New Light on the Quantum Hall System

Emil Johansson Bergholtz

Licentiate Thesis in
Theoretical Physics

Department of Physics
Stockholm University

September 2005

Abstract

The fractional quantum Hall effect (FQHE) is one of the most peculiar and theoretically demanding problems in modern physics. A two-dimensional electron gas subject to a strong perpendicular magnetic field exhibits exotic behavior such as quantized conductance and fractionally charged excitations. The theoretical difficulties that arise are mainly due to the non-perturbative nature of the problem. However, if the system is studied in a geometry with periodic boundary conditions (*i.e.* a torus or a cylinder) there is a free parameter—the radius. In this thesis we exploit the possibility of varying this parameter and claim that the FQHE can be understood as a one-dimensional spin chain with short-range interactions. Our most detailed analysis concerns the metallic state at $\nu = 1/2$. We obtain an analytical solution in terms of non-interacting neutral fermions, "composite fermions", valid for the low-energy sector of the electron gas on the thin torus. Furthermore, we provide strong evidence that our exact solution is adiabatically connected to the two-dimensional bulk case. A simple picture of the FQHE emerges: the gapped fractions originate from simple crystals in the thin limit while the gapless states are separated from this by a phase transition. Hence, this approach provides a natural framework for understanding the common features, as well as the differences between various filling fractions, ν. Moreover, it has the potential of giving a deeper understanding of the microscopic origin of composite fermions—*i.e.*, the quasi-particles that are commonly believed to explain the FQHE by mapping the problem onto the well-understood integer quantum Hall effect (IQHE).

Acknowledgements

First and foremost I thank my supervisor Anders Karlhede. It has been a great privilege to work with him on such an interesting project. Especially his ability to bring order to my sometimes disordered thoughts is highly appreciated. I would also like to acknowledge Hans Hansson for feedback on our project as well as other illuminating discussions. Of course, all the people in the corridor that makes it easier to go to work in the morning also deserve credit. At last I thank Olle for some computer help and Emma for proof reading this thesis and contributing with an illustration.

Contents

Accompanying papers

Paper 1

"Density Matrix Renormalization Group Study of a Lowest Landau Level Electron Gas on a Thin Cylinder"

Emil J. Bergholtz and Anders Karlhede,
Unpublished, cond-mat/0304517 (2003).

Paper 2

"Half-Filled Lowest Landau Level on a Thin Torus"

Emil J. Bergholtz and Anders Karlhede,
Phys. Rev. Lett. **94**, 026802 (2005).

Paper 3

"One-Dimensional Theory of the Quantum Hall System"

Emil J. Bergholtz and Anders Karlhede,
Preprint (2005).

Chapter 1

Introduction

The area of condensed matter physics covers a vast and expanding part of modern physics. Some of the most interesting subjects, in my view, deal with macroscopic quantum phenomena. Here, the radical implications of quantum mechanics lead to exotic behavior such as superfluidity—quantum liquids that can flow upwards—and superconductivity—currents that flow without resistance! Another famous example in which a macroscopic number of particles occupies the same quantum state, thus giving rise to a new kind of matter, is Bose-Einstein condensation (BEC). The quantum Hall effect (QHE), which we focus on in this thesis, is a macroscopic quantum phenomenon containing many fascinating topics. The effect takes place in two-dimensional cooled electron gases in very strong magnetic fields—typically 1-30 Tesla. Partly due to the extreme conditions needed to see the effect, it was not discovered until 1980 when von Klitzing (Nobel Prize in 1985), Dorda and Pepper [1] surprised the physics community with their discovery of the so-called integer quantum Hall effect (IQHE). The experimental signatures of the effect were plateaus of quantized conductance. This phenomenon, although unexpected, is rather easy to explain theoretically as a consequence of the quantum mechanical motion of electrons in magnetic fields in presence of disorder [2, 3]. Thus, there is an explanation based on single particle dynamics.

In 1982 Tsui, Störmer and Gossard [4] revealed the even more unexpected fractional quantum Hall effect (FQHE) while analyzing purer samples at even lower temperatures. The origin of this effect turns out to be much more intricate—it is caused by collective quantum mechanical phenomena rather than single particle properties. The basic (but far from complete) theoretical understanding of the FQHE was given by Laughlin [5], who wrote down a microscopic wave function for some of the most pronounced states and also predicted the existence of fractionally charged quasi-particles. Laughlin, Tsui and Störmer shared the Nobel Prize in 1998 "for their discovery of a new form of quantum fluid with fractionally charged excitations".

The fractional quantum Hall effect covers a vast field of topics such as exotic edge states, spin phenomena, bilayer systems, anisotropic states in higher Landau levels and, perhaps most notably, fractionally charged excitations as well as fractional statistics. We will, however, focus on the basic theoretical problem of the FQHE; *i.e.* to find the low energy states of a gas of strongly interacting (and completely spin polarized) electrons in the lowest Landau level.

The quantum Hall system is, theoretically, most often studied either on a sphere or in the plane using circular symmetry while periodic boundary conditions are, by far, the most frequently used in other branches of condensed matter physics. What geometry the system is studied in is of course just a matter of convenience in the thermodynamic limit. We choose the torus/cylinder[1] because it has the advantage that it maps the system onto a one-dimensional spin chain with lattice spacing $1/R$, where R is the radius of the cylinder. However, the usefulness of this mapping may seem doubtful since the range of interaction ($\propto R$ measured in units of the lattice spacing) tends to infinity as the system size grows. This may be the reason why this approach has not been much exploited earlier[2]. In this thesis however, we claim that the FQHE can be understood as a one-dimensional spin chain with short-range interactions.

We study the quantum Hall system on a torus as a function of its (smallest) radius, R. When the radius is short, the range of the electron-electron interaction becomes short, and we get a systematic expansion of the quantum Hall system around a simple case—the thin torus. Here, we show that many of the characteristics of various filling fractions survive in this limit and that this approach provides a natural framework for understanding the common features as well as the differences between various filling fractions, ν. Moreover, it gives a deeper understanding of the microscopic origin of the composite fermions—*i.e.*, the quasi-particles that Jain [7] introduced to explain the FQHE by relating the problem to the well understood integer quantum Hall effect (IQHE).

Our most detailed analysis concerns the metallic state at $\nu = 1/2$. We obtain an analytical solution of the low energy sector on the thin torus, in terms of non-interacting, one-dimensional, neutral fermions (dipoles). The ground state, which is homogeneous, is the Fermi sea obtained by filling the negative energy states, and the excited states are gapless neutral excitations out of this one-dimensional sea. This establishes a case where the strongly correlated electrons in the lowest Landau level give rise to a low-energy sector of non-interacting particles—"composite fermions". Furthermore, we provide strong evidence (exact diagonalization of small systems, density matrix renormalization group (DMRG) [8] and renormalization group (RG) arguments) that our exact solution is adiabatically connected to the bulk case. This suggests a one-dimensional Luttinger liquid [9] description, with possible observable effects in transport experiments, of the bulk state, where it develops continuously from the state on a thin torus as the radius increases.

The scientific discoveries in this thesis are contained in three accompanied papers. The first includes work based on my undergraduate thesis, where I applied a numerical recipe, DMRG, to quantum Hall systems. Partly due to my lack of expertise in writing optimal computer programs we were restricted to study a certain limit of these systems—the "thin" limit, that came to be the subject of this thesis and probably also of future research. While the FQHE fractions (at that time) seemed rather boring in this limit, we totally unexpectedly found a peculiar quantum fluid in the half-filled lowest Landau level. This

[1] In this thesis we will often switch back and forth between torus and cylinder geometry. This is just a matter of convenience; the cylinder is easier to work with and most of the physics is more easily explained in the cylinder case. However, with the torus we get rid of unwanted boundary effects and have an additional symmetry that can be exploited.

[2] An important exception to this is the work by Haldane and Rezayi who studied the Laughlin states on a stretched and squeezed cylinder in 1994 [6].

state showed striking similarities to the bulk $\nu = 1/2$ state and also proved to be very insensitive to the range of the interaction [10]. This was further analyzed in the second paper, where we obtained an analytical understanding of the low-energy sector of this theory in terms of non-interacting particles, composite fermions. We also conjectured that our solution would evolve into a Luttinger liquid description of the bulk state [11]. The third paper contains an investigation of our conjecture, that we believe seals the case, and also a unified view of all fractions (gapped and gapless). Here, a very simple picture of the FQHE emerges; the gapped fractions originate from simple crystals in the electrostatic limit ($R \to 0$), while the gapless states may be studied as instabilities in the one-dimensional setting, as seems to be the case at $\nu = 1/2$ [12].

The outline of the thesis is as follows. After this introduction we review the basics of the quantum Hall effect in chapter two. We give a short description of the experiments, the observed phenomena as well as the current theoretical understanding of these. We hope to make it evident that there are some key points missing in the understanding of the FQHE, even though the mainly accepted theory is both elegant and successful[3].

We present a one-dimensional model of the QHE in the third chapter and in the fourth chapter we study the thin/short-range limit of this model. The fifth chapter is devoted to the half-filled lowest Landau level, and we derive the exact solution on the thin torus and argue how this can account for the special features of this system. In chapter six we outline how our approach can answer other questions about the physics in the lowest Landau level at various filling fractions.

[3] A very nice and critical view of the current understanding of the quantum Hall system is given by Dyakonov [13].

Chapter 2

Quantum Hall basics—theory and experiments

Here we review the classical and quantum Hall effects as well as the basic theoretical understanding of them. We also point out why we think that the commonly accepted theories of the FQHE do not yet have, what one might call, a sound theoretical basis.

2.1 The classical Hall effect

The classical Hall effect was discovered by Edwin Hall already in 1879 [14]. He observed that when a metal plate (in the $xy-$plane) was placed in a perpendicular magnetic field, $\mathbf{B} = B\hat{z}$, and a current was driven in the $x-$direction, a potential difference—the Hall voltage, V_y—appeared in the $y-$direction (see Fig 2.1). This effect is easily explained by classical electromagnetism. The Lorentz force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ acts on a charged particle moving in a magnetic field. Since the force is perpendicular to the magnetic field and to the velocity (*i.e.*, to the current, I) there must be a potential difference V_y between the edges in the $y-$direction. It follows that V_y is proportional to B, thus if one plots the the Hall voltage, V_y, (or the Hall resistance $R_H = V_y/I$) against the magnetic field, one expects a straight line. This behavior is confirmed experimentally in conducting materials such as metals.

