Lyapunov Vectors for Large Systems

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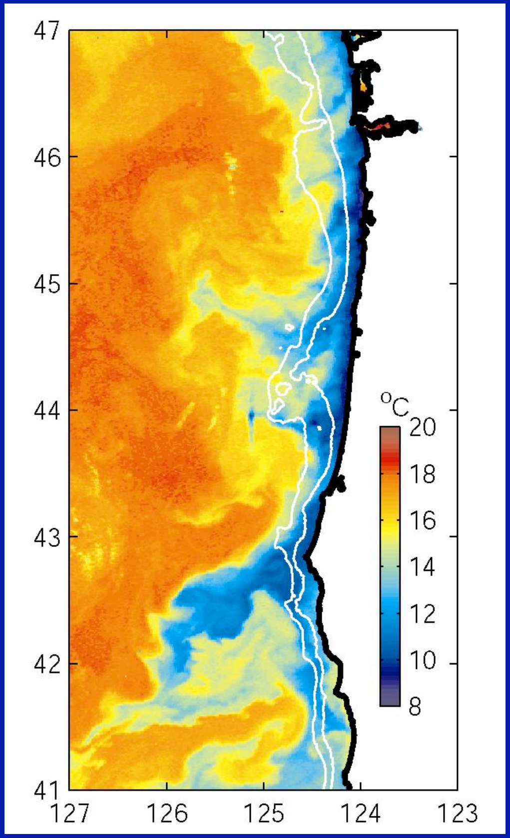
Coastal Transition Zone

Oregon CTZ SST



24 Aug 2003

Off Oregon and California, CTZ includes shelf, slope, adjacent ocean interior

Complex flows in CTZ govern shelf/ocean exchange

CTZ flow strongly influenced by continental slope—not well resolved in basin scale models

Natural coastal domain includes CTZ and extends 200-300 km offshore and alongshore 41º-47ºN

Coastal sea-level response to large-scale winds

Forced and damped first-order wave equation

$$\zeta_t + c\zeta_x + r\zeta = \tau$$

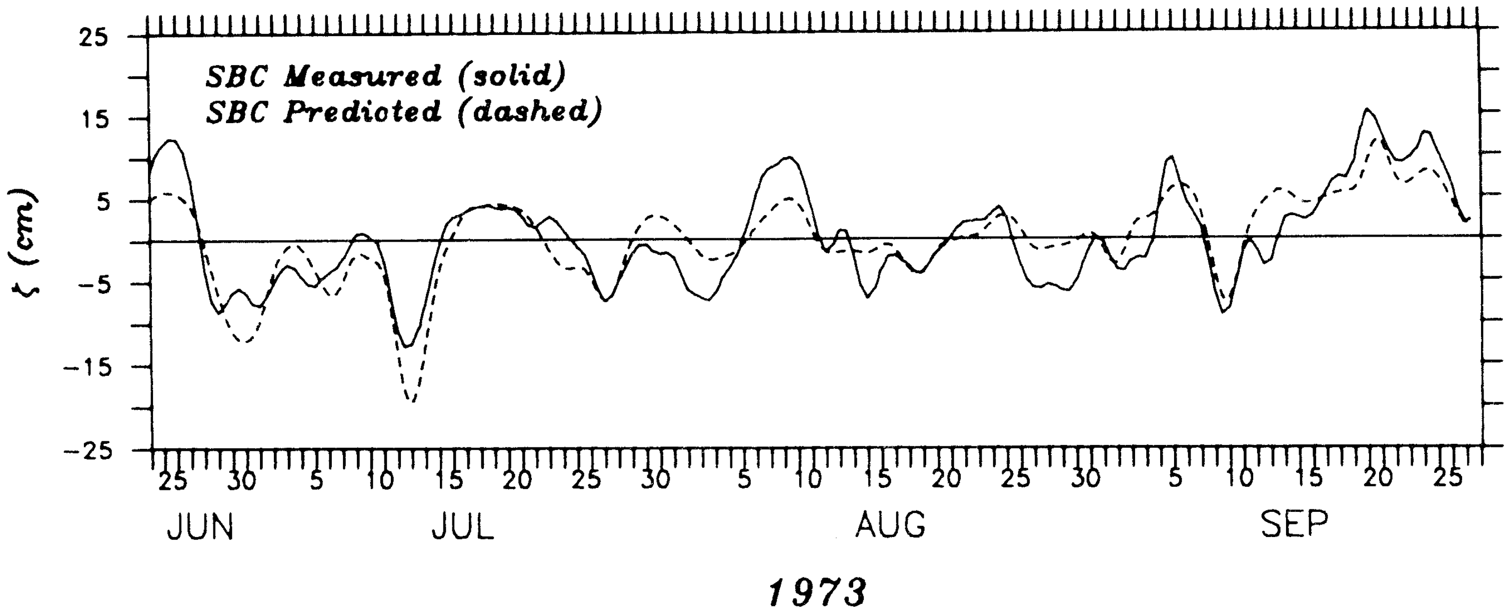
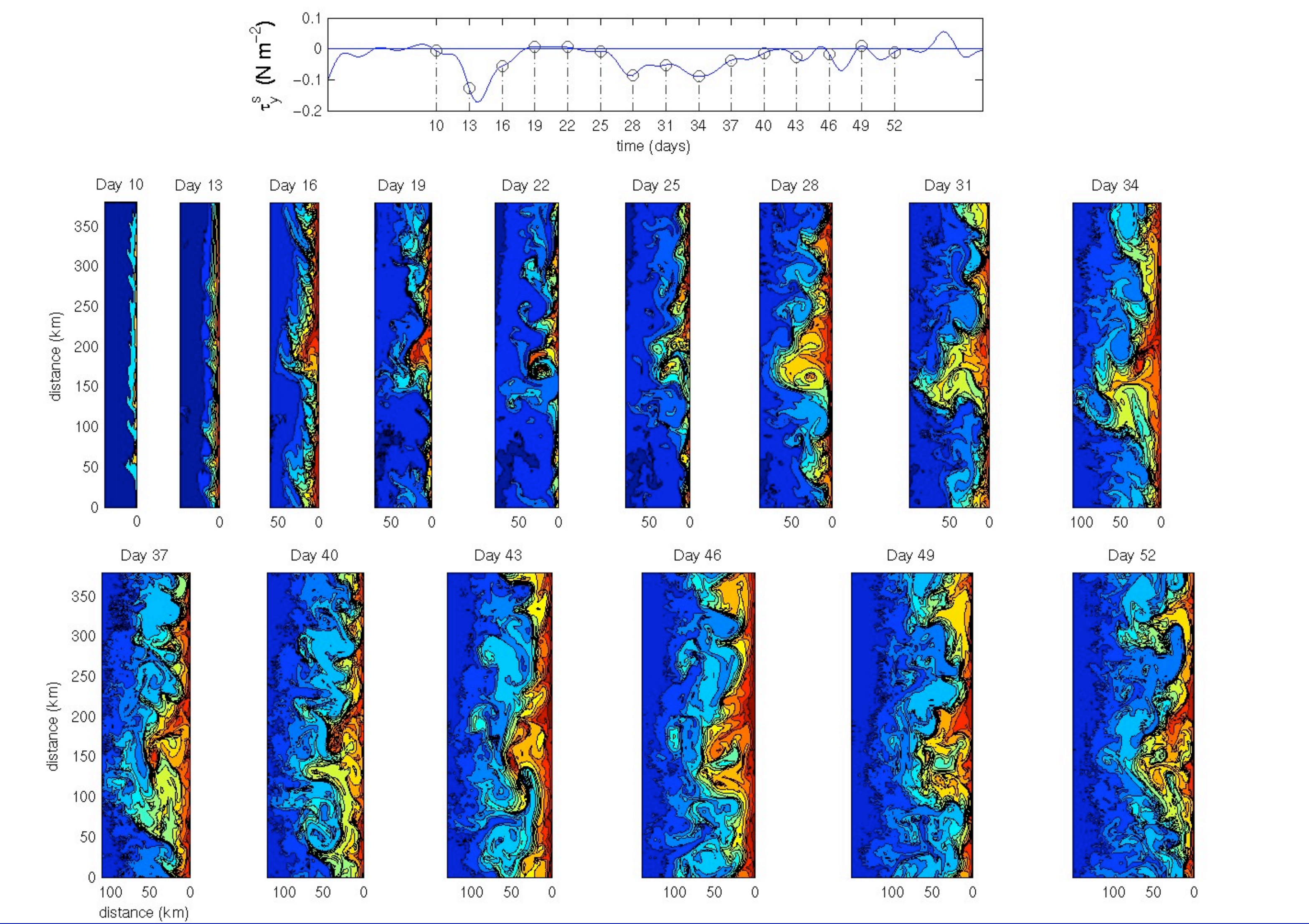


