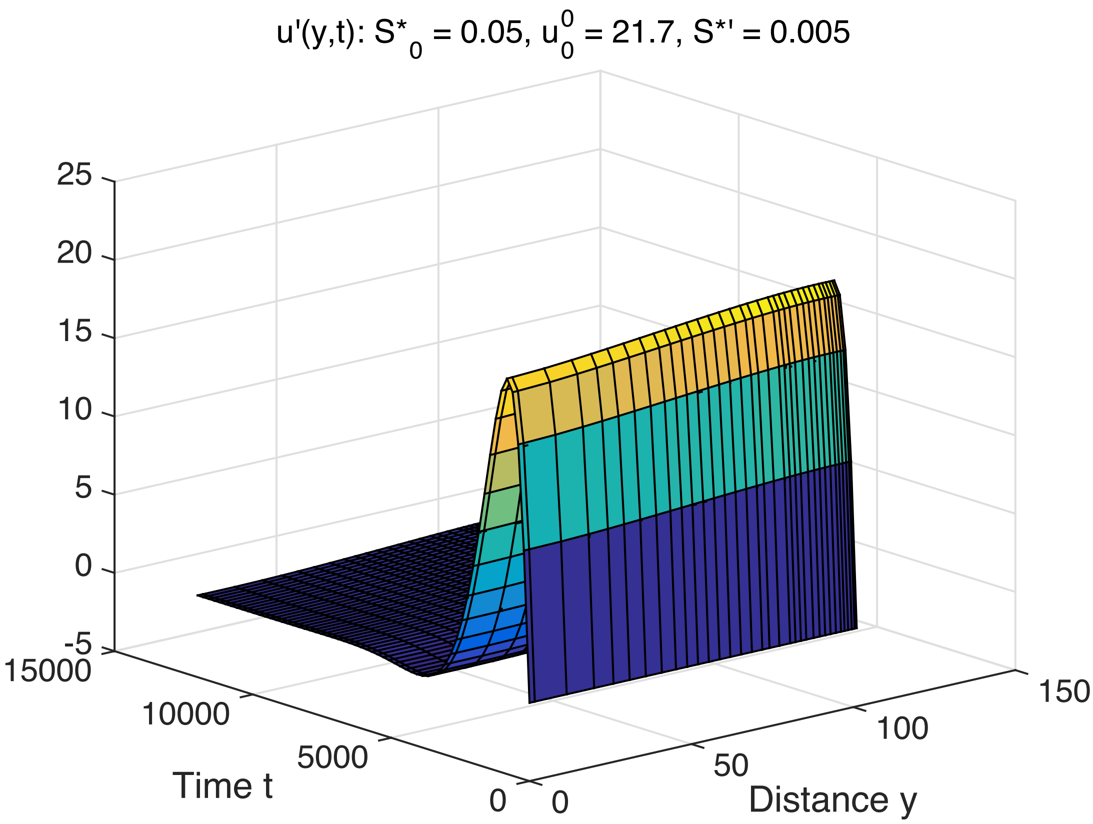
$$d = 0.025,\quad g = 980,\quad \mu_f / \rho_f = 0.15$$

Dimensionless parameters

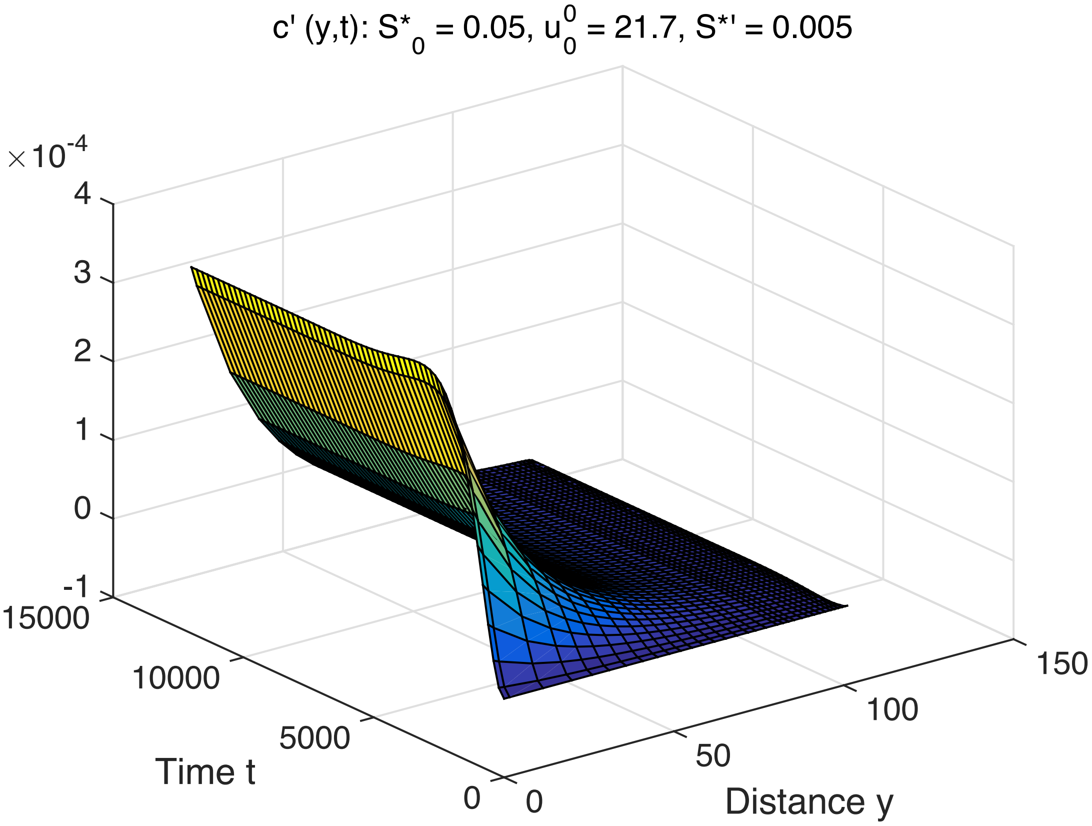
$$u_0^0 = 21.7,\ T^0 = 22.3,\ \sigma = 2200,\ \alpha = 20$$

$$H = 101,\ \kappa = 0.41,\ \varepsilon = 0.01,\ \beta = 1.35 \times 10^{-4}$$

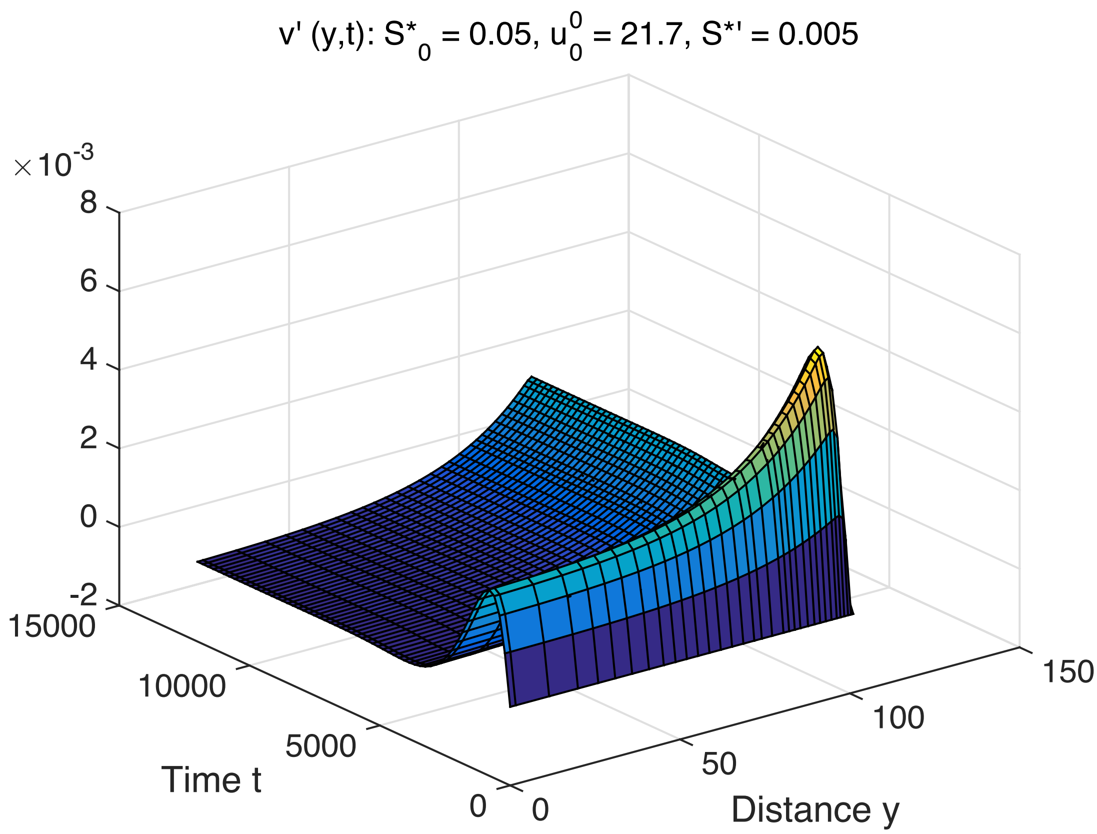
Solve the system using Matlab “pdepe”.

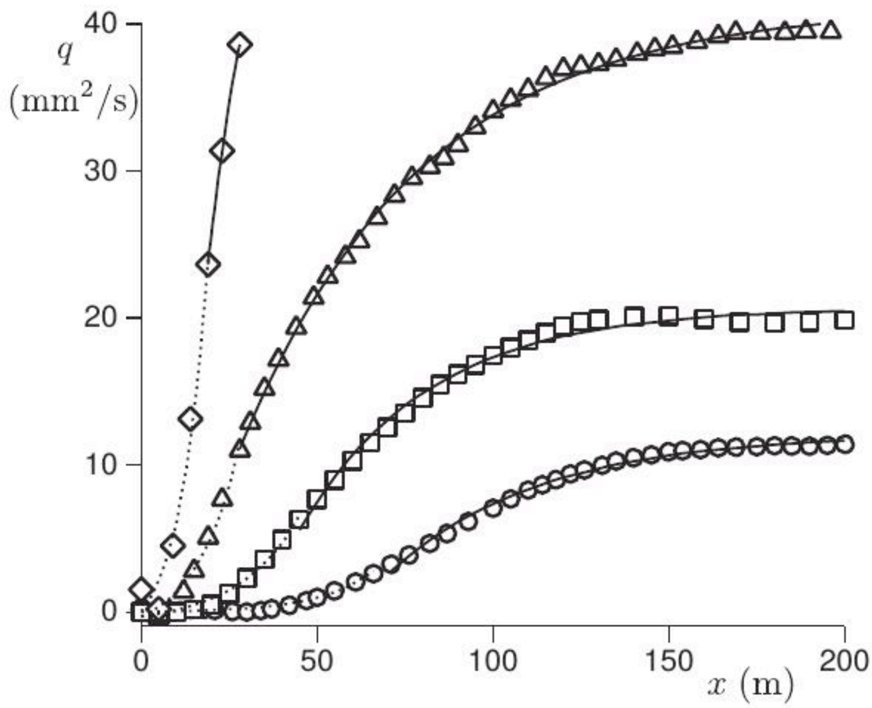


u'(y,t): S*$_0$ = 0.05, u_0^0 = 21.7, S*' = 0.005
25
20
15
10
5
0
-5
15000
10000
5000
0
Time t
0
50
100
150
Distance y



c' (y,t): S*$_0$ = 0.05, u_0^0 = 21.7, S*' = 0.005
$\times 10^{-4}$
4
3
2
1
0
-1
15000
10000
5000
0
Time t
0
50
100
150
Distance y





B. Andreotti, P. Claudin, O. Pouliquen, Geomorphology 123, 343–348 (2010)

What's been done?

A two-phase continuum theory for saltation, which can also accommodates turbulent and collisional suspension, has been formulated.

The theory permits boundary-value problems to be phrased and solved for steady, uniform flows and initial-boundary-value problems to be phrased and solved both for unsteady, uniform flows and steady, developing flows.

The solutions reproduce the distributions and feature of flows seen in laboratory flows, including the characteristic times for the adjustment of the different mechanisms of suspension to changes in conditions.

At least one of these adjustments occurs in a non-monotone way.