2.2 The quantum Hall experiment

The integer quantum Hall effect is observed at low temperatures in two dimensional electron gases placed in very strong perpendicular magnetic fields, $\mathbf{B} = B\hat{z}$. In contrast to its classical analog, R_H, the Hall resistance $R_{xy} = V_y/I$ is not a linear function of B. Instead, it is observed to jump stepwise between regions where it is constant (*i.e.*, between plateaus). If one also measures the longitudinal resistance $R_{xx} = V_x/I$, one finds that it vanishes when R_{xy} is on a plateau and shows large peaks whenever R_{xy} jumps, see Fig 2.2. Furthermore, the plateaus are observed at the values

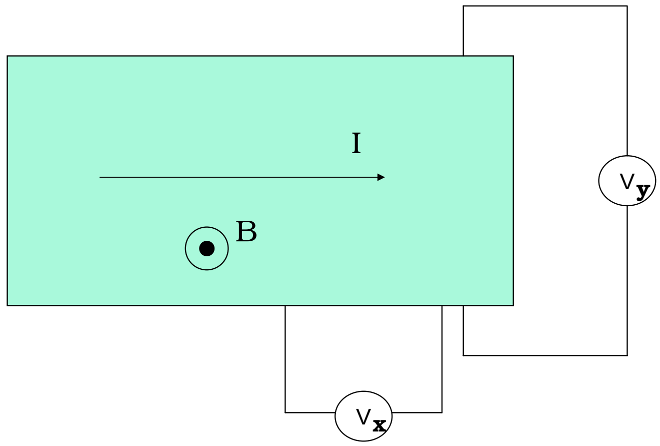

Figure 2.1: *The setup for the Hall experiment. A current is driven through a Hall bar (e.g., a metal or a two-dimensional electron gas). This gives rise to a potential difference, V_y, due to the Lorentz force, $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, acting on the charge carriers. Hence, the Hall resistance $R_H = V_y/I$ is expected to be proportional to B. This classical relationship is however modified under the extreme conditions of the QHE, see Fig. 2.2*

$$R_{xy} = \frac{R_K}{n}, \quad n = 1, 2, 3, \ldots \;, \tag{2.1}$$

where $R_K = h/e^2 = 25812.807\ \Omega$ is the so-called von Klitzing constant. It should be noted that R_{xy} is universal and independent of experimental details such as sample geometry, size, disorder and so on. Also, the quantization of the resistance is so enormously precise that it has become a laboratory resistance standard.

Later on, when the samples became even cleaner and the temperatures reached in the laboratories decreased even further, a new quantum Hall effect emerged—the fractional effect. New plateaus are observed at

$$R_{xy} = \frac{R_K}{\nu}, \tag{2.2}$$

with fractional $\nu = p/q$—mostly with odd q. Pronounced plateaus are found at $\nu = \frac{1}{3}, \frac{1}{5}, \frac{2}{5}, \ldots$ and we will later identify ν as the filling fraction, see chapter three[1].

[1] For a given sample $\nu \propto \frac{1}{B}$, thus $R_{xy} \propto B$ as in the classical effect. However, disorder and quantum mechanics causes deviations from this by forming plateaus.

There is also another type of exotic quantum fluid in the quantum Hall regime, that has received considerable attention. These are the metallic (gapless) states at *e.g.*, $\nu = 1/2$. Here, R_{xx} is finite and sample dependent as $T \to 0$, whereas R_{xy} is unquantized.

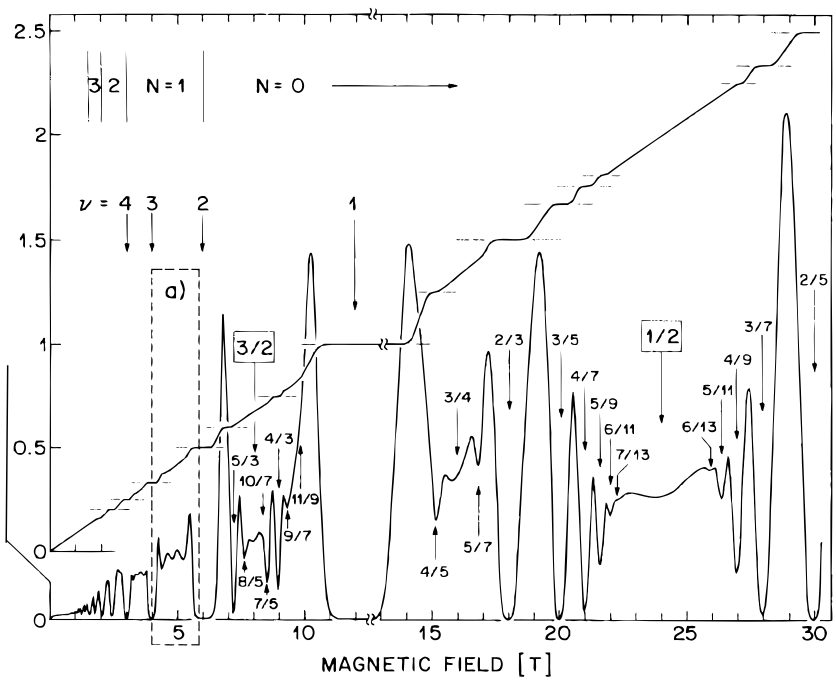

Figure 2.2: R_{xy} *(the approximately straight curve above) and* R_{xx} *are shown as functions of the magnetic field,* B*. At the quantum Hall fractions,* R_{xx} *drops quickly whereas* R_{xy} *forms a plateau (see e.g.* $\nu = 2/3$ *and* $\nu = 1$*). The compressible states are characterized by a finite longitudinal resistance,* R_{xx}*, and the transverse resistance,* R_{xy}*, remains unquantized (see e.g.* $\nu = 3/4$ *and* $\nu = 1/2$*). The results are taken from Willett et al. (1987) [15]. Today, many more states and details are observed [16].*

In the early 1980's it was very hard to fabricate pure enough samples to be able to see the effect, hence only a few fractions were observed initially. Nowadays however, they are fabricated by standard techniques and new fractions are constantly appearing as the semiconductor technology advances. This gives continuous new input to the subject and an opportunity for theorists to predict and explain the results of future experiments.

2.3 Understanding the quantum Hall experiment

The physics of the quantum Hall effect takes place in the so-called quantum Hall regime. To begin with the temperature has to be low. Otherwise the physics would not be dominated by the low energy states and quantum mechanics would not matter. It is also an experimental fact that the phenomena take place in strong magnetic fields. Of course, in the low field limit the electrons behave as an ordinary electron gas.

Theory predicts that there are (at least) two more criteria that must be fulfilled to render the quantum Hall effect possible. First, disorder has to be present. If not, the plateaus cannot exist. There also has to be a finite gap in the energy spectrum in order to make it possible that R_{xx} vanishes as $T \to 0$. In a real sample, there is of course always disorder, thus the first criterion is fulfilled. It is also straight forward to show that there is an energy gap in the case of the integer effect—this follows from just filling the quantum states with lowest kinetic energy. However, in the fractional effect, the emergence of a gap is completely caused by electron-electron interactions and it is a hard problem to analyze this strongly correlated quantum many-body problem.

To see this qualitative difference, of the origins of the integer and fractional effects, we can consider non-interacting electrons moving in a magnetic field (see chapter three for details). The kinetic energies become

$$E_n = (n + 1/2)\hbar\omega_c, \qquad n = 0, 1, \ldots \tag{2.3}$$

where $\omega_c \equiv \frac{eB}{mc}$ is the cyclotron frequency. These energy states, the so-called Landau levels [17], are hugely degenerate and in the large B limit[2] the low energy many particle states are obtained by filling the lowest available single particle energy states. At filling $\nu = p$, this just corresponds to filling the p lowest Landau levels. The number of states, N_s, within each Landau level equals the number of magnetic flux quanta, $\Phi_0 = \frac{hc}{e}$, penetrating the sample and in general we define the filling factor as

$$\nu = \frac{N_e}{N_s} = \frac{N_e\Phi_0}{BA}, \tag{2.4}$$

where N_e is the number of electrons in the sample and A is the area of the sample. Ignoring the electron-electron interaction, this readily explains the gap in the integer effect since there is an energy, $\Delta E = \hbar\omega_c$, to be payed when moving an electron to a higher Landau level. Thus the IQHE is a phenomenon that can be explained by considering the single particle states. However, in the fractional case we see that there is a huge degeneracy of many particle states with exactly the same kinetic energy. Thus, it becomes clear that the gap must be caused by the formation of special quantum states that effectively minimizes the energy of the electron-electron interaction.

2.4 Laughlin's theory

While other authors (*e.g.*, [4, 18]) tried to explain the fractional quantum Hall effect by the formation of crystalline states, such as Wigner crystals with triangular symmetry, early calculations of such states showed nothing special in the spectrum at $\nu = 1/3$, or at any of the other observed FQHE fractions. Instead Laughlin put forward a strikingly new explanation in his famous paper from 1983 [5]. He proposed that electrons at filling $\nu = 1/(2m+1)$ condense into a new kind of matter—an incompressible homogenous quantum fluid. Furthermore, he wrote down an extremely good trial wave function for these states and

[2] Electron-electron repulsion can, of course, be neglected in this limit compared to the large kinetic energy. This is true even though the electron-electron interaction also turns out to depend on B.

predicted the existence of fractionally charged excitations, which should appear when the filling slightly deviates from $\nu = 1/(2m+1)$.

Expressed in complex coordinates, $z = x + iy$, Laughlin's wave function, at filling fraction $\nu = \frac{1}{2m+1}$, takes the following form in the plane:

$$\Psi_{\frac{1}{2m+1}}(z_1, z_2, \ldots, z_N) = \prod_{i<j}(z_i - z_j)^{2m+1} e^{-\frac{1}{4}\sum_i |z_i|^2}. \tag{2.5}$$

To be in the lowest Landau level the form of Ψ is actually very restricted. The second part of the wave function has to be there and the first part has to be an antisymmetric polynomial in the electron coordinates $z_1, z_2, \ldots, z_N$. Moreover, the power of the polynomial is determined by the filling fraction, ν, and the polynomial also has to be homogenous to ensure a well-defined angular momentum. At $\nu = 1$ (the filled lowest Landau level) there is of course a unique solution to this:

$$\Psi_1(z_1, z_2, \ldots, z_N) = \prod_{i<j}(z_i - z_j) e^{-\frac{1}{4}\sum_i |z_i|^2}, \tag{2.6}$$

Thus, in general the problem of finding the low energy states is reduced to finding a polynomial that minimizes electron-electron repulsion. In this light, Laughlin's construction seems almost obvious.

Laughlin's ground state proves to be a very good approximation to the ground state of a large class of repulsive electron-electron interactions[3] and it is the exact solution to a certain type of short range interaction [19, 20]. Laughlin's theory is the basis of the understanding of the FQHE today and the $\nu = 1/3$ and $1/5$ states are observed and behave as Laughlin predicted[4]. One intuitive way to understand why Laughlin's wave functions are as good as they really are is to note that at precisely these fractions is it possible to write down fermionic (antisymmetric) wave functions that vanish faster when two particles approaches each other than at other fractions. Thus an effective minimization of the energy is possible at these fractions.