FIG. 18. Time series of measured and predicted ζ at SBC.

(Halliwell and Allen, 1984)

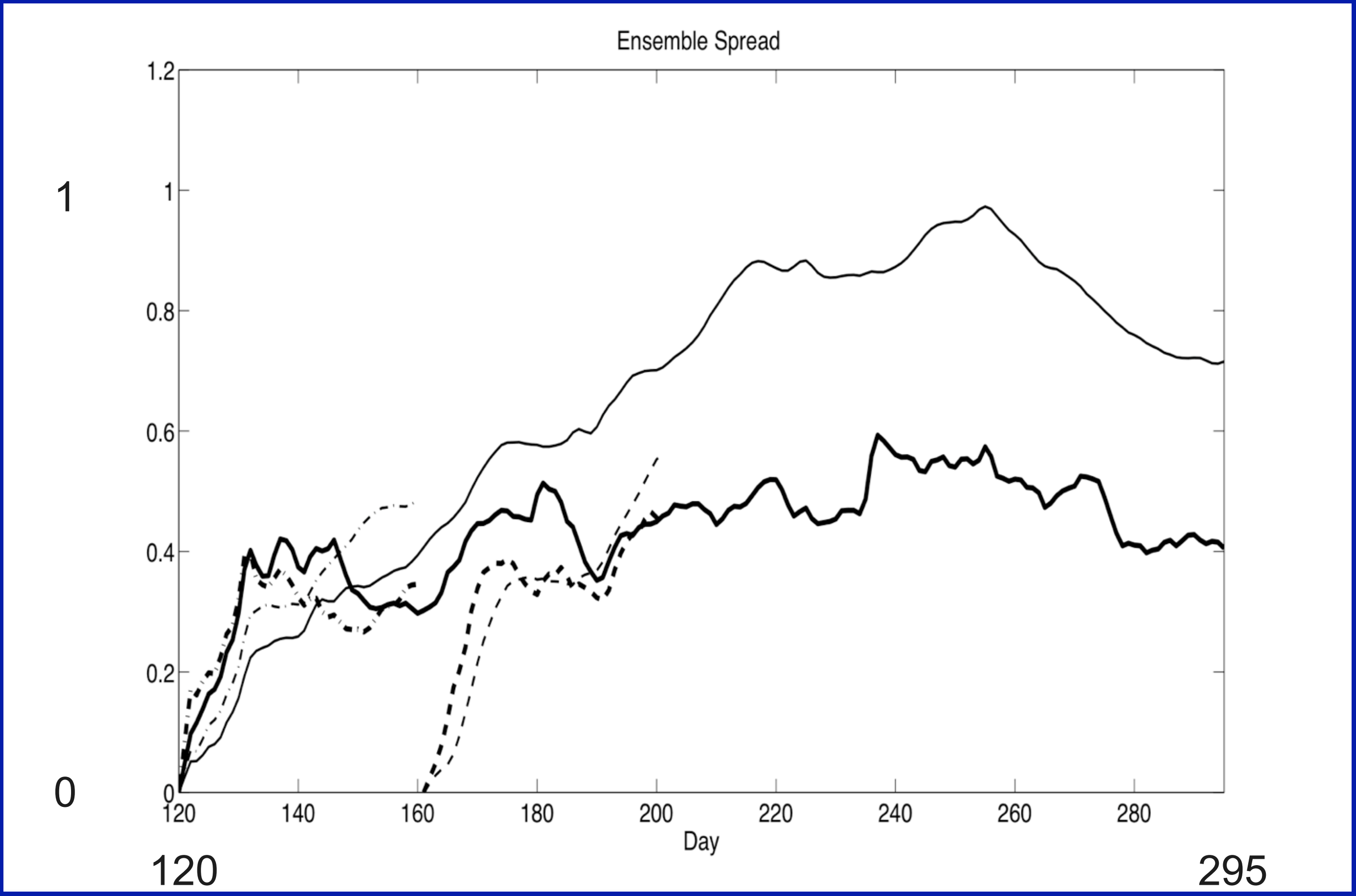
Numerical modeling - instabilities of coastal upwelling jet



(see Durski and Allen, JPO, 2005)

Coastal ocean model ensemble – wind-analysis error

Normalized ensemble spread



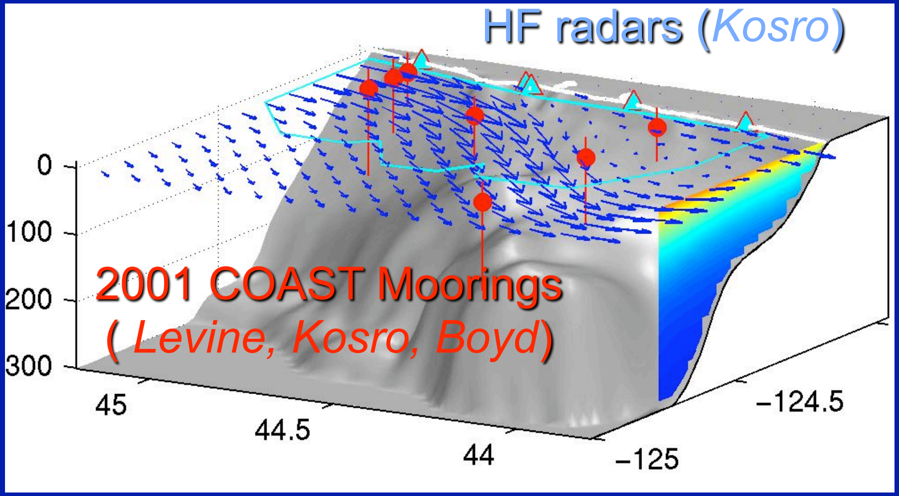
Day

(Kim, Samelson, and Snyder, submitted)

Data Assimilation

- Representer-based variational DA on finite time intervals (alternate form of 4DVAR)
- Strong-constraint (correct initial conditions) and weak-constraint (correct ICs plus forcing and dynamical errors)
- DA and dynamics: instabilities and CTZ-eddy/shelf-current interactions
- Test TL & Adj ROMS + Inverse Ocean Modeling (IOM) System (Chua & Bennett)

Previous work:
Optimal Interpolation w/ POM
(Oke et al. JGR 2002;
Kurapov et al. 2005abc)



Instability and disturbance growth

- Steady flow: standard normal modes, exponential growth
- Time-periodic flow: Floquet-vector normal modes, product of time-periodic function and exponential growth
- Nonperiodic flow: Lyapunov-vector normal modes, asymptotic exponential growth

LVs are generalized normal modes: intrinsic, invariant, consistent

(Lyapunov vector = “Covariant Lyapunov vector” = “characteristic vector”)

Lyapunov exponents and vectors

(Oseledets, 1968)

Oseledets theorem

From Scholarpedia

Valery Oseledets (2008), Scholarpedia, 3(1):1846. doi:10.4249/scholarpedia.1846

Curator: Dr. Valery Oseledets, Lomonosov Moscow State University, Russia

Contents

Lyapunov exponents and vectors

(Oseledets, 1968)

Let $A_0, A_1, \ldots, A_t, \ldots$ be a sequence of nonsingular $m \times m$ matrices satisfying $\frac{1}{t}\log\|A_t\| \to 0$, and let $A(t) = A_{t-1}\cdots A_0$ for $t = 1, 2, \ldots$. Suppose there is a number $c > 0$ such that $\|A(t)\| \le \exp(ct)$. The Lyapunov exponent of a nonzero vector $e \in R^m$ is defined by

$$\chi(e) = \limsup \frac{1}{t} \log \|A(t)e\|. \tag{1}$$

More generally, let e^k be a subspace of R^m of dimension k and $\lambda(t, e^k)$ the absolute value of the determinant of the linear transformation $e^k \to A(t)e^k$ defined by the matrix $A(t)$. In particular, $\lambda(t, e^1) = \dfrac{\|A(t)e\|}{\|e\|}$ for a nonzero vector $e \in e^1$. The Lyapunov exponent of the subspace is defined by $\chi(e^k) = \limsup \frac{1}{t}\log\lambda(t, e^k)$. This reduces to the definition (1) for the one-dimensional case.

The function $\chi(e)$ attains at most m distinct values $\chi_1 < \chi_2 < .. < \chi_r$ for some $r \le m$. Let L^i be the subspace defined by the condition $\chi(e) \le \chi_i, 0 \ne e \in L^i$. We have that $L^0 = 0 \subset L^1 \subset \ldots \subset L^r = R^m$ and that $\chi(e) = \chi_i$ for $0 \ne e \in L^i \setminus L^{i-1}, i = 1, ..r$. The number $k_i = \dim(L^i) - \dim(L^{i-1})$ is called the multiplicity of the value χ_i. The sequence A(t) is said to be **Lyapunov regular** if $\sum_{i=1}^{r} k_i\chi_i = \lim \frac{1}{t}\log|\det A(t)|$.

Lyapunov exponents and vectors

(Oseledets, 1968)

Theorem 1. If the sequence $A(t)$ is Lyapunov regular, then the Lyapunov exponents of all orders are exact, i.e., $\boxed{\chi(e^k) = \lim \frac{1}{t} \log \lambda(t, e^k).}$

Theorem 4. The function $t \to A(t, x)$ is Lyapunov regular as $t \to \pm\infty$ for μ-almost every x. There is a measurable $\boxed{\text{splitting } R^m = E^{k_1(x)} \oplus \ldots \oplus E^{k_r(x)}}$ such that $\lim\limits_{t \to \pm\infty} \frac{1}{t} \log \|A(t, x)e\| = \chi_i(x)$ for $0 \neq e \in E^{k_i(x)}$ and $\dim(E^{k_i(x)}) = k_i(x)$. If $e^k \subset E^{k_i(x)}$ then $\lim\limits_{t \to \pm\infty} \frac{1}{t} \log \lambda(e^k) = k\chi_i(x)$ uniformly over $e^k \subset E^{k_i}(x)$. Furthermore, $\lim\limits_{t \to \pm\infty} \frac{1}{t} \log \sin(\angle(E^{k_i}(T^t x), E^{k_j}(T^t x)) = 0, i \neq j$ and $\chi_i(Tx) = \chi(x), k_i(Tx) = k_i(x), E^{k_i(Tx)}(Tx) = A(x)E^{k_i(x)}(x)$. The subspaces $E^{k_i(x)}$ are called Oseledets subspaces.

(Theorem 3 is the multiplicative ergodic theorem)

Lyapunov exponents and vectors

(Oseledets, 1968)

Theorems 1-3:

Exponents (exponential growth rates; nested subspaces; MET)

Theorem 4:

Vectors (time-invertible case; splitting; vectors if multiplicity = 1)

Lyapunov exponents and vectors

Get vectors from intersections of subspaces (which can be obtained, e.g., via Gram-Schmidt orthogonalization) for forward-time and backward-time limits, not from eigenvalue problem.

Lyapunov exponents and vectors

Get vectors from intersections of subspaces (which can be obtained, e.g., via Gram-Schmidt orthogonalization) for forward-time and backward-time limits, not from eigenvalue problem.

Problem:

For kth LV in N-dimensional system, must compute intersection of subspaces of dimension k and N-k+1.

If N is large, this is a big computation.

Lyapunov vectors and singular vectors

- Lyapunov vectors: fundamental (invariant) time-dependent normal modes for instability of time-dependent flow
- Singular vectors: optimal transient growth in specified norm over fixed time interval

Singular vectors (Optimal perturbations)

"Singular vectors" are the eigenvectors of:

$$L(t_2, t_1)^T N L(t_2, t_1) \xi_j(t_1; t_1, t_2) = \sigma_j^2 N \xi_j(t_1; t_1, t_2)$$

Here L is the linear propagator, N is the norm, and t_1 and t_2 are the initialization and optimization times.

The long-time limit singular vectors are related to the Lyapunov vectors.

Long-time singular vectors and Lyapunov vectors

"Forward" SVs:

(initialize at t; optimize at t_2)

$$\hat{\xi}_j(t) = \lim_{t_2 \to \infty} \xi_j(t; t, t_2)$$

$$\hat{\xi}_j(t) = \sum_{i=j}^{N} \hat{q}_{ji}\phi_i(t)$$

"Backward" SVs:

(initialize at t_1; optimize at t)

$$\hat{\eta}_j(t) = \lim_{t_1 \to -\infty} \xi_j(t; t_1, t)$$

$$\hat{\eta}_j(t) = \sum_{i=1}^{j} \hat{p}_{ji}\phi_i(t)$$

=> forward/backward SVs provide bases for Lyapunov subspaces

The trick (geometrically)

So, forward/backward SVs provide bases for Lyapunov subspaces…and SVs are orthogonal.

Thus, we can use *k-1* backward SVs to find the *N-k+1* Lyapunov subspace, by taking the complement of the (*k-1*)-SV subspace.

Analytically, this is expressed in terms of Kronecker-delta relations for the SV inner-products; to find the LV, we end up solving for the null vector of a *k* x *k* matrix.

Efficient recovery of Lyapunov vectors from singular vectors (with C. Wolfe)

- Standard method: For *k*-th LV, find intersection of *k*-th backward SV subspace with (*N* - *k* + 1)-th forward SV subspace; => N + 1 SV's required to obtain LV in *N*-dimensional system

- New method: *2k-1* SV's required to obtain first *k* LV's in *N*-dimensional system. Note that often *k* << *N*.

Wolfe, C. L., and R. M. Samelson, 2007. An efficient method for recovering Lyapunov vectors from singular vectors. Tellus, 59A, 355-366.

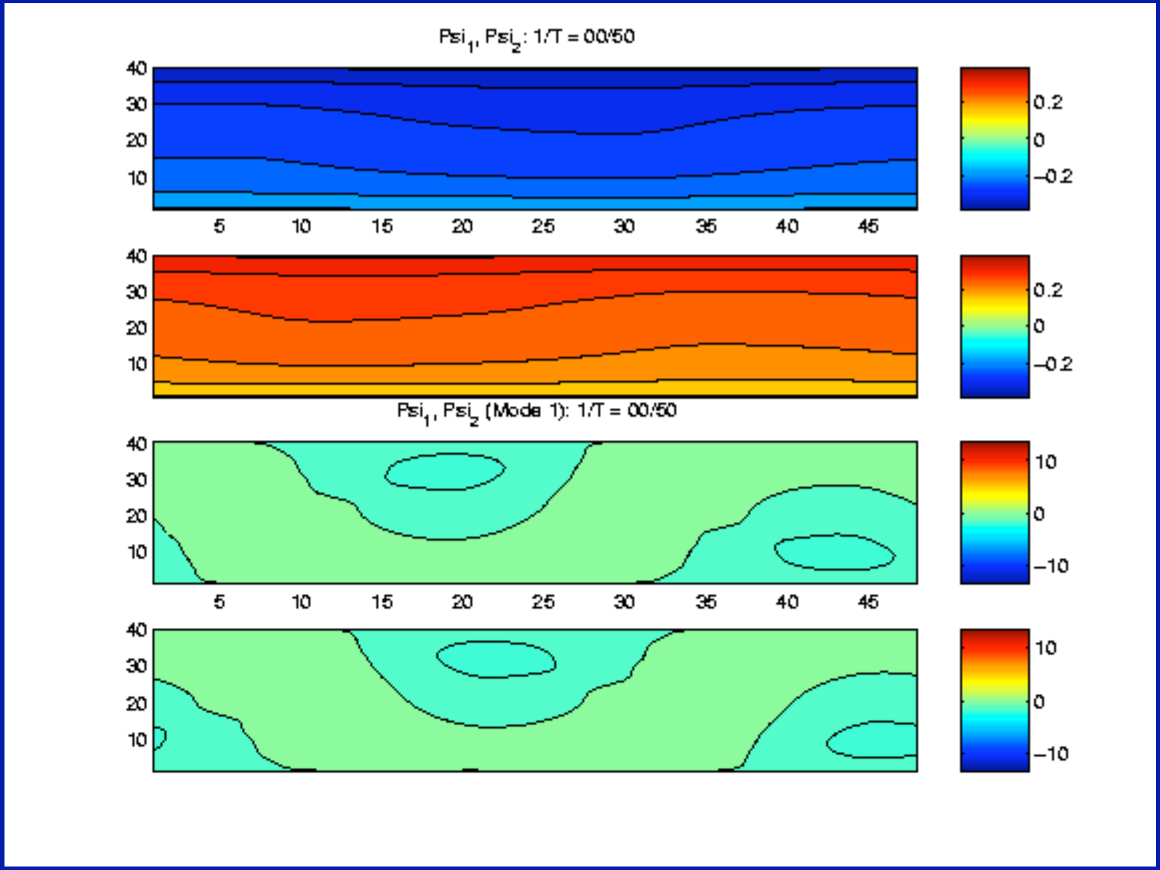
Unstable baroclinic wave-mean oscillation

Two-layer, quasigeostrophic, periodic-channel, pseudo-spectral model, ~4000 DOF

(Essentially, a pair of coupled 2D vorticity equations; PDEs in (*x*,*y*,*t*).)

Basic oscillation

FV #1



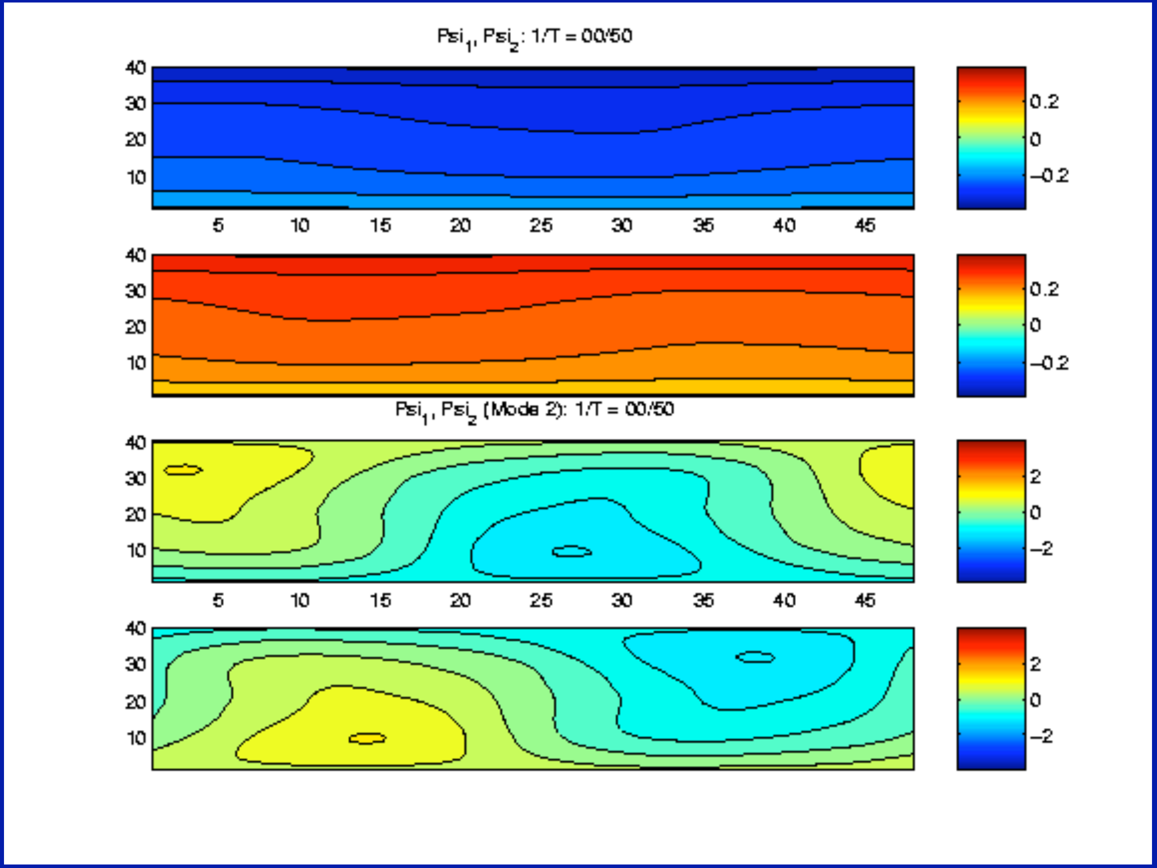
Unstable baroclinic wave-mean oscillation

Two-layer, quasigeostrophic, periodic-channel, pseudo-spectral model, ~4000 DOF

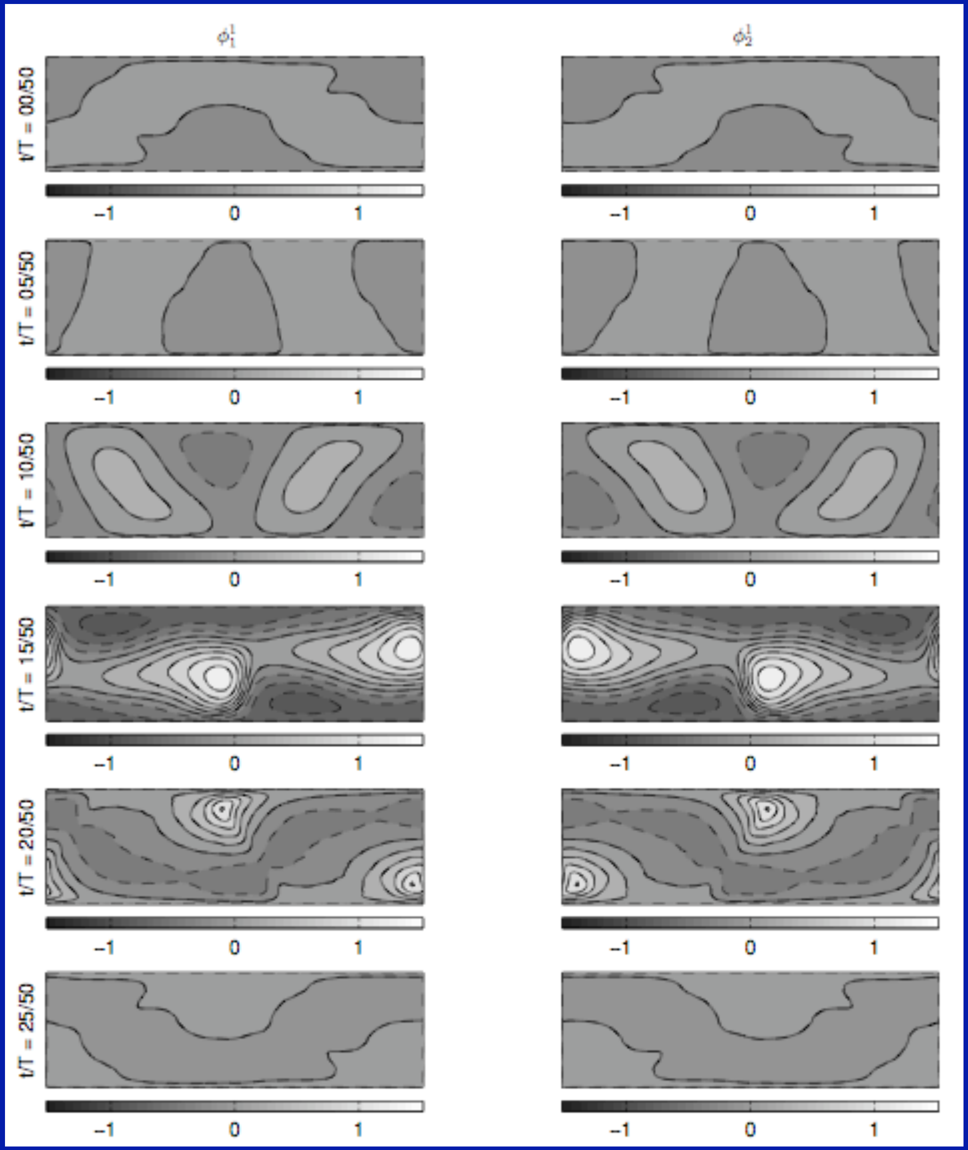
(Essentially, a pair of coupled 2D vorticity equations; PDEs in (*x*,*y*,*t*).)

Basic oscillation

FV #2



Lyapunov (Floquet) vector #1



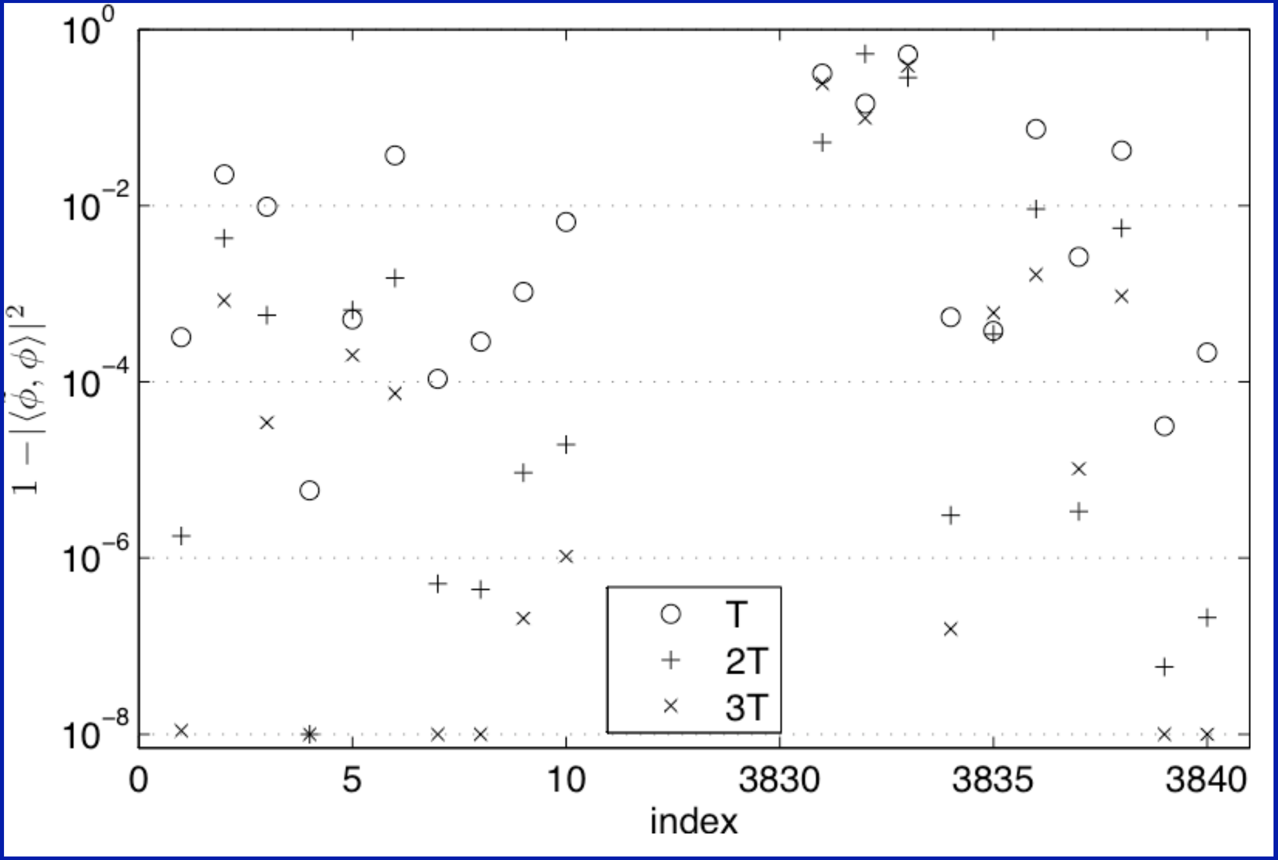
Recovery of LVs (FVs) from SVs

Relative error in estimate of 10 leading (and 10 trailing) LVs from recoveries using 20 SVs

vs.

LV index

N = 3840 DOF



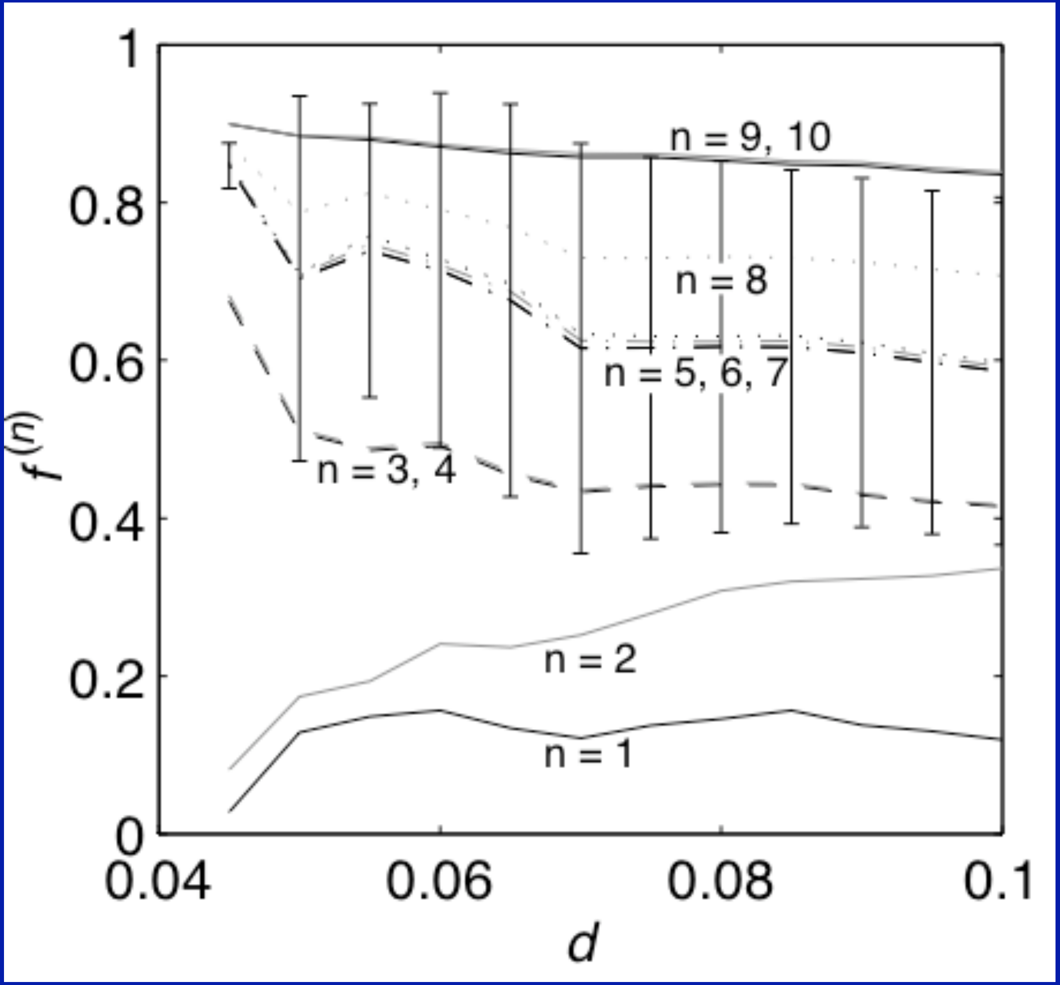
LV (FV) description of local chaotic attractor structure

Variance of chaotic attractor near-returns for leading *n* LVs

vs.

Near-return distance

In contrast: Leading-*n*-SV explained variance less than 0.1



Notes on LVs for baroclinic wave-mean oscillation

- 3840 (or more) DOF; but only a few (3-4) unstable modes

- Lyapunov exponents are small compared to local growth and
  decay rates for FV structure function (i.e., “DOS violation”)

- Convergence to LVs in 1-3 T = 2-6 baroclinic lifecycles

- Is the analogy to the atmosphere valid?

  …multiply DOF by ratio of Earth area/domain area

  => get something like the “500” unstable
  unstable baroclinic atmospheric modes

  …but, error doubling rates in atmosphere seem much larger
  than these Lyapunov exponents, i.e., similar to baroclinic
  instability growth rates

End of results

Now, onward to pure speculations….

LVs for coastal ocean model

Not yet computed, but expect separation between slow, large-scale instabilities in the offshore regime, and fast, small-scale instabilities in the onshore (shelf) regime.

Opportunities to learn about dynamics and to contribute to construction of good data assimilation scheme.

Follow-on question: interactions between shelf and offshore.

A good problem!

LVs from forecast/analysis cycle of global NWP model?

Not yet computed, but numerically feasible using SV algorithm!

Some brief preliminary computations were done by Carolyn Reynolds (NRL, Monterey, USA) with the NOGAPS model. These suggested (partial) convergence of the needed backward and forward SVs after 10 days (but need to filter inertial instabilities at tropopause, etc.…):

Similarities for
Evolved SVs for /data/reynoldc/predict/2007010100/LVFp240m216/histsv
Evolved SVs for /data/reynoldc/predict/2007010100/LVFp240m240/histsv

	1	2	3	4	5	6	7	8	9	10	
1	95	4	0	0	0	0	0	0	0	0	99
2	4	84	6	0	0	3	0	3	0	0	100
3	1	5	76	10	0	0	6	0	0	0	98
4	0	0	9	72	0	16	0	0	1	0	98
5	0	6	2	11	0	64	5	9	0	0	97
6	0	0	0	1	95	0	0	0	0	2	98
7	0	0	3	1	0	1	32	42	15	0	94
8	0	0	0	3	0	10	31	42	10	0	96
9	0	0	2	0	0	1	23	1	58	1	86
10	0	0	0	0	0	2	0	1	13	4	20
	100	99	98	98	95	97	97	98	97	7	

SIMILARITY(1:2)= 88.60 AVE DIAG ELEM. = 46.30

Compute “THE” LVs of the global atmosphere from multi-decadal reanalysis???

LVs are invariant => only need to do it ONCE!

For example, for 0000 UTC 1 January 1990. Then evolve by linearization to get later (earlier) structures….

We would know the “DIMENSION OF THE ATTRACTOR” and the “STRUCTURE OF THE TANGENT SPACE”!

We would have “THE BASIS” for the optimal DA subspace!

BUT…

Compute “THE” LVs of the global atmosphere from multi-decadal reanalysis???

BUT…

- Accuracy requirements would be prohibitive for linearized evolution to later (earlier) structure (amplification over years of baroclinic growth rate of 1/days)

- Convergence? LV convergence limited by LE differences; error growth can be faster: must link together many overlapping calculations (=> Is there a ‘shadowing theorem’?)

- Atmosphere not really recurrent (Hunt – ergodicity; Kalnay – space-time localization of bred vectors)

On the other hand, if it can’t be done, can the “chaotic attractor” idea of synoptic variability be sustained?

Finally, a more modest question….

Is this SV-based intersection algorithm for computing LVs equivalent to the Ginelli et al. algorithm (or, what is the relation between the two)?

ECODYC10 Discussion points

A. Computing Lyapunov Vectors
("Covariant", "characteristic")

B. Are global Lyapunov Vectors the right linear objects
for analysis of multi-scale systems like the atmosphere?

- Mr. Magoo vs. The Ballerina, or
How I Violated DOS And Lived To Tell About It

- A toy model for computation of "THE" LVs
of the global atmosphere

A. Computing Lyapunov vectors (“Covariant”, “characteristic”)

How do the Ginelli et al. forward-backward algorithm and the Wolfe-Samelson SV-intersection algorithm compare?

1. Are they equivalent? (No, they seem to be different.)

2. Is one more efficient than the other, and, if so, which one?

3. Can a mathematical result be proven, i.e., convergence?

Compute *n-th* Lyapunov Vector ϕ_n from *2n-1* Singular Vectors:

1. Compute *n* limiting backward SVs or *n-th* order GS orthogonalized basis η_j
2. Compute *n-1* limiting forward SVs ξ_j
3. Compute inner products and form *n* x *n* matrix ***D***; solve for zero eigenvector

SVs are the eigenvectors of:

$$L(t_2,t_1)^T N L(t_2,t_1)\xi_j(t_1;t_1,t_2) = \sigma_j^2 N \xi_j(t_1;t_1,t_2)$$

$$\boldsymbol{\phi}_n = \sum_{j=1}^{n} \langle \hat{\boldsymbol{\eta}}_j, \boldsymbol{\phi}_n \rangle \hat{\boldsymbol{\eta}}_j, \qquad y_k^{(n)} = \langle \hat{\boldsymbol{\eta}}_k, \boldsymbol{\phi}_n \rangle \qquad k = 1, 2, \ldots, n,$$

$$\mathbf{D}^{(n)} \boldsymbol{y}^{(n)} = 0, \qquad D_{kj}^{(n)} = \sum_{i=1}^{n-1} \langle \hat{\boldsymbol{\eta}}_k, \hat{\boldsymbol{\xi}}_i \rangle \langle \hat{\boldsymbol{\xi}}_i, \hat{\boldsymbol{\eta}}_j \rangle \qquad k, j \leqslant n.$$

Wolfe, C. L., and R. M. Samelson, 2007. An efficient method for recovering Lyapunov vectors from singular vectors. Tellus, 59A, 355-366.

Compute *n-th* Lyapunov Vector ϕ_n by forward-backward algorithm:

1. Compute the *n-th* order GS orthogonalized basis at time *t*; at each time-step *s* < t, store a *KxK* (?) upper triangular matrix R_s from decomposition of matrix of Jacobian-iterated and orthogonalized GS basis vectors.

2. Choose a vector in the *n-th* order GS subspace that has a non-zero component of the *n*-th GS vector but is otherwise arbitrary, and integrate backwards using the inverses R_s^{-1} of the upper triangular matrices R_s, until the vector emerges that grows most rapidly (decays most slowly) in backward time.

3. This vector is the desired ϕ_n, since it must be the vector in the *n-th* order GS subspace that grows most slowly in forward time.

Ginelli, F., et al., 2007. Characterizing dynamics with covariant Lyapunov vectors, Physical Review Letters, 99, 130601.

B. Are global Lyapunov Vectors the right linear objects for analysis of multi-scale systems like the atmosphere?

LVs are invariant => only need to do it ONCE! Then evolve by linearization to get later (earlier) structures…. We would know the "DIMENSION OF THE ATTRACTOR" and the "STRUCTURE OF THE TANGENT SPACE"!

BUT…

- Atmosphere not really recurrent (Hunt – ergodicity; Kalnay – space-time localization of bred vectors)
- Convergence without recurrence? (Is there a 'shadowing theorem'?)
- Is slow long-term growth or fast short-term growth more important? (Mr. Magoo vs. The Ballerina)

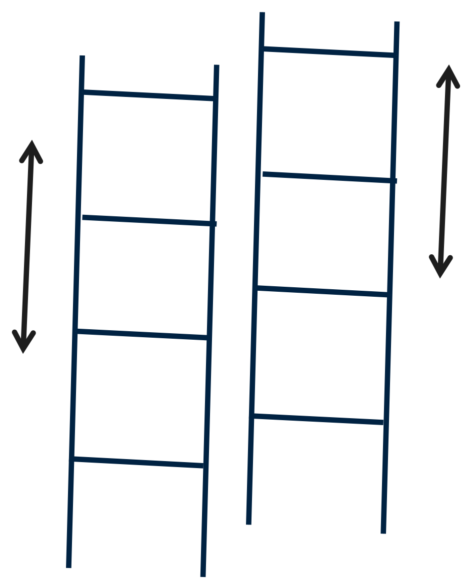
Mr. Magoo vs. The Ballerina
or
How I Violated DOS And Lived To Tell About It

Which is more important:

slow long-term (LV) growth?

fast short-term (SV/LV-DOS) growth?

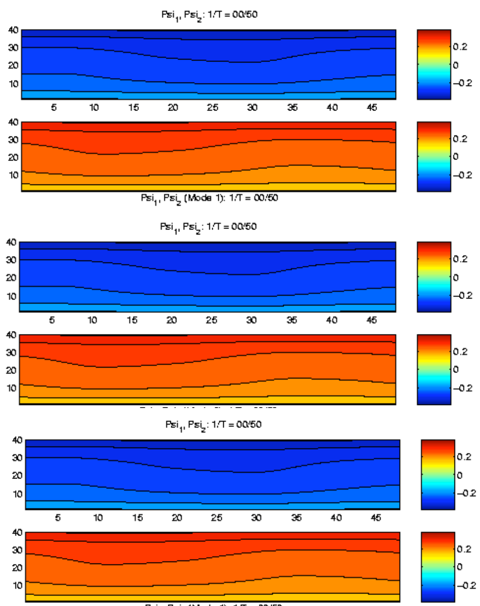
and, do LVs capture short-term growth?



(images from www.entertainmentearth.com and www.postershop.com)

A toy model for computation of "THE" LVs of the global atmosphere

Consider a large stack of non-interacting unstable (chaotic?) baroclinic wave-mean oscillations with random initial phases and random (hard-sphere?) variations in relative position
("All storms look the same; but their position and timing is random")



Taken as a whole, recurrence time is "infinite," but LVs of each oscillation are known exactly (e.g., FVs plus relative displacements)!

1. What does direct LV analysis of the whole give?

2. Is there a general approach for localizing the LV analysis to recover approximately the exact results?

Note: Can use weakly-nonlinear wave-mean models (ODEs) instead of strongly nonlinear numerical simulations