11
Symmetry, topology, and information

The capacity of topologically robust quantities to distinguish different phases even in absence of local order parameters is the central subject of this book. The topological degeneracy in the fractional quantum Hall effect thus reflects the topological order present there, which does not require the breaking of any local symmetry. This has supplemented the idea of the symmetry distinction of different phases, the bedrock of the Landau-Ginzburg–Wilson paradigm of phases and transitions between them, as outlined in the opening chapters.

What then about the interplay between those ideas? On one hand, there is the question of their compatibility, e.g. can topologically ordered phases also exhibit conventional symmetry breaking? This item we have already touched upon in the context of quantum Hall ferromagnets in Chapter 3.7. On the other hand, one can ask whether there exist any phenomena which constitutively rely on a combination of ingredients from symmetry and topology. The first part of this chapter is devoted to taking a closer look at that question.

The second part addresses how quantum information concepts are useful to understand quantum wavefunctions, particularly those arising in topological states. A major impetus for work on non-Abelian states is the goal of quantum memories and computers, as described in Chapter 9. There has also been a useful flow in the opposite direction, and we sketch one way to quantify information in a single quantum wavefunction: the entanglement entropy with respect to a bipartition of Hilbert space. This turns out to help place topological states, particularly those with fractional particles, in a broader context, and has also led to a number of revolutionary numerical techniques. This chapter is of necessity more of a survey of an actively evolving area than the preceding ones, and we encourage readers who wish to delve more deeply to consult the reviews cited. We close with a few general comments on the continuing search for topological phases, both in real materials and in the mind's eye.

11.1 Symmetry-protected topological phases

One of the central phenomena in topological condensed matter physics – the topological insulator – requires a helping of symmetry to exhibit its topology: as explained in Chapter 3, it is the presence of time reversal symmetry that eliminates the scattering between the counterpropagating edge states, and hence leads to the quantised transport coefficient.

In one dimension, we have so far encountered a variety of different topological systems hosting a number of interesting phenomena. These include the chains involving the names Peierls, Su-Schrieffer-Heeger, Majumdar-Ghosh, Haldane/AKLT and Kitaev. Going beyond their individual properties, this discussion is devoted to identifying the more formal structure underpinning their existence, with a focus on genesis, distinctiveness and stability of their topological properties. Indeed, it turns out that non-trivial topological properties in $d = 1$ can essentially only occur in the presence of symmetries; in their absence, all states are topologically identical. In the following, we outline the theory underpinning this symmetry protected topology.

The notion of symmetry-protected topological phases in one dimension was advanced in particular in the context of studying the $S = 1$ Heisenberg chain with nearest-neighbour interactions, also known as the Haldane chain (Pollmann et al., 2012). A soluble relative of this model, the $S = 1$ AKLT chain, was introduced in the context of Klein models in Chapter 5.2.4. There, its basic phenomenology was discussed – a non-degenerate, short-range entangled state exhibiting fractional spin $S = 1/2$ edge states exponentially localised on a lengthscale set by the bulk gap. It had been noted early on that there existed deformations of this model which connected it to a trivial band insulator without encountering a gap closing *en route* (Anfuso and Rosch, 2007). An account placing this material in a broad information-theoretic context is available in (Zeng et al., 2015)

11.1.1 Symmetry fractionalisation

The central observation for the notion of symmetry protected topology in $d = 1$ is that a restriction on the paths through the space of Hamiltonians to ones respecting certain symmetries does provide a notion of topological stability to the Haldane phase. The mathematical framework for capturing the underlying idea, which now goes by the name of symmetry fractionalisation, involves representation theory, in particular the projective representations of the appropriate symmetry groups.

The central ingredient is simply stated: the action of a symmetry can act *independently* on the two edges of the chain, provided the bulk is gapped. This allows for the representations of the symmetry at the edge to acquire a relative phase, which under certain conditions there may take on only a discrete set of possible values. These discrete possibilities can then not be *continuously* deformed into one another, and they are hence topologically stable. When the underlying symmetry is removed, however, this structure disappears entirely.

The remainder of this section fleshes out these statements following the account of Verresen et al. (2017); it applies these insights to a family of Kitaev chains, christened α-chains, to provide a concrete unifying framework for a number of previously encountered models, and to generate insights into the overall richness of the resulting classification. We start by explaining the simplest setting before adding various generalisations until we are in a position to discuss the α-chains in general.

We consider a chain of length L with open boundary conditions and a local Hilbert space, $\mathcal{H}_i$, of dimension d, such that the total Hilbert space, $\mathcal{H} = \otimes_i \mathcal{H}_i$, has dimension d^L

Let the system Hamiltonian H be symmetric with respect with a global symmetry group G. The action of this group on states in Hilbert space is encoded by a set of unitary matrices U; we identify the set of representations provided by these unitary matrices with the group itself, allowing us to write $U \in G$ as a shorthand.

Next, assume that we are dealing with a so-called on-site symmetry, i.e. one whose members can be written as a tensor product over unitaries acting on individial sites i:

$$U = \otimes_i U_i \ . \tag{11.1}$$

We restrict our attention to the case of the symmetry G not being broken, so that the ground state in the presence of periodic boundary conditions is unique; the action of G must therefore be trivial in the bulk as it cannot convert different ground states into one another. This still leaves the possibility of its action being non-trivial at the edges, provided that there is (as is the case in the AKLT chain) an edge state degeneracy.

We thus define two operators, $U_{L,R}$ to act on the left and right edges of the system, respectively, such that $U = U_L U_R$. Again, as noted for the edge states of the AKLT chain, the support of $U_{L,R}$ will extend into the bulk by a distance set by the (inverse) bulk gap, so that in the limit of a long chain, $L \to \infty$, their support will be disjoint.

The essence of symmetry fractionalisation is that $U_{L,R}$ are, *individually*, symmetries of H, i.e. that $[U_L, H] = [U_R, H] = 0$. This can be seen by decomposing $H = H_L + H_R$ with the support of H_L chosen such that it is disjoint with that of U_R, and similarly for H_R and U_L. Then, $0 = [U, H_L] = [U_L U_R, H_L] = [U_L, H_L] U_R$. Since $U(R)$ is invertible, it follows that

$$[U_L, H_L] = [U_L, H_L + H_R] = [U_L, H] = 0 \ , \tag{11.2}$$

as desired.

To extract the projective nature of the resulting edge representations, we continue to restrict ourselves to the simplest setting, namely a 'bosonic' system, i.e.

one in which operators acting with disjoint support commute; and we consider a commutative pair of symmetry operations $U, V \in G$, i.e. $[U, V] = 0$, such that $UVU^{-1}V^{-1} = \mathbb{I}$. Then

$$\mathbb{I} = (U_L U_R)(V_L V_R)(U_L^{-1} U_R^{-1})(V_L^{-1} V_R^{-1}) = (U_L V_L U_L^{-1} V_L^{-1})(U_R V_R U_R^{-1} V_R^{-1}) \quad (11.3)$$

Since the sole action of one of these factors must be proportional to the identity in its region of support, it follows that

$$(U_L V_L U_L^{-1} V_L^{-1}) = \exp(i\alpha); \ (U_R V_R U_R^{-1} V_R^{-1}) = \exp(-i\alpha) \ . \quad (11.4)$$

A nontrivial value of $\exp(i\alpha) \neq 1$ implies that the symmetry operations are represented *projectively* at the edges.

The dimension of such a projective representation has an immediate physical interpretation. A d-dimensional projective representation is associated with a d-dimensional edge mode. The AKLT chain should thus go along with a $d = 2$-dimensional projective representation of the appropriate protecting symmetry.

This also means that a $d = 1$-dimensional representation is trivial in that it does not host a protected edge state. This is reflected in the twin facts that, firstly, such a situation does not permit non-trivial values of α and, secondly, that the accompanying phase factor in the projective representation case can be gauged away, leaving behind a trivial non-projective symmetry representation via $\tilde{U}_L = \exp(i\alpha)U_L$.

Thus, the values of α for products in Eq. 11.1.1 do not fix the phases of the representation of the $U_{L,R}$ entirely. Like the magnetic field corresponding to different gauge choices of vector potential, there is a gauge-invariant content to these phases, and it is this which is used to group SPT phases into classes.

To use this as a basis for a topological classification scheme, one needs to determine which values of α can be deformed into one another continuously, and which cannot – the latter can then be said to have topological stability. Clearly, this is the case if the values are *discrete*, as this forbids a continuous deformation between them.

Perhaps the simplest instance is provided for a group consisting of a pair U, V of $\mathbb{Z}_2$ symmetries, i.e. $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. As U^2 is just the identity, $U_{L,R}^2$ can only be simple phase factors, which therefore commute with $V_{L,R}$. As $V_L^2 U_L^2 = V_L U_L^2 V_L = \exp(i\alpha) V_L U_L V_L U_L = \exp(2i\alpha) V_L^2 U_L^2$, it follows that $\exp(2i\alpha) = 1$, so that α can only take on two values, $\alpha = 0$ or π.

The object encoding a general classification scheme of SPT phases is then supplied by algebraic topology. The quantity in question is the second group cohomology group with coefficients in $U(1)$, denoted by $H^2(G, U(1))$. (Group cohomology is an abstract mathematical structure analogous to the cohomology of differential forms in Chapter 2.) For the example above, $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. The symmetry group $SO(3)$ turns out to have the same property, $H^2(SO(3), U(1)) = \mathbb{Z}_2$, identifying half-integer and integer spins as topologically distinct. In particular, this implies that the subgroup of π-rotations also protects the Haldane phase. We note that the case of the Floquet 0π paramagnet, discussed in Section 10.5.2 only has a single $\mathbb{Z}_2$ symmetry; there, it is the temporal aspect of the drive – which also underpins the possibility of period doubling – which supplies the remaining ingredient.

At the same time, in the absence of a non-trivial symmetry group G, it is clear that this classification scheme yields only one, the topologically trivial, outcome. This observation underpins the the statement that topological stability in one spatial dimension is predicated on symmetry protection.

An instance which does not yield a discrete set of outcomes is provided by $H^2(\mathbb{Z}\times\mathbb{Z}, U(1)) = U(1)$. This amounts to the possibility of a *continuous* set of phases: any given value would therefore not correspond to a topologically stable class. However, if one is considering a periodic system with a unit cell containing degrees of freedom with a finite-dimensional Hilbert space, the corresponding symmetry group will be finite dimensional, or a compact Lie group, both of which yield discrete outcomes and thence permit a topologically stable outcome.

The discussion of the previous paragraphs applies to unitary on-site symmetries (in particular excluding spatial symmetries such as translations) for 'bosonic' systems. The word bosonic refers to systems where operators acting on different edges of the system commute. Both of these conditions are restrictive in the sense that there are generic physical situations which violate them. The first is provided by the case of anti-unitary symmetries; and the second for fermionic systems, discussed in the following section, where operators with support on spatially disjoint regions need not commute on account of the *anticommutation* properties of Fermions: the phases arising due to quantum statistics *can* be probed non-locally.

The case of anti-unitary symmetries, T, such as the time-reversal symmetry discussed in Section 3.3, is quite analogous to the above discussion. The symmetry again fractionalises over the left and the right edge, where it thus acts independently.

Considering the case where $T = UK$, where U is an on-site symmetry and K is complex conjugation, and restricting ourselves to the case $T^2 = 1$, the ensuing treatment then makes use of the operator $\overline{U} = TUT$, to obtain $\mathbb{I} = (\overline{U}_L U_L)(\overline{U}_R U_R)$. Thence, $\overline{U}_L U_L = \exp(i\kappa)$, which in particular implies that $\overline{U}_L$ is proportional to the inverse of U_L, and hence commutes with it. It follows from complex conjugation that $\exp(i\kappa) = \exp(-i\kappa)$, so that $\kappa = 0$ or π. It follows that there are only two – hence topologically distinct – possibilities, $\overline{U}_L U_L = \pm 1$.

11.1.2 Fermionic symmetry fractionalisation

The above exposition has explicitly relied on the possibility of defining sets of operators on two ends of a chain which commute with each other, the gap of the bulk acting as an effective barrier, keeping the gapless modes localised at the edge. The issue is that quantum statistics is not strictly local in this sense, and single fermion