We end this section by stating that, on the cylinder, which we study in this thesis, the Laughlin state takes the form [6]

$$\Psi_{\frac{1}{2m+1}}(z_1, z_2, \ldots, z_N) = \prod_{i<j}(e^{iz_i/R} - e^{iz_j/R})^{2m+1} e^{-\frac{1}{2}\sum_i y_i^2}. \tag{2.7}$$

2.5 Jain's composite fermion picture of the FQHE

The Laughlin fractions were found experimentally and could be explained very elegantly. However, nature proved much more complicated and interesting. Shortly after the first discovery of the fractional effect at $\nu = 1/3$ [4], a whole zoo of new states emerged (see *e.g.* Figure 2.1). An early explanation for the new fractions was given by the Haldane-Halperin hierarchy model [21, 22], in which quasi-particles existing in the vicinity of a given Laughlin state condense into a new Laughlin state with the electrons replaced by the quasi-particles.

[3] This statement is based on exact diagonalization studies on small systems, where ground states are shown to have very large overlaps with this state, typically ≥ 0.99.

[4] At even lower fractions, such as $\nu = 1/9$, the states are instable to crystallization, thus regular lattices are preferred ground states.

This construction may be iterated many times and provides an intuitive way of understanding the new states. However, these ideas had the drawback that they could not be tested or motivated by *e.g.* calculating overlaps (there were no microscopic wave functions to compare with).

Inspired by Laughlin's wave function engineering, Jain continued in this fashion to explain a large class of observed QH states[7, 23]. He noted that the Laughlin state may be written as

$$\Psi_{\frac{1}{2m+1}}(z_1, z_2, \ldots, z_N) = \prod_{i<j}(z_i - z_j)^{2m}\Psi_1, \tag{2.8}$$

where Ψ_1 is the completely filled lowest Landau level (2.6). Generalizing this, he proposed that the ground state wave functions at $\nu = \frac{p}{2mp+1}$ can be written as

$$\Psi_{\frac{p}{2mp+1}}(z_1, z_2, \ldots, z_N) = \mathcal{P}_{LLL}\prod_{i<j}(z_i - z_j)^{2m}\Psi_p, \tag{2.9}$$

where $\mathcal{P}_{LLL}$ is a projection operator that projects the state onto the lowest Landau level and Ψ_p is the state with the p lowest Landau levels filled. This bizarre construction of wave functions, involving completely unphysical higher Landau levels that are then projected away[5], is justified by exact diagonalization studies on small systems for various m and p.

Jain went on noting that the filling fraction, ν, may be written in the form

$$\frac{1}{\nu} = 2m + \frac{1}{p}. \tag{2.10}$$

Guided by this, Jain came up with a novel idea—the concept of composite fermions. He proposed, in 1989, that the FQHE is the manifestation of the IQHE of composite fermions. The integer p should be interpreted as the filling factor $\nu^* \equiv p$ for the composite fermions. The even integer $2m$ is thought of as the number of magnetic flux quanta "attached" to each electron. Different m give the so called Jain sequences. The most pronounced sequence is obtained for $m = 1$, with $\nu = 1/3, 2/5, 3/7, \ldots$

At the same time, independent progress was made by Zhang, Hansson and Kivelson [25] and by Read [28] who constructed effective field theories for the fractional quantum Hall effect. This was further developed by Lopez and Fradkin [26] and by Kalmeyer and Zhang [27], resulting in a field theory (of Chern-Simmons type [29]) where a singular gauge transformation turn the electrons into "composite fermions" while partly canceling the external magnetic flux.

An alternative way of writing (2.10) is in terms of an effective magnetic field, B^*, as seen by the hypothetical composite fermions:

$$B^* = B - 2m\nu B = B - 2mN_e\Phi_0. \tag{2.11}$$

This looks indeed like each electron has "attached" and neutralized $2m$ elementary flux quanta.

This gives a mapping of the fractional quantum Hall effect (FQHE) onto the integer quantum Hall effect (IQHE). This mapping includes an ad hoc attachment of magnetic flux quanta to the electron, which in turn reduces the effective

[5]The projection is a terrible beast to cope with, see *e.g.* Girvin in [24].

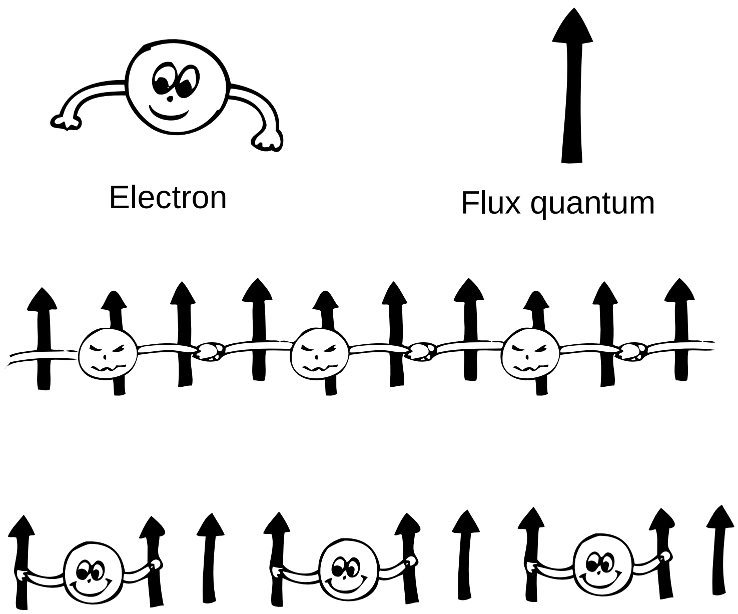

Figure 2.3: *A caricature of the formation of composite fermions [30]. Here, every electron at $\nu = 1/3$ magically "binds" two magnetic flux quanta, leaving one third of the magnetic field as seen by the composite fermions, thus mapping the $\nu = 1/3$ system of strongly interacting electrons onto a $\nu^* = 1$ system of free composite fermions.*

magnetic field. Thus it reduces the number of quantum states in each Landau level, hence changes the filling fraction.

2.6 Compressible states

In the quantum Hall regime there is also another type of interesting quantum fluid states that has received considerable attention—the metallic states, observed by Jiang et al. in 1989 [31]. Here, R_{xx} is finite and sample dependent as $T \to 0$, whereas R_{xy} is unquantized. Such states are found at even denominator fractions such as $\nu = 1/2$ and $1/4$.

These compressible (*i.e.* gapless) states are also thought to be described by the composite fermion concept, but with a rather different outcome. Take the $\nu = 1/2$ state for example: this state is gapless and thus not a quantum Hall state. However, it is a limiting case of two sequences of gapped QH states approaching $\nu = 1/2$ from above and from below (see Jain's sequences above)[6]. With $m = 1$ in (2.10) and (2.11) we see that the effective filling, ν^*, tends to infinity while the effective magnetic field, B^* vanishes. Hence the system is not mapped onto the IQHE. Instead the system is mapped onto free (composite) fermions! Thus the composite fermion theory predicts that the ground state is a free Fermi gas (*i.e.*, with a Fermi surface) in zero magnetic field.

The physics of the half-filled Landau level and its vicinity ($\nu = 1/2 \pm \epsilon$) was

[6] In the same way the $\nu = 1/4$ state is a limiting case for the $m = 2$ sequence and so on.

successfully described by Halperin, Lee and Read (HLR) [32], who introduced a mean field theory where the magnetic field is canceled by a smeared out statistical field. This also results in composite fermions moving in zero magnetic field, which implies that the state becomes a two-dimensional free fermion gas with a Fermi surface. This picture has been spectacularly confirmed by surface acoustic wave experiments (SAW) [33], which shows a Fermi surface as predicted by HLR, as well as by ballistic transport measurements showing particles moving with increased cyclotron radii near $\nu = 1/2$ [34, 35, 36], see Fig 2.4.

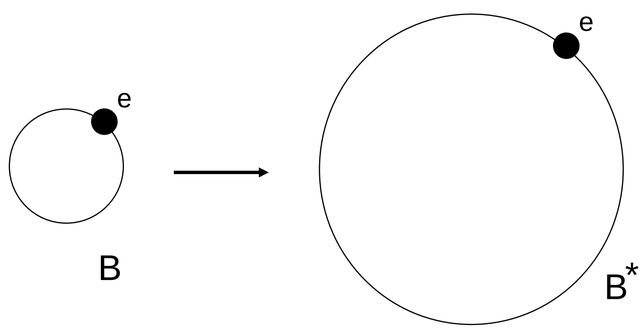

Figure 2.4: *Ballistic transport. Particles near $\nu = 1/2$ move in circles (cyclotron orbits) determined by an effective magnetic field $B^* < B$. Note that this can alternatively be seen as an effect of reduced charge $e^* < e$, which is the most natural interpretation of our findings (see chapter five).*

The theory of the $\nu = 1/2$ state has later been further developed by several authors [37, 38, 39, 40], and a description of the state in terms of neutral dipoles has been proposed. However, the microscopic origin of these dipoles remained a mystery (at least in our opinion).

Rezayi and Read [41] carried on in the spirit of Laughlin and Jain and proposed a microscopic wave function for the $\nu = 1/2m$ state. The Laughlin state at $\nu = 1/2m$ ($\Psi_{\frac{1}{2m}}$ in (2.5)) is symmetric under particle exchange, thus it describes bosons. A natural solution to this is to multiply the state with a Slater determinant to fix the statistics while keeping the short-range behavior that proved so successful for the quantum Hall states. Rezayi and Read multiplied the bosonic $\nu = 1/2m$ Laughlin state by a determinant of plane waves. This again takes the state outside the lowest Landau level, hence a projection is needed. The final result is

$$\Psi_{\frac{1}{2m}}(z_1, z_2, \ldots, z_N) = \mathcal{P}_{LLL} Det[e^{i\mathbf{k}_i \cdot \mathbf{r}_j}] \prod_{i<j} (z_i - z_j)^{2m} \prod_i e^{-\frac{1}{4}|z_i|^2} \qquad (2.12)$$

where $\{\mathbf{k}_i\}$ is a set of parameters or 'momenta'. As we will see later the choice of these momenta depends on the geometry of the system.

2.7 Unsolved mysteries

There is no doubt that the agreement between the theory of the quantum Hall system and experiments is impressive. However, we feel that there are important questions regarding the physics in the lowest Landau level that remain to be answered. Here, we discuss some of these questions[7].

A profound mystery concerns the origin of the composite fermions. There is no real understanding of why the strongly correlated electron system in the lowest Landau level, at varies filling fractions, becomes a system of weakly interacting composite fermions—no controlled microscopic derivation of these quasi-particles really exists. In fact, nobody has come up with a satisfactory definition of what a composite fermion is. Jain states that composite fermions are *"electrons bound to an even number of flux quanta"* [7]. Obviously, an electron cannot just swallow magnetic flux—the flux quantas are not objects that can travel by themselves, as Figure 2.3 suggests. Also, the relation between Jain's wave functions (2.9), that have the big advantage of being verified numerically for small systems, and the composite fermions is not very precise. Before projection, one can vaguely argue that is looks like there are objects filling an integer number of Landau levels. However, after projection, these arguments fail since the Landau level structure is projected away.

A simple minded application of the composite fermion concept to understand the excitations in the system gives an energy scale that is completely wrong. If the excitations are interpreted as composite fermions lifted into higher Landau levels, in direct analogy to the IHQE, then all low energy excitations have the same scale, $\hbar\omega_c$, and the difference between two consecutive energy levels are the same. At $\nu = 2/5$, the excitation energy becomes $\Delta E = \hbar\omega_c/5$, at $\nu = 1/3$ it becomes $\Delta E = \hbar\omega_c/3$ and so on. On the other hand, it is known that the gap in the FQHE is caused by electron-electron interactions. Thus the scale of excitations is rather $\frac{e^2}{4\pi\epsilon\ell}$, where ℓ is the typical length between the electrons. By noting that *e.g.* the dependence of B is very different in these expressions, we conclude that the scale of the excitations is unrealistic in this approach. Although this problem can be fixed in the wave function approach, it plagues the Chern-Simons theory and it certainly changes the interpretation of the excitations. Another counter-intuitive consequence of the composite fermion picture is that these quasi-particles carry the same electric charge as the electrons, whereas the observed excitations in the system are fractionally charged.

Even though Laughlin provided intuitive arguments for why there is a gapped quantum Hall state at *e.g.* $\nu = 1/3$ by writing down wave functions, there exist no such arguments or simple wave functions at *e.g.* $\nu = 2/3$, which is an equally developed state. In general, the quantum Hall system, at $\nu < 1$, possess particle-hole symmetry—there are simple relations between correlation functions, energies *etc* at filling ν and $1 - \nu$. However, this symmetry is absent in the theoretical explanations of the effect.

[7] An excellent review of the current understanding of the quantum Hall system, in a critical spirit, is given by Dyakonov in [13].

Attempts to resolve many of these problems have been made. Perhaps most notably, the Hamiltonian approach due to Murthy and Shankar is a such attempt [37]. However, this approach has its own problems and we think it is fair to say that the question marks have not turn into exclamation marks yet.

We end this section by quoting Dyakonov's judgement of the composite fermion approach: *"it is no more (but also no less) than an amazingly successful guess"* [13].

Chapter 3

The one-dimensional nature of the quantum Hall system

In this chapter we solve the so-called Landau problem [17] of a single charged particle moving in two-dimensions in presence of a perpendicular magnetic field (we do this on a cylinder for simplicity). This simple exercise provides us with a good feeling for the quantum Hall problem; it gives the full explanation of the gap causing the IQHE, as well as the insight that an isolated Landau level (hence also the FQHE) effectively is a one-dimensional system. In the latter part of the chapter we put the system on a torus and formalize this insight while deriving a one-dimensional lattice model of the FQHE. This part necessarily becomes a bit more technical (the algebra of translations in a magnetic field is cumbersome).

3.1 Quantum mechanics in strong B-fields

In this section we study the quantum states of a single electrically charged particle in a two-dimensional system with a perpendicular magnetic field [24]. Such a system is illustrated by Figure 3.1 and the states we seek correspond to the states where just one particle is present on the surface of the cylinder. The choice of geometry is just a matter of convenience, and we use the cylinder (or torus) just to get periodic boundary conditions. In the end we are interested in the thermodynamic limit, where the boundary effects are unimportant. However, this choice turns out to be crucial in our approach to the quantum Hall system.

In order to find the single particle Hamiltonian we use the concept of minimal coupling and choose an appropriate gauge for the vector potential. A good choice is the so-called Landau gauge:

$$\mathbf{A} = By\hat{x}, \tag{3.1}$$

which gives a magnetic field in the direction of $\hat{z}$ because $\nabla \times \mathbf{A} = \mathbf{B} = -B\hat{z}$. We can now write the Hamiltonian as

$$H = \frac{1}{2m}((p_x + \frac{eB}{c}y)^2 + p_y^2) \tag{3.2}$$

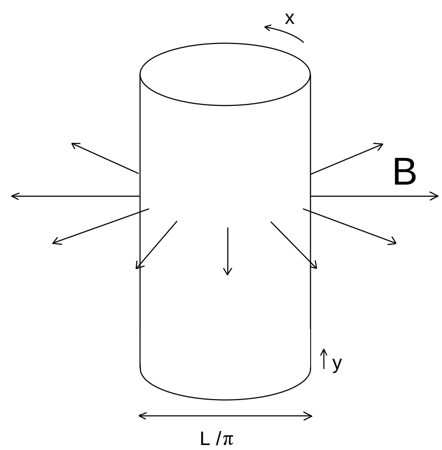

Figure 3.1: *A cylinder with a magnetic field* **B** *perpendicular to the electron layer.*

using minimal coupling. As one can see from this expression, the Hamiltonian is not invariant under translation along the y-axis, even though the physics is (B-field only in the $\hat{z}$-direction).

Let us now consider this system on a cylinder as shown in figure 3.1. We take advantage of the translation symmetry in the x-direction and write the wave function as

$$\psi(x,y) = e^{ikx}\xi(y), \tag{3.3}$$

where we have used separation of variables as an ansatz. Written in this form it is an eigenstate of p_x and hence we can make the substitution $p_x \to \hbar k$ in the Hamiltonian. The system also gives us a constraint on k. Imposing periodic boundary conditions

$$\psi_k(x,y) = \psi_k(x+L,y) \tag{3.4}$$

gives

$$k = \frac{2\pi p}{L}, \qquad p = 0, \pm 1, \ldots, \tag{3.5}$$

where L is the circumference of the cylinder. We then arrive at an effective one-dimensional Schrödinger equation

$$h_k\xi(y) = E^k\xi(y) \tag{3.6}$$

where

$$h_k \equiv \frac{1}{2m}(p_y^2 + (\hbar k + \frac{eB}{c}y)^2). \tag{3.7}$$

This is a familiar equation, namely the one-dimensional harmonic oscillator. We can therefore use our knowledge from elementary quantum mechanics [42]. The eigenvalues are

$$E_n^k = (n+1/2)\hbar\omega_c, \qquad n = 0, 1, \ldots, \tag{3.8}$$

where $\omega_c \equiv \frac{eB}{mc}$ is the cyclotron frequency. Note that the energy is independent of k.

Equation (3.8) tells us that the number n gives the energy of a single particle state. Since n is an integer, it means that there are discrete energy levels; these are the so-called Landau levels [17]. The gap responsible for the IQHE occurs when an integer number of Landau levels are filled, since there is a gap to the states with increasing n. However, in the FQHE the number of occupied Landau levels is fractional. Hence the origin of the gap is much more complicated and is completely caused by electron-electron interactions.

The eigenfunctions are

$$\psi_{nk}(\vec{r}) = \frac{1}{\sqrt{\pi^{1/2}2^n n! L}} e^{ikx} H_n(y + k\ell^2) e^{-\frac{1}{2\ell^2}(y+k\ell^2)^2}, \tag{3.9}$$

where H_n is the nth Hermite polynomial ($H_0 = 1$). Note that the eigenfunctions are centered at $y = Y_k = -k\ell^2$, *i.e.* the single particle wave functions are centered at different $y-$coordinates depending on the momentum of the state. In this thesis we focus on the lowest ($n = 0$) Landau level. The so-called magnetic length

$$\ell \equiv \sqrt{\frac{\hbar c}{eB}} = \frac{257\text{Å}}{\sqrt{\frac{B}{1\ tesla}}}$$

is the characteristic length of the system and roughly reflects the mean distance between the electrons. In the quantum Hall regime ℓ is typically of the order $100\text{Å} = 10^{-8}m$.

From figure 3.2 we see that the area per state in the lowest Landau level is $2\pi\ell^2$, thus the density of states is $n_s = \frac{1}{2\pi\ell^2}$. Given the number of electrons n_e per unit area in the system we can define the filling fraction ν as

$$\nu \equiv \frac{n_e}{n_s} = 2\pi\ell^2 n_e = \frac{hc}{eB} n_e = \frac{n_e \Phi_0}{B}, \tag{3.12}$$

where B/Φ_0 is the number of flux quanta, $\Phi_0 = \frac{hc}{e}$, penetrating the sample per unit area.

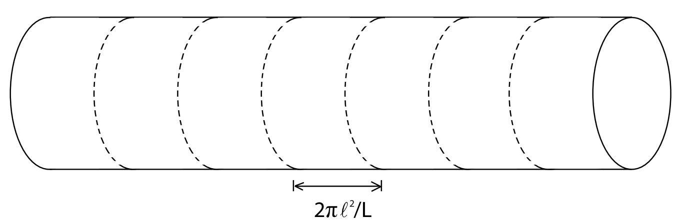

Figure 3.2: *An illustration of how the single particle states are distributed on the cylinder. The dotted lines are at $Y_k = -k\ell^2$ around which the states ψ_k are centered. This provides an exact mapping of the two-dimensional electron gas in a single Landau level onto a one-dimensional system.*

3.2 The one-dimensional model

In this section, we switch back to the torus geometry and put the one-dimensional model of the quantum Hall system on firmer grounds. This necessarily becomes a bit more technical (e.g., the states are periodic versions of the cylinder states and momentum is defined a bit differently). However, the main results are illustrated by figures and can be understood from these, together with the rather comprehensive and self-contained figure captions.

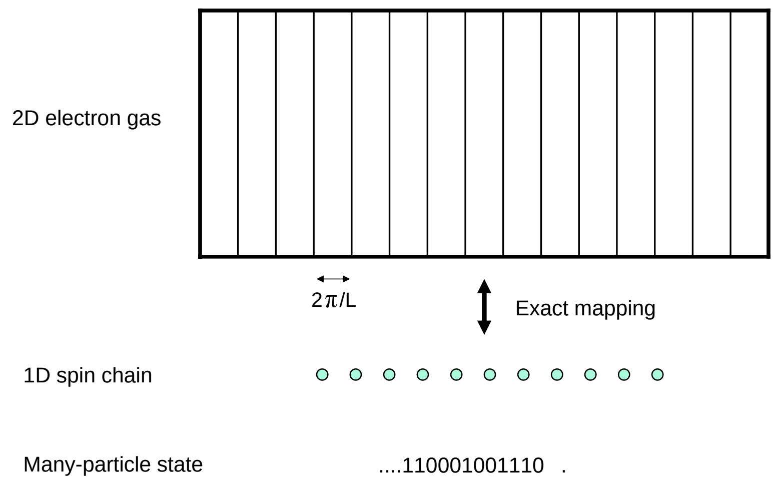

Figure 3.3: *The mapping of a single Landau level onto a one-dimensional lattice model. The single particle states are centered around lines that can be thought of as the lattice sites in the one-dimensional model, where each site is either occupied or empty. Furthermore, the system can, for general interactions, be thought of as a 1D spin chain. This is illustrated in Figure 3.4 in the case of two-body interactions.*

We map the two-dimensional quantum Hall problem onto a one-dimensional lattice model following [45]. We consider an electron confined to the lowest Landau level on a torus with lengths L_1, L_2 in the x and $y-$directions respectively. We set the magnetic length to one, $\ell = \sqrt{hc/eB} = 1$, and from now on we label the states, ψ_n, by integers n. In Landau gauge,

$$\vec{A} = By\hat{x}, \tag{3.13}$$

the magnetic translation operators become

$$t_1 = e^{(L_1/N_s)\partial_x}, \quad t_2 = e^{(L_2/N_s)(\partial_y + ix)} \quad , \tag{3.14}$$

where N_s is the number of flux quanta through the surface of the torus. We define

$$T_\alpha = \prod_i^{N_e} t_{\alpha,i}, \quad \alpha = 1, 2 \ , \tag{3.15}$$

where $t_{\alpha,i}$ translates electron i in the $\alpha-$direction. These operators commute with the Hamiltonian for the charged particle coupled to $\vec{A}$. Wave functions are required to be periodic up to a phase (we use $\phi_\alpha = 0$ in [11, 12]),

$$T_\alpha^{N_s}\Psi = e^{i\phi_\alpha}\Psi, \tag{3.16}$$

leading to $L_1L_2 = 2\pi N_s$ and $T_1T_2 = e^{2\pi i/N_s}T_2T_1$. With

$$\psi_0 = \frac{1}{\pi^{1/4}L_1^{1/2}}\sum_{n=-\infty}^{\infty} e^{inL_2x}e^{-(y+nL_2)^2/2}, \tag{3.17}$$

we obtain the t_1 eigenstates $\psi_m = t_2^m\psi_0$, $t_1\psi_m = e^{i2\pi m/N_s}\psi_m$, $m = 0, 1, ...N_s-1$. The states ψ_m are just periodic versions[1] of the single particle states (3.9) with $n = 0$. Thus they span the lowest Landau level. ψ_m are centered along the lines $y = -2\pi m/L_1$. This provides a mapping of the lowest Landau level onto a one-dimensional lattice model, where m number the sites and the lattice constant is $2\pi/L_1 \equiv 1/R$.

We introduce field operators

$$\hat{\Psi}(\mathbf{r}) = \sum_k \psi_k(\mathbf{r})c_k, \quad \{c_n^\dagger, c_m\} = \delta_{mn}, \tag{3.18}$$

where $c_m^\dagger$ creates an electron in state ψ_m. The electron density becomes $\hat{\rho}(\mathbf{r}) = \hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})$ and the electron-electron interaction is

$$\begin{aligned} H = \frac{1}{2}\int\int :\hat{\rho}(\mathbf{r}_1)V(\mathbf{r}_1-\mathbf{r}_2)\hat{\rho}(\mathbf{r}_2): d^2r_1d^2r_2 = \\ = \frac{1}{2}\sum_{k_1k_2k_3k_4} V_{k_1k_2k_3k_4}c_{k_1}^\dagger c_{k_2}^\dagger c_{k_3}c_{k_4}, \end{aligned} \tag{3.19}$$

where :: means that the operator product is normal ordered[2]. The matrix elements are

$$V_{k_1k_2k_3k_4} = \int\int \psi_{k_1}^\dagger(\mathbf{r}_1)\psi_{k_2}^\dagger(\mathbf{r}_2)V(\mathbf{r}_1-\mathbf{r}_2)\psi_{k_3}(\mathbf{r}_2)\psi_{k_4}(\mathbf{r}_1)d^2r_1d^2r_2. \tag{3.20}$$

As a consequence of momentum conservation, the sum in (3.19) goes over three independent variables. Another sum is eliminated by assuming translation invariance and the electron-electron interaction Hamiltonian becomes

$$H_{ee} = \sum_n\sum_{k>m} V_{km}c_{n+m}^\dagger c_{n+k}^\dagger c_{n+m+k}c_n, \tag{3.21}$$

where $V_{km} = V_{k,-m} \geq 0$ for a repulsive interaction. The physics of the interaction can be understood by dividing H_{ee} into two parts: V_{p0}, the electrostatic repulsion (including exchange) between two electrons separated p lattice constants, and $V_{m+p,m}$, the amplitude for two particles separated a distance p to hop symmetrically to a separation $p+2m$ and vice versa. This is illustrated in Figure 3.4.

[1] These states can be written in terms of the elliptic theta functions, see [45].

[2] All creation operators are moved to the left.

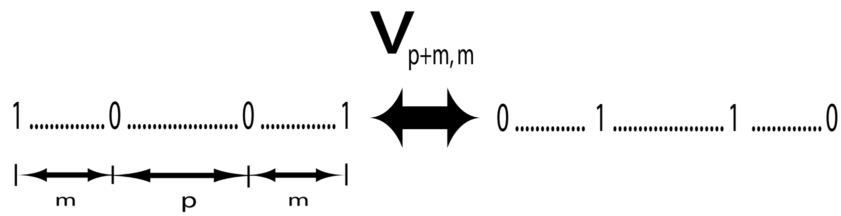

Figure 3.4: *This illustrates the action of a general two-body interaction on the torus/cylinder. The hopping comes with strength $V_{m+p,m}$ and the symmetry is imposed by momentum conservation—the position in the $y-$direction gives the momentum in the $x-$direction. Note that the $m=0$ terms should be included and can be thought of as the electrostatic repulsion (although the exchange term is included in this). However, the $p=0$ hops are forbidden by the Pauli principle.*

For a short-range real space electron-electron interaction of the form $V(\mathbf{r}) = \nabla^2\delta(\mathbf{r})$ one finds[3]

$$V_{km} = (k^2 - m^2)e^{-2\pi^2(k^2+m^2)/L_1^2}. \tag{3.22}$$

When the torus becomes thin, *i.e.* when L_1 decreases, the distance $2\pi/L_1$ between the single particle states increases, hence fewer terms in (3.21) contribute. For the $\nabla^2\delta$-interaction one finds that the range of the interaction is of the order of six magnetic lengths.

We consider the electron gas at filling fraction $\nu = p/q$, where p and q are integers (with no common divisor), and assume that the number of electrons νN_s is an integer. The many particle states can be chosen as T_1 eigenstates with momentum K_2 mod(N_s) (in units of $2\pi/N_s$). T_2 translates the system in the y-direction and changes K_2 by νN_s—*i.e.* by the number of particles. Since T_2 commutes with the Hamiltonian, all energy eigenstates are q-fold degenerate. In general, all states are characterized by a two-dimensional vector $K_\alpha = 0, \ldots, \frac{N_s}{q} - 1$. Hence the $\frac{N_s!}{(N_s-N_e)!N_e!}$ states can be divided into $\frac{N_s^2}{q^2}$ different sectors, which makes it possible to diagonalize larger systems.

At a fixed filling fraction, the electrostatic repulsion strives to keep the particles apart, whereas the hopping terms favour maximally hoppable states. To find the low energy states is in general a very complicated problem. However, in a few special cases the problem turns out to be solvable[4]. Furthermore, this approach makes the powerful techniques of one dimension available, which might be used to shed more light on the subject.

[3]This result is obtained with free boundary conditions in the $y-$direction.

[4]To date, we know that the $L_1 \to 0$ case for general filling fractions and the $\nu = 1/2$ case on a thin, but finite, torus are solvable. See the following chapters for details.

Chapter 4

The thin limit

A general $N-$particle state in the lowest Landau level is a linear combination of states characterized by the positions (or, equivalently, the momentas) $\{k_1, \ldots, k_N\}$ around which they are centered. Here, we show that the ground state of an interacting electron gas on a cylinder, for general filling fractions, approaches a regular crystal as $R \to 0$. This is achieved by analyzing the nature of the ground states of repulsive real-space interactions and by considering the wave functions that are known to describe the bulk physics well. Furthermore, we study particle-hole excitations and show how the concept of fractional charge emerges naturally in this limit.

4.1 Real-space interactions

Here we address the issue of the ground states for various filling fractions, $\nu < 1$, in the limit of extremely thin cylinders. We argue that, for reasonable interactions, the ground states will be regular lattices and that all hopping terms will be irrelevant.

Following Westerberg and Hansson [43], we Fourier expand the potential in the $x-$direction

$$V(\vec{r}) = \sum_{m=-\infty}^{\infty} V^m(y) e^{imx/R}. \tag{4.1}$$

The matrix elements in (3.21) become

$$V_{km} = \frac{e^{-\frac{m^2}{2R}}}{\sqrt{8\pi}} \int_{-\infty}^{\infty} V^m(y) e^{-\frac{(y-k)^2}{2R}} dy. \tag{4.2}$$

This shows that the $m \neq 0$ terms are exponentially damped in the $R \to 0$ limit. Thus

$$\lim_{R\to 0} H_{ee} = \sum_n \sum_k V_{k0} n_{n+k} n_n, \tag{4.3}$$

where $n_k = 0, 1$. This is nothing but the one-dimensional lattice gas, and all energy eigenstates have fixed charges and the ground states are regular lattices of some sort. For the type of interactions that we consider, like the Coulomb or short-range interactions considered above, we find regular lattices with unit cells containing p electrons and q sites at filling $\nu = p/q < 1$. At $\nu = 1/3$ the unit cell is 100, at $\nu = 2/9$ it becomes 100001000 and so on.

4.2 Wave functions

4.2.1 Laughlin's wave functions

We follow Rezayi and Haldane [6] and note that the Laughlin states (3.9) take the form

$$\Psi_{\frac{1}{2m+1}}(z_1, z_2, \dots, z_N) = \prod_{i<j}(e^{iz_i/R} - e^{iz_j/R})^{2m+1} e^{-\frac{1}{2}\sum_i y_i^2}, \tag{4.4}$$

while the single particle states[1] are

$$\psi_k(x_i, y_i) \propto e^{ikx_i/R} e^{-(y_i+k/R)^2/2} = e^{ikz_i/R} e^{-y_i^2/2} e^{-k^2/2R^2} \tag{4.5}$$

on the cylinder. In order to see what happens in the $R \to 0$ limit we can simply expand Ψ in terms of the single particle states. All terms in this expansion are multiplied with a factor $e^{\sum_j k_j^2/2R^2}$. Thus the term with maximal $\sum_j k_j^2$ will dominate when $R \to 0$, regardless of what combinatorial factor this term may be multiplied by in this expansion [6].

To see how this works we look at the $\nu = 1/3$ Laughlin state for three particles. We write the Laughlin state as

$$\begin{aligned}
\Psi_{\frac{1}{3}}(z_1, z_2, z_3) &= \prod_{i<j}^{3}(e^{iz_i/R} - e^{iz_j/R})^3 e^{-\frac{1}{2}\sum_i y_i^2} = \\
= \prod_{i<j}^{3}(e^{3iz_i/R} - 3e^{2iz_i/R}e^{iz_j/R} &+ 3e^{iz_i/R}e^{2iz_j/R} - e^{3iz_j/R}) e^{-\frac{1}{2}\sum_i y_i^2} = \\
&= \mathcal{A}[e^{6iz_1/R}e^{3iz_2/R} + 3e^{4iz_1/R}e^{5iz_2/R} + 3e^{6iz_2/R}e^{2iz_1/R}e^{iz_3/R} \\
&+6e^{5iz_1/R}e^{3iz_2/R}e^{iz_3/R} + 15e^{4iz_2/R}e^{3iz_1/R}e^{2iz_3/R}] e^{-\frac{1}{2}\sum_i y_i^2},
\end{aligned} \tag{4.6}$$

where $\mathcal{A}$ is an anti-symmetrization operator[2]. Identifying these terms with the single particle states gives

$$\begin{aligned}
\Psi_{\frac{1}{3}}(z_1, z_2, z_3) &= \mathcal{A}[\psi_6(z_1)\psi_3(z_2)\psi_0(z_3)e^{(6^2+3^2+0^2)/2R^2} \\
+3\psi_4(z_1)\psi_5(z_2)\psi_0(z_3)e^{(4^2+5^2+0^2)/2R^2} &+ 3\psi_2(z_1)\psi_6(z_2)\psi_1(z_3)e^{(2^2+6^2+1^2)/2R^2} \\
+6\psi_5(z_1)\psi_3(z_2)\psi_1(z_3)e^{(5^2+3^2+1^2)/2R^2} &+ 15\psi_3(z_1)\psi_4(z_2)\psi_2(z_3)e^{(3^2+4^2+2^2)/2R^2}].
\end{aligned} \tag{4.7}$$

Clearly, the first term dominates as $R \to 0$. Thus

$$\lim_{R\to 0} \Psi_{\frac{1}{3}}(z_1, z_2, z_3) = \mathcal{A}[\psi_6(z_1)\psi_3(z_2)\psi_0(z_3)]. \tag{4.8}$$

[1] The label k is an integer here. The $x-$momentum is k/R.

[2] *E.g.*, $\mathcal{A}[f(x)g(y)] = f(x)g(y) - f(y)g(x)$ and so on. Note, however, that there are three variables in each term in this example, even though $e^{0z_3/R} = 1$ is not written explicitly.

This state is nothing but a regular lattice of electrons occupying every third site. Generalizing this to an $N-$particle Laughlin state at $\nu = \frac{1}{2m+1}$ gives

$$\lim_{R\to 0} \Psi_{\frac{1}{2m+1}}(z_1, z_2, \ldots, z_N) = \\ = \mathcal{A}[\psi_{(2m+1)(N-1)}(z_1)\psi_{(2m+1)(N-2)}(z_2)\cdots\psi_0(z_N)]. \tag{4.9}$$

Hence, a lattice of N electrons, equidistantly separated by $2m$ empty lattice sites is obtained. It is actually not very hard to see why this is true. The first thing to observe is that a configuration with maximal $\sum_j k_j^2$ should contain the highest possible $k-$value, *i.e.* the highest possible power of $e^{iz_i/R}$. This value turns out to be $(2m+1)(N-1)$. Once this is assured, the next highest $k-$value should be maximized. Again, by looking at the Laughlin state one concludes that this value is at most $(2m+1)(N-2)$ and so on.

We adopt a chemical notation for these states in terms of unit cells 10_q, that includes one electron followed by q empty sites. Thus, for the Laughlin states we simply have

$$\lim_{R\to 0} \Psi_{\frac{1}{2m+1}}(z_1, z_2, \ldots, z_N) \sim (10_{2m})^N \equiv \underbrace{1\underbrace{0\cdots0}_{2m\ zeros}\ \cdots 10\cdots 0}_{N\ unit\ cells}. \tag{4.10}$$

4.2.2 Jain's wave functions

This basically works as the Laughlin fractions. However, it is more technically involved and I will just sketch how it works. On the cylinder it is favourable to write the Jain states as

$$\Psi_{\frac{p}{2np+1}}(z_1, z_2, \ldots, z_N) = \ = \mathcal{P}_{LLL}\Psi_p \prod_{i<j}(e^{iz_i/R} - e^{iz_j/R})^{2n}. \tag{4.11}$$

Now, we consider the first part of this, $\mathcal{P}_{LLL}\Psi_p$, as an operator acting on the bosonic Laughlin state at filling $\nu = \frac{1}{2n}$. The bosonic state approaches a state with an electron at every $2n$:th position by the arguments in the preceding section. The operator, $\mathcal{P}_{LLL}\Psi_p$, turns out to have the effect (in the $R \to 0$ limit) that it inserts an extra zero after every p:th electron. Thus we conclude that

$$\lim_{R\to 0} \Psi_{\frac{p}{2np+1}}(z_1, z_2, \ldots, z_N) \sim (10_{2n}(10_{2n-1})^{p-1})^{N/p}$$

$$\equiv \underbrace{\overbrace{1\underbrace{0\cdots0}_{2n\ zeros}\ \overbrace{1\underbrace{0\cdots0}_{2n-1\ zeros}\ \cdots 10\cdots 0}^{p\ cells\ with\ 2n-1\ zeros}}^{One\ unit\ cell}\cdots}_{N/p\ unit\ cells}. \tag{4.12}$$

Clearly, the Laughlin states are a special case of the above (with $p = 1$). We also note that these unit cells are exactly the same as we obtained for the real-space interactions above. Thus we conclude that the Laughlin-Jain wave functions describe ground states also on the very thin cylinder.

4.2.3 The Rezayi-Read state

The $R \to 0$ limit of the Rezayi-Read state is not uniquely defined because of the variational parameters $\{\mathbf{k}_i\}$ present in

$$\Psi_{\frac{1}{2m}}(z_1, z_2, \ldots, z_N) = \mathcal{P}_{LLL} Det[e^{i\mathbf{k}_i \cdot \mathbf{r}_j}] \prod_{i<j} (e^{iz_i/R} - e^{iz_j/R})^{2m} \prod_i e^{-\frac{1}{2}y_i^2}. \quad (4.13)$$

However, for reasonable sets of momenta, $\{\mathbf{k}_i\}$, the limit turns out to include terms like

$$\lim_{R \to 0} \Psi_{\frac{1}{2m}}(z_1, z_2, \ldots, z_N) \sim 0110011001100110\ldots + \ldots . \quad (4.14)$$

This is certainly not a ground state as $R \to 0$, but rather looks like the seed for shortest range hopping, V_{21}, in the sense that many new states are created when V_{21} is applied. This is in sharp contrast to the $R \to 0$ limit of the bosonic $\nu = 1/2$ Laughlin state $(101010\ldots)$ which is annihilated by V_{21}. Thus, it seems like the physics of the bulk $\nu = 1/2$ state is separated from the thin limit by at least one quantum phase transition. In the next chapter, we will study the half-filled Landau level on the thin but finite torus and obtain an understanding of this behavior of the Rezayi-Read state, which is different from that of the gapped states.

4.3 Excitations

It is known that the low energy excitations in the quantum Hall system are fractionally charged particles and holes. In the thin limit these have a rather trivial manifestation. At filling $\nu = \frac{p}{2pm+1}$, the $R \to 0$ limit of the quasi-hole is created by removing 10_{2m-1} somewhere in the system (a quasi-electron is created by the reverse operation). The fractional charge of this operation is easily determined by Schrieffers counting argument [46]: By removing 10_{2m-1} at $2pm+1$ different locations and adding $2m$ unit cells $10_{2m}(10_{2m-1})^{p-1}$ to keep the number of sites fixed, $2pm+1$ quasi-holes, each with charge $e^* = e\frac{(2pm+1)-2pm}{2pm+1} = \frac{e}{2pm+1} = \frac{e}{q}$, are created. This generalizes to other fillings $\nu = p/q$, wiz

$$e^* = \frac{e}{q}. \quad (4.15)$$

Our studies show that these excitations agree with the $R \to 0$ limit of the particle-hole excitations proposed within the Laughlin-Jain wave function approach. Furthermore, these conclusions are supported by exact diagonalization studies, where we find that the low energy excitations are consistent with this interpretation: At $\nu = 6/18 = 1/3$ and $L_1 = L_2$, we find that the lowest band of excitations contains precisely those states that can be interpreted as the particle-hole excitations above.

It is interesting to consider what happens when several quasi-holes are created in *e.g.* a Laughlin state. At $\nu = \frac{1}{2m+1}$ a quasi-hole is created by inserting 10_{2m-1} somewhere in the string of unit cells 10_{2m}. Repeating this while keeping the particles as far apart as possible, one eventually reaches the filling $\nu = \frac{2}{4m+1}$. If even more cells 10_{2m-1} are inserted one eventually arrives at filling $\nu = \frac{3}{6m+1}$ and so on. Thus all the Jain fractions $\nu = \frac{p}{2pm+1}$ are constructed in this way.

Furthermore, this shows that the single, well defined, operation of inserting 10_{2m-1} have a meaning that depends on what state it is inserted into. Near filling $\nu = \frac{p}{2pm+1}$ this is interpreted as as a quasi-hole with charge $e^* = \frac{e}{2pm+1}$. In fact, this provides an explicit construction of the Haldane-Halperin hierarchy model on the thin torus.

Chapter 5

The half-filled lowest Landau level

Here, we present an exact solution for the low energy sector of the half-filled lowest Landau level on a thin, but finite, torus. The ground state is a filled one-dimensional Fermi sea of neutral dipoles, and supports gapless excitations. Thus it provides an explicit example of how composite fermions emerge. The dipole description on the thin torus has striking similarities with the bulk description of the $\nu = 1/2$ state and we argue that our solution is connected to this.

5.1 An exact solution

We truncate the interaction in (3.21) and keep only the two shortest range electrostatic terms and the shortest range hopping term

$$\begin{aligned} H = \quad & \sum_n [V_{10}\hat{n}_n\hat{n}_{n+1} + V_{20}\hat{n}_n\hat{n}_{n+2} \\ & -V_{21}(c_n^\dagger c_{n+1}c_{n+2}c_{n+3}^\dagger + H.c.)] \quad , \end{aligned} \tag{5.1}$$

where $\hat{n}_k = c_k^\dagger c_k$. This provides a good approximation of the interaction on a thin torus as discussed below and in [11].

$$1001 \overset{V_{21}}{\longleftrightarrow} 0110$$

Figure 5.1: *This hopping, together with the two shortest range electrostatic terms, is included in our model. The hopping turns out to be the crucial part of the model while the electrostatic terms are less important.*

The crucial part in (5.1) turns out to be the hopping term V_{21}. However, we will begin by considering the electrostatic part of the Hamiltonian at the special point $V_{10} = 2V_{20}$. The electrostatic part, $H|_{V_{21}=0}$, has eigenstates with fixed charges, $|n_1 n_2 ... n_{N_s}\rangle$, where $n_i = 0, 1$ and $|1\rangle = c^\dagger|0\rangle$. The energies are

$$E_0 = \frac{V_{10}}{2}(\frac{N_s}{2} + n_{111} + n_{000}) \quad . \tag{5.2}$$

Here, n_{111} (n_{000}) is the number of 3-strings[1], *i.e.* strings consisting of three nearby electrons (holes) in $n_1 n_2 ... n_{N_s}$. Thus there is a degenerate ground state manifold $\mathcal{H}_0$ consisting of all states where at most two electrons or two holes are next to each other. The energies in $\mathcal{H}_0$ follow by induction. All states without 3-strings for N_s sites are constructed by inserting an electron and a hole in $N_s - 2$ states without 3-strings. The energy of two particles and two holes (*i.e.* $\nu = 1/2$, $N_s = 4$) is always V_{10} and by considering the increase in energy under the constraint above, the ground state energy follows directly. The excitations are 3-strings of either electrons or holes and each 3-string has energy $V_{20} = V_{10}/2$. The energy of a 3-string follows by considering the change in energy of one or several 3-strings when moving one constituent.

We define a subspace $\mathcal{H}'$ of the full Hilbert space by requiring each pair of sites $(2p-1, 2p)$ to have charge one. Note[2] that $\mathcal{H}' \subset \mathcal{H}_0$. We expect that $\mathcal{H}'$ is the low-energy sector under fairly general conditions since it contains the maximally hoppable state $|100110011001....\rangle$ and it has a low electrostatic energy by construction. Furthermore H preserves the subspace $\mathcal{H}'$ thus any other ground state candidate may not mix with the states in $\mathcal{H}'$. The Hamiltonian (5.1) can be exactly diagonalized in $\mathcal{H}'$ giving non-interacting neutral fermions (*c.f.* composite fermions).

There are two possible states for a pair of sites in $\mathcal{H}'$:

$$|\downarrow\rangle \equiv |01\rangle, \quad |\uparrow\rangle \equiv |10\rangle \tag{5.3}$$

and it is natural to introduce the spin operators

$$s_p^+ = c_{2p-1}^\dagger c_{2p}, \quad s_p^- = c_{2p}^\dagger c_{2p-1}. \tag{5.4}$$

On states in $\mathcal{H}'$, s^+, s^- describe hard core bosons—they commute on different sites but anti-commute on the same site

$$\begin{gathered}[s_i^-, s_j^+] = [s_i^-, s_j^-] = [s_i^+, s_j^+] = 0, \quad i \neq j \\ \{s_i^-, s_i^+\} = 1, \quad \{s_i^-, s_i^-\} = \{s_i^+, s_i^+\} = 0,\end{gathered} \tag{5.5}$$

and H is the nearest neighbor spin 1/2 XY-chain,

$$H = \frac{\alpha N_s}{2} + V_{21}[\sum_{p=1}^{N_s/2} s_{p+1}^+ s_p^- + s_{p+1}^- s_p^+]. \tag{5.6}$$

Expressing the (hard core) bosons in terms of fermions d using the Jordan-Wigner transformation,

$$s_p^- = K_p d_p, \quad K_p = e^{i\pi \sum_{j=1}^{p-1} d_j^\dagger d_j}, \tag{5.7}$$

[1] A string of length $k \geq 3$ is counted as $k-2$ strings and periodic boundary conditions are assumed

[2] Acting with T_2 gives an equivalent grouping of the sites $(2p, 2p+1)$ instead—and a corresponding subspace $\mathcal{H}'_T$.

the Hamiltonian (5.1) is simply that of free fermions[3],

$$H = \frac{V_{10}N_s}{4} + V_{21}[\sum_{p=1}^{N_s/2-1} d_{p+1}^\dagger d_p + d_1^\dagger K_{N_s/2} d_{N_s/2} + H.c.], \tag{5.8}$$

when restricted to $\mathcal{H}'$. Thus, after a Fourier transformation,

$$d_j = \frac{1}{\sqrt{N_e}} \sum_k e^{ijk} \tilde{d}_k, \tag{5.9}$$

where $k = -\pi + n\frac{2\pi}{N_e}$, $n = 1, \ldots, N_e$, the Hamiltonian becomes

$$\tilde{H} = \frac{V_{10}N_s}{4} + 2V_{21} \sum_k \cos k \; \tilde{d}_{k+1}^\dagger \tilde{d}_k. \tag{5.10}$$

The ground state is obtained by filling all the negative energy states (*i.e.* all states with $|k| > \frac{\pi}{2} \Leftrightarrow \cos k < 0$):

$$|\, GS \,\rangle = \prod_{|k| > \frac{\pi}{2}} \tilde{d}_k^\dagger |\, 101010 \ldots \rangle. \tag{5.11}$$

This state has energy

$$E/N_s = \frac{V_{10}}{4} - \frac{V_{21}}{\pi} \tag{5.12}$$

per site. The excitations are neutral particle-hole excitations out of this Fermi sea. They have a natural interpretation in terms of dipoles and in the limit $N_s \to \infty$ the excitations become gapless. All of the electric charge sits in the Fock vacuum, $|\, 101010 \ldots \rangle$, while the quasi-particles, $d^\dagger$, that build up the ground state are neutral. Note that the number of "composite fermions" equals half the number of original electrons[4], *i.e.* $N_e/2$. Furthermore, the vacuum is the $L_1 \to 0$ limit of the bosonic Laughlin state, thus this construction very closely resembles the Rezayi-Read state (2.12).

A natural issue to address is whether the obtained state is homogenous or not. The density operator is

$$\begin{aligned} \rho(\vec{r}) &= \sum_{m,n} \psi_m^*(\vec{r}) \psi_n(\vec{r}) c_m^\dagger c_n \\ &\propto \sum_{m,n} e^{i(n-m)x} e^{-\frac{1}{2}((y+n)^2 + (y+m)^2)} c_m^\dagger c_n, \end{aligned} \tag{5.13}$$

thus $\langle c_m^\dagger c_n \rangle$ gives information about the electron density of the ground state. We start by noting that $c_m^\dagger c_n$ takes a state in $\mathcal{H}'$ out of this subspace if m and n are located at different pairs of sites (on different spin sites). Thus $\langle c_m^\dagger c_n \rangle = 0$ in this case. The linear density, $\langle\, c_n^\dagger c_n \,\rangle$, and the case of different electron sites on the same spin site ($\langle\, c_{2n-1}^\dagger c_{2n} \,\rangle = \langle\, s_n^z \,\rangle$) follows from the fact that we are indeed at $\nu = 1/2$ and that the nearest neighbor spin 1/2 XY-chain has $\sum_n s_n^z |\, GS \,\rangle = 0$ for even number of sites and $\sum_n s_n^z |\, GS \,\rangle = \pm |\, GS \,\rangle$ for odd[5].

[3]The K-factor can be ignored for $N_s \to \infty$, see [44].

[4]If N_e is odd, the number of fermions is not unique—either $\frac{N_e+1}{2}$ or $\frac{N_e-1}{2}$.

[5]This also relies on the assumption that the state is an eigenstate of T_2^2, which makes sure that the excess spin is smeared out.

This shows that the ground state is homogenous for every even N_e and that the amplitude of the charge density wave is $1/N_s = 1/2N_e$ when N_e is odd. Hence, in the thermodynamic limit $\langle c_m^\dagger c_n \rangle = \frac{1}{2}\delta_{mn}$, and the state is homogeneous.

This solves the problem in $\mathcal{H}'$ and, by action of T_2, in $\mathcal{H}'_T$. It remains to consider the states outside $\mathcal{H}'$ and $\mathcal{H}'_T$. In [11] we argue that these states are separated from our solution by a finite energy gap. Intuitively, this makes sense since $\mathcal{H}'$ contains the maximally hoppable state $|01100110011....110\rangle$. Furthermore we note that whereas $\mathcal{H}'$ is invariant under H, other states in $\mathcal{H}_0$ may mix with states not in the ground state manifold. Even though our arguments are strong and we have confirmed this numerically, we still lack a strict proof that our solution constitutes the low energy sector of (5.1).

5.1.1 Luttinger liquid description on the thin torus

We now consider the stability of the solution we have obtained for H in (5.1) at the solvable point, $V_{10} = 2V_{20}$; $V_{21} \geq 0$, and argue that it develops into an interacting Luttinger liquid for physically reasonable interactions. For the real space short range interaction $V(\mathbf{r}) = \nabla^2\delta(\mathbf{r})$, $V_{10} = 2V_{20}$ corresponds to $L_1 = 2\pi/\sqrt{2\ln 2} = 5.3$. The hopping term included in (5.1) is then $V_{21} = \frac{3}{16}V_{10}$, whereas the leading ignored terms are small: $V_{30} = \frac{9}{264}V_{10}$ and $V_{31} = \frac{1}{64}V_{10}$. This is close to the solvable point.

Of course the interaction between real-life electrons are not of the short-range type described above. If one instead considers a more realistic Coulomb interaction, $V(\vec{r}) = \frac{e}{4\pi\epsilon r}$, one finds that the electrostatic interaction does not fall of exponentially with distance but is long range. However, this is not a big problem because at a fixed filling, ν, the physics of these interactions can be captured by an effective short-range electrostatic interaction (*i.e.* with new renormalized coefficients). We conclude that our model will remain a good approximation on the thin torus for a larger class of interactions. This conclusion is supported by extensive numerical calculations for both Coulomb [12] and short-range interactions [10].

By the arguments above we note that our model is good but by no means exact for a physical interaction. However, we know that the states not in $\mathcal{H}'$ are separated from the low energy states by a gap in our model. Hence small perturbations of the Hamiltonian around the solvable point lead, in perturbation theory, to an effective spin 1/2 Hamiltonian in $\mathcal{H}'$. The low energy sector at the solvable point (H in (5.1) with $V_{10} = 2V_{20}$) is contained in the spin 1/2 Hilbert space $\mathcal{H}'$. To obtain the general effective Hamiltonian explicitly is non-trivial. However, the electrostatic terms, V_{m0}, preserve $\mathcal{H}'$ and simply become

$$\delta H_{el.stat} = \sum_{i,n}[(2V_{2n,0} - V_{2n-1,0} - V_{2n+1,0})s_i^z s_{i+n}^z]. \tag{5.14}$$

Note that the coupling looks like a second derivative (with a minus sign in front). However, these generated terms are small near the solvable point and will as such be suppressed in this region. The hopping terms, V_{mn}, will, in general, contribute in second order perturbation theory with terms that are quadratic and quartic in the spin operators.

The generated terms are spin operators of quadratic and higher order. They all have small coefficients since there is a gap to states not in $\mathcal{H}'$ (the matrix elements for transitions to states in $\mathcal{H}'_T$ vanish). All terms are irrelevant in the sense of the renormalization group, except for $s_i^z s_{i+n}^z$ which makes the non-interacting fermion theory develop into a Luttinger liquid with interaction parameter $K \neq 1$ (the free fermions obtained at the solvable point have $K = 1$) [47]. These perturbations are small, thus we remain in the XY-phase. On the other hand, if the perturbations become large, we would end up in the Ising phase and the spectrum would become gapped[6].

Based on the renormalization group argument, and strongly supported by DMRG calculations as well as by exact diagonalization studies, we conclude that the $\nu = 1/2$ system on a thin torus is a Luttinger liquid for a finite range of L_1 and that the generation of $s_i^z s_{i+n}^z$ terms indicates that the interaction parameter that determines the decay of correlation functions is shifted from its free value at the solvable point.

5.2 The connection to the bulk state

The obtained solution on the thin torus has striking similarities to what is expected from theory and experiment for the bulk state. Based on this, we conjectured [11] that this state develops continuously, without a phase transition, to the bulk state as $L_1 = 2\pi R \to \infty$.

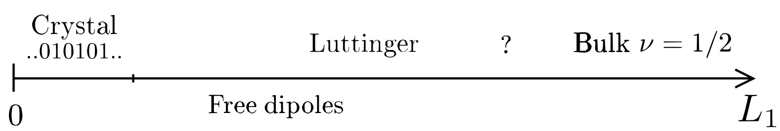

Figure 5.2: *Phase diagram for the half-filled Landau level on a torus, where one circumference, L_1, varies while the other is infinite. For very short L_1 a crystalline state determined by electrostatics alone is obtained. At $L_1 \sim 5$ magnetic lengths, there is a phase transition to a system non-interacting neutral one-dimensional fermions (dipoles). Based on the similarities of the state at short L_1 and the bulk state, we conjecture that there is no phase transition between this state and the bulk $\nu = 1/2$ system [11].*

To investigate our conjecture, we have performed exact digonalization studies of small system for various N_e and L_1 using an unscreened Coulomb potential. The obtained ground states are then compared with the Rezayi-Read state (2.12), that is expected to describe the bulk state well, by calculating overlaps [12]. For $L_1 \leq 5.3$ we find a simple crystal $1010\ldots$ independent of $N_e \leq 10$. At $L_1 \sim 5.3$ there is a sharp transition into a new state that we identify as our Luttinger liquid solution, obtained above, by comparing quantum numbers and calculating overlaps.

As L_1 is increased further, there is a number of different transitions into new states—details depend sensitively on N_e but all transitions are much smoother than the one at $L_1 \sim 5.3$. However, as shown[7] in Figure 5.3, we find that each

[6]This is exactly what happens on the very thin torus. The electrostatic terms (5.14) become large and we end up in a ferromagnetic/crystalline state with a gap.

[7]We choose to present results for odd N_e. This is due to the fact that the state obtained in exact diagonalization corresponds to a linear combination of two Rezayi-Read state with different Fermi seas, but the same quantum numbers, when N_e is even.

of these states corresponds to a given set of momentas in the Rezayi-Read state. Furthermore, these Fermi seas of momenta develop in a natural and systematic way. Starting from an elongated sea, which we identified as the exact solution, a single momenta is moved at each level-crossing, terminating in a symmetric Fermi sea when $L_1 = L_2 = \sqrt{4\pi N_e}$.

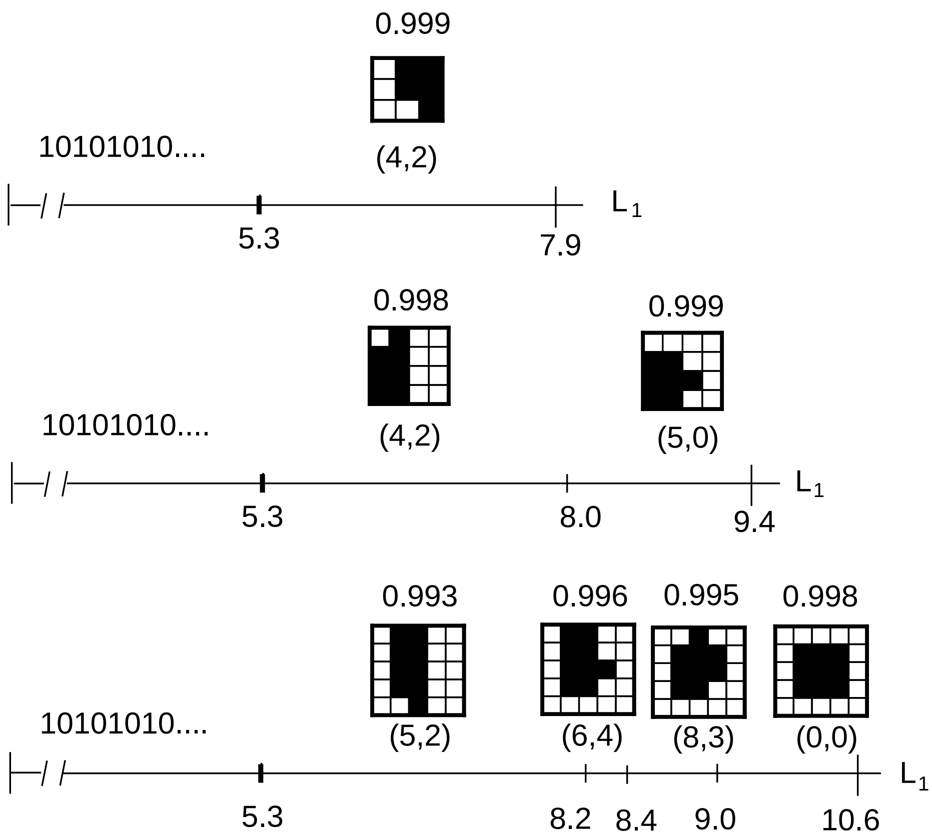

Figure 5.3: *Phase diagrams for the $\nu = 1/2$ state as a function of L_1. The results are obtained in exact diagonalization, using unscreened Coloumb interaction. Overlaps with the Rezayi-Read state with the displayed Fermi seas of momenta are shown for 5, 7 and 9 electrons. The quantum numbers (K_1,K_2) are displayed below each Fermi sea. The phase diagram is symmetric about $L_1 = L_2 = \sqrt{4\pi N_e}$ (up to a $x \leftrightarrow y$ shift in the Fermi sea).*

Our interpretation of this is that the transitions at $L_1 > 5.3$ are not to new phases but rather to similar states—all of them are after all described by the Rezayi-Read state. Thus, our exact solution evolves into the bulk state as $L_1 \to \infty$, without a phase transition.

It is tempting to speculate on what consequences our findings may have. We have showed that the $\nu = 1/2$ system is a Luttinger liquid for L_1 slightly larger than 5.3 and $L_2 \to \infty$. As L_1 increases there are transitions to similar states. This suggests that the system is still a Luttinger liquid—at least as long as L_1 is finite. The big question is what happens as $R \to \infty$? Is it still a gapless one-dimensional system or does it become a two-dimensional Fermi sea as commonly believed? If it remains one-dimensional this would change the interpretation of the system and it may also lead to observable effects in transport measurements, such as non-linear $I - V$ characteristics.

Chapter 6

Our view of the quantum Hall system

In this chapter, we argue that the quantum Hall system can, at various filling fractions, be understood as a one-dimensional spin chain with short-range interactions. The gapped fractions develop continuously from $R = 0$ to $R \to \infty$ whereas the gapless fractions are separated by a phase transition and seem to be understandable as instabilities of the one-dimensional setting[1]. We also briefly discuss a possible way of determining the nature of some recently observed states.

6.1 The quantum Hall system as a one-dimensional spin chain with short-range interactions

We have argued that the Laughlin-Jain wave functions are still ground states in the thin limit, $R \to 0$. Thus the physics of the fractional quantum Hall effect exists also on the thin torus. Since the other observed QH states also have a gap to all excitations, it is very plausible that these also exists in the thin limit.

At the half-filled Landau level, we have shown that an exact solution of the interacting electron gas on the thin cylinder evolves into a Luttinger liquid description of the ground state as R increases. This state is separated from the thin limit by a phase transition, and we expect this to be true in general for the gapless states (obviously there is a gap at all fractions, determined by electrostatics, when $R \to 0$).

The bulk excitations also have a simple manifestation in the thin limit and the hierarchy model as well as the emergent Landau level structure can be understood on the thin torus.

The mapping of a single Landau level onto a one-dimensional system is exact, and in a sense obvious. What is non-trivial is that the short range spin chain is relevant—after all the range of interaction tends to infinity in the bulk case. However, the findings above shows that the thin case is indeed relevant. Thus,

[1] A third possibility occurs for very small ν. Here, the one-dimensional crystal structure must evolve into a two-dimensional Wigner crystal. This case is, however, beyond the scope of this thesis.

we claim that the quantum Hall system, in the lowest Landau level, can be understood as a one-dimensional spin chain with short-range interactions.

6.2 Why are the $\nu = 1/2$ and $\nu = 1/3$ states so different?

It is illuminating to consider the action of the shortest range hopping, V_{21}, on the $R \to 0$ states. Whereas the $\nu = 1/2$ state is annihilated by the hopping in this limit, the gapped $\nu = 1/3$ is actually the maximally hoppable state! Hence there is no competition between electrostatics and hopping in the latter case and the hopping can be turned on adiabatically—the ground state will mix in components other than the Tao-Thouless state, $|\ldots 100100100\ldots\rangle$, as the hopping strength is increased. In the former case, there is a direct conflict between hopping and electrostatics on the very thin torus—this results in two regions separated by a phase transition. For small R electrostatics dominates and the ground state is a pure crystal, $|\ldots 10101010\ldots\rangle$, with every other site occupied. Then, suddenly hopping becomes dominant and the state shifts to a homogenous quantum fluid originating from the maximally hoppable state $|\ldots 011001100110\ldots\rangle$ (*i.e.* a transition from the Ising to the XY-phase). This gives a clue to why some fractions are gapped while others are gapless. Hopefully, the powerful techniques of one dimension can be used to answer these kinds of questions in more generality.

It is tempting to consider the possibility of mapping other fractions than $\nu = 1/2$ onto spin chains. The mapping of the low energy sector at $\nu = 1/2$ onto an $s = 1/2$ spin chain, would then presumably generalize into a mapping of $\nu = 1/3$ onto an $s = 1$ chain and so on[2]. This suggests that the Haldane conjecture for the gaps in spin chains might also apply to the two-dimensional electron gas in a strong magnetic field [48, 49].

6.3 New states

Recently, minima in R_{xx} were observed by Pan et al [16] at new fractions such as $\nu = 5/13$, $3/8$, $4/11$ and $6/17$ which cannot be identified as Jain states. Rather, they have unit cells $(10_2)_2 1010_2 10$ and $10(10_2)_p$, $p = 2, 3, 5$. While Jain and others [50, 51] have tried to explain some of these states as a fractional effect of composite fermions, their origin and nature is not fully settled yet.

Since both the gapless and the gapped Hