Quantum Geometry

recall : Hamiltonian $H = H(\lambda)$

e'system: $H(\lambda)|n(\lambda)\rangle = E_n(\lambda)|n(\lambda)\rangle$

gauge pot. A_λ :

- diag el.'s : $\langle n(\lambda)|A_\lambda|n(\lambda)\rangle = i\langle n(\lambda)|\partial_\lambda n(\lambda)\rangle$
- off-diag. el.'s : $n \neq m$

$$\langle n(\lambda)|A_\lambda|m(\lambda)\rangle = -i\frac{\langle n(\lambda)|\partial_\lambda H|m(\lambda)\rangle}{E_n(\lambda) - E_m(\lambda)}$$

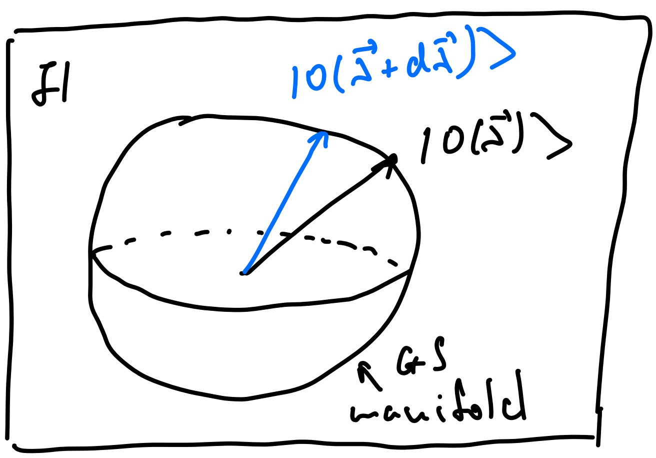
focus now on:

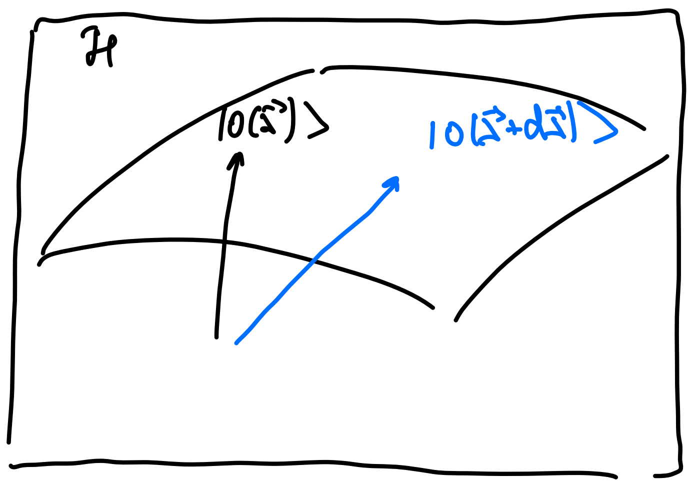
i) ground states (GS) : $H|0(\vec\lambda)\rangle = E_{GS}(\vec\lambda)|0(\vec\lambda)\rangle$

ii) $\vec\lambda \in \mathbb{R}^d$ multi-dimensional : $(\vec\lambda)_\beta = \lambda_\beta$

$$(\vec\nabla_\lambda)_\beta = \frac{\partial}{\partial\lambda_\beta} = \partial_\beta$$

- observation : $|0(\vec\lambda)\rangle$ lives on a surface in $\mathcal{H}$-space parameterized by $\vec\lambda$





Q: what is the distance ds^2 b/w $|0(\vec\lambda)\rangle$ & $|0(\vec\lambda+d\vec\lambda)\rangle$?

$ds^2 = 1 - |\langle 0(\vec{\lambda}) | 0(\vec{\lambda} + d\vec{\lambda})\rangle|^2$ probability of excitations

$= \chi_{\alpha\beta}\, d\lambda_\alpha\, d\lambda_\beta + \mathcal{O}(|d\lambda|^3)$

Taylor expand in $d\lambda$

$ds^2 \geq 0 \Rightarrow$ no linear terms

$\chi_{\alpha\beta}$: geometric tensor

$\rightarrow$ can measure prob. of exc. ds^2 in a quench exp.

$\lambda_\alpha \longrightarrow \lambda_\alpha + d\lambda_\alpha$:

probability amplitude:

$a_n = \langle n(\vec{\lambda} + d\vec{\lambda}) | 0(\vec{\lambda})\rangle \approx \langle \partial_\alpha n(\vec{\lambda}) | 0(\vec{\lambda})\rangle d\lambda_\alpha$

$= - \langle n(\vec{\lambda}) | \partial_\alpha 0(\vec{\lambda})\rangle d\lambda_\alpha = i \langle n | A_\alpha | 0\rangle d\lambda_\alpha$

prob. to excite system:

$$ds^2 = \sum_{n \neq 0} |a_n|^2 = \sum_{n\neq 0} \langle 0|A_\alpha|n\rangle\langle n|A_\beta|0\rangle d\lambda_\alpha d\lambda_\beta + \mathcal{O}(|d\lambda|^3)$$

$$\approx \sum_n \langle 0|A_\alpha|n\rangle\langle n|A_\beta|0\rangle d\lambda_\alpha d\lambda_\beta - \langle 0|A_\alpha|0\rangle\langle 0|A_\beta|0\rangle d\lambda_\alpha d\lambda_\beta$$

$$= \left(\langle 0|A_\alpha A_\beta|0\rangle - \langle 0|A_\alpha|0\rangle\langle 0|A_\beta|0\rangle\right) d\lambda_\alpha d\lambda_\beta$$

$= \langle 0|A_\alpha A_\beta|0\rangle_c\, d\lambda_\alpha d\lambda_\beta$ connected GS correlation function of gauge pot.

$\Rightarrow \chi_{\alpha\beta} = \langle 0|A_\alpha A_\beta|0\rangle_c$ defines geom. tensor in terms of gauge pot.

→ in terms of GS wavefn:

$$\chi_{\alpha\beta} = \langle \partial_\alpha 0 | \partial_\beta 0 \rangle_c = \langle 0 | \overleftarrow{\partial}_\alpha \partial_\beta | 0 \rangle_c$$

$$= \langle \partial_\alpha 0 | \partial_\beta 0 \rangle - \langle \partial_\alpha 0 | 0 \rangle \langle 0 | \partial_\beta 0 \rangle$$

Remarks:

1) $\chi_{\alpha\beta}$ is invariant under the global $U(1)$ gauge transf.

$$|n(\vec{\lambda})\rangle \longrightarrow e^{i\phi_n(\vec{\lambda})} |n(\vec{\lambda})\rangle \qquad (HW)$$

2) $\chi_{\alpha\beta} = \chi^*_{\beta\alpha}$ is hermitian

but only symmetric part of χ contributes to the distance b/w the GS's:

$$\chi_{\alpha\beta}\, d\lambda_\alpha\, d\lambda_\beta = \frac{\chi_{\alpha\beta} + \chi_{\beta\alpha}}{2} d\lambda_\alpha\, d\lambda_\beta + \underbrace{\underbrace{\frac{\chi_{\alpha\beta} - \chi_{\beta\alpha}}{2}}_{\substack{\text{anti-symm} \\ \text{under} \\ \alpha \leftrightarrow \beta}} \underbrace{d\lambda_\alpha\, d\lambda_\beta}_{\substack{\text{symmetric} \\ \text{under} \\ \alpha \leftrightarrow \beta}}}_{=0}$$

def: Fubini-Study metric tensor

$$g_{\alpha\beta} := \frac{1}{2}\left(\chi_{\alpha\beta} + \chi_{\beta\alpha}\right) = \frac{1}{2} \langle 0 | A_\alpha A_\beta + A_\beta A_\alpha | 0 \rangle_c$$

$$= \frac{1}{2} \langle 0 | [A_\alpha, A_\beta]_+ | 0 \rangle_c$$

↳ anti-commutator

$$= \mathrm{Re}\, \langle 0 | A_\alpha A_\beta | 0 \rangle_c$$

$g_{\alpha\beta}$ determines distance b/w quantum states def: Berry curvature

$$F_{\alpha\beta} := i\left(\chi_{\alpha\beta} - \chi_{\beta\alpha}\right) = -2\,\mathrm{Im}\,\chi_{\alpha\beta}$$

$$= i\,\langle 0|[A_\alpha, A_\beta]|0\rangle_c$$

$F_{\alpha\beta}$ determines the quantum geometry & topology
i.e. geometry & topology of the GS-manifold

def: Berry connection

$$A_\alpha := \langle 0|\mathcal{A}_\alpha|0\rangle = i\langle 0|\partial_\alpha|0\rangle = i\langle 0|\partial_\alpha 0\rangle$$

GS expectation value of gauge pot $\mathcal{A}_\alpha$

$$\Rightarrow F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

-> relation b/w Berry connection & the phase of the GS wavefn.:

for $\psi_{GS}(\vec{r};\vec{\lambda}) = e^{i\phi(\vec{\lambda})}\,|\psi_{GS}(\vec{r};\vec{\lambda})|$

$$\Rightarrow A_\alpha = -\int d\vec{r}\,|\psi_{GS}(\vec{r};\vec{\lambda})|^2\,\partial_\alpha\phi = -\partial_\alpha\phi$$

integral of Berry connection A_α over a closed path C on the GS-manifold gives the total phase accumulated by the GS along the path: Berry's phase

↳ Aharonov-Bohm effect

$$\varphi_B := \oint_C \partial_\alpha\phi\, d\lambda_\alpha = -\oint_C A_\alpha\, d\lambda_\alpha \overset{\text{Stokes}}{=} \int_S F_{\alpha\beta}\, d\lambda_\alpha \wedge d\lambda_\beta$$

wedge product
-> generalization of $\times$-product to arbitrary d

Example: 2LS, $\vec{\lambda} = (\theta, \varphi)$

$$H = \vec{h}(\theta,\varphi)\cdot\vec{\sigma}, \quad \vec{h}(\theta,\varphi) = h\begin{pmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ \sin\theta\end{pmatrix}$$

$$\Rightarrow |0\rangle = \begin{pmatrix}\cos\theta/2\\ e^{i\varphi}\sin\theta/2\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}\sin\theta/2\\ -e^{i\varphi}\cos\theta/2\end{pmatrix}$$

- geometric tensor: 2×2 matrix $\chi = \begin{pmatrix}\chi_{\theta\theta} & \chi_{\theta\varphi}\\ \chi_{\varphi\theta} & \chi_{\varphi\varphi}\end{pmatrix}$

$$\chi_{\theta\theta} = \langle\partial_\theta 0|\partial_\theta 0\rangle - \langle\partial_\theta 0|0\rangle\langle 0|\partial_\theta 0\rangle = \ldots = \frac{1}{4} = g_{\theta\theta}$$

$$\chi_{\varphi\varphi} = \ldots = \frac{1}{4}\sin^2\theta = g_{\varphi\varphi}$$

$$\chi_{\theta\varphi} = \ldots = \frac{i}{4}\sin\theta = \chi^*_{\varphi\theta} \quad ; \quad g_{\theta\varphi} = 0 = g_{\varphi\theta}$$

- Berry curvature

$$F_{\theta\varphi} = \partial_\theta A_\varphi - \partial_\varphi A_\theta = \ldots = -\frac{1}{2}\sin\theta = -F_{\varphi\theta}$$

Rmks

→ Fubini-Study metric of GS-manifold for the 2LS is equivalent to the metric of a sphere of radius $r = 1/2$

→ for the excited state manifold, χ is the same but F has opposite sign (HW)

Q: How do we measure $\chi_{\alpha\beta}$ & $F_{\alpha\beta}$?

Linear Response Theory

- consider equilibrium system described by Hamiltonian H_0
- equilibrium expt. value of observable $\mathcal{O}$:

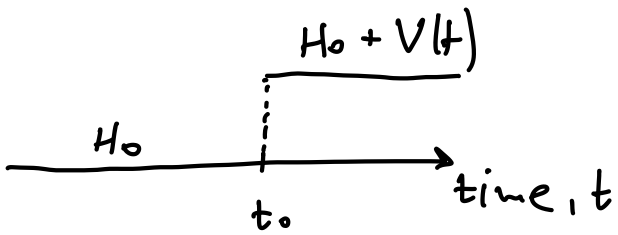
$$\langle \mathcal{O} \rangle = \mathrm{tr}(\rho_0 \mathcal{O}) = \sum_n \frac{e^{-\beta E_n}}{Z} \langle n | \mathcal{O} | n \rangle$$

where $H_0 |n\rangle = E_n |n\rangle$

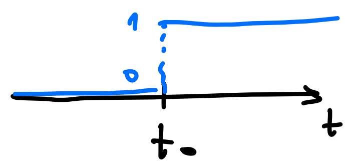
& $\rho_0 = \frac{e^{-\beta H_0}}{Z} = \sum_n \frac{e^{-\beta E_n}}{Z} |n\rangle\langle n|$ density matrix

& $Z = \mathrm{tr}\, \rho_0 = \sum_n e^{-\beta E_n}$ partition fn

- turn on perturbation $V(t)$ at time t_0



$$H(t) = \underset{\text{large}}{H_0} + \underset{\text{small}}{V(t)}\, \Theta(t > t_0)$$

$\Theta(t > t_0) :=$ 

→ expectation value changes in time after quench:

$$\langle \mathcal{O}(t) \rangle = \mathrm{tr}(\rho(t)\, \mathcal{O})$$

where e'states evolve following Schrödinger's eq.:

$$i\partial_t |n(t)\rangle = H(t)\, |n(t)\rangle$$

. change reference frame to eliminate H_0 - term
(recall V is a weak perturbation)
↳ interaction picture

lab frame (Schrödinger's picture)

$$H(t) = H_0 + V(t)\,\Theta(t>t_0)$$

$$U(t,t_0) = \mathcal{T} e^{-i\int_{t_0}^{t} ds\, H_0 + V(s)}$$

new frame (interaction picture)

$$\tilde{H}(t) = \Theta(t>t_0)\, e^{-itH_0}\, V(t)\, e^{+itH_0}$$

$$=: \Theta(t>t_0)\, \tilde{V}(t)$$

$$\tilde{U}(t,t_0) = \mathcal{T} e^{-i\int_{t_0}^{t} ds\, \tilde{V}(s)}$$

$$\approx \mathbb{1} - i\int_{t_0}^{t} ds\, \tilde{V}(s) + \dots$$

$\Rightarrow$ go to int. picture

$$\langle \mathcal{O}(t)\rangle \overset{\downarrow}{=} \langle \tilde{\mathcal{O}}(t)\rangle = \mathrm{tr}\left(\tilde{\rho}(t)\, \tilde{\mathcal{O}}(t)\right)$$

$$= \mathrm{tr}\left(\tilde{U}(t,t_0)\, \rho_0\, \tilde{U}^\dagger(t,t_0)\, \tilde{\mathcal{O}}(t)\right)$$

$$\approx \mathrm{tr}\Big(\underbrace{\Big(\mathbb{1} - i\int_{t_0}^{t} ds\, \tilde{V}(s)\Big)\rho_0\Big(\mathbb{1} + i\int_{t_0}^{t} ds\, \tilde{V}(s)\Big)}_{\approx \rho_0 - i\int_{t_0}^{t} ds\, [\tilde{V}(s), \rho_0]}\tilde{\mathcal{O}}(t)\Big)$$

$$= \underbrace{\mathrm{tr}\left(\rho_0\, \tilde{\mathcal{O}}(t)\right)}_{= \mathrm{tr}\left(e^{-itH_0}\mathcal{O} e^{+itH_0}\rho_0\right)} - i\int_{t_0}^{t} ds\, \underbrace{\mathrm{tr}\left([\tilde{V}(s),\rho_0]\,\tilde{\mathcal{O}}(t)\right)}_{= \mathrm{tr}\left(\rho_0\, [\tilde{\mathcal{O}}(t), \tilde{V}(s)]\right)}$$

$[\rho_0, H_0] = 0$

$$\overset{\text{tr is cyclic}}{=} \mathrm{tr}\left(\mathcal{O}\rho_0\right) = \langle \mathcal{O}\rangle$$

tr is cyclic

$$= \langle \mathcal{O}\rangle - i\int_{t_0}^{t} ds\, \langle [\tilde{\mathcal{O}}(t), \tilde{V}(s)]\rangle + \mathcal{O}(\|V\|^2)$$

$\rightarrow$ linear response relative to equil. expct. value $\langle \mathcal{O}\rangle$

$$\langle \mathcal{O}(t)\rangle - \langle \mathcal{O}\rangle = i\int_{-\infty}^{t} dt'\, \Theta(t'>t_0)\, \langle [\tilde{V}(t'), \tilde{\mathcal{O}}(t)]\rangle$$

Kubo formula

<u>example</u>: response of magnetization of a system to a weak applied / external magnetic field

external perturbation: $V(t) = -\vec{m} \cdot \vec{h}(t)$

AC magnetic field: $\vec{h}(t) = \vec{h}_0 e^{-i\omega t}$

magnetization: $\vec{m} = g \, \mu_0 \sum_j \vec{S}_j$

gyromagn. ratio — Bohr magneton — Pauli vector of spin j: $\vec{S}_j = \frac{1}{2} \vec{\sigma}_j$

<u>linear response regime</u>: response is proportional to intensity of perturbation:

$$M_\alpha(t) := \langle \tilde{m}_\alpha(t) \rangle = \epsilon_{\alpha\beta} \, h_\beta(t)$$

magnetic susceptibility

from Kubo formula with $\mathcal{O} = m_\alpha$

$$\langle \tilde{m}_\alpha(t) \rangle - \langle m_\alpha \rangle = i \int_{-\infty}^{0} dt' \, \Theta(t > t_0) \langle [-\tilde{m}_\beta(t') \, h_\beta(t'), \tilde{m}_\alpha(t)] \rangle$$

$$= \underbrace{i \int_{-\infty}^{0} dt' \, \Theta(t' > t_0) \langle [\tilde{m}_\alpha(t'), \tilde{m}_\beta(t)] \rangle e^{-i\omega(t'-t)}}_{= \epsilon_{\alpha\beta}(\omega)} \underbrace{e^{-i\omega t} \, h_{0,\beta}}_{= h_{0,\beta}(t)}$$

susceptibility at $t = 0$, $t_0 \to -\infty$, along $\alpha = \beta$

$$\epsilon(\omega) = i \int_{-\infty}^{0} dt' \, e^{-i\omega t'} \langle [\tilde{m}(0), \tilde{m}(t')] \rangle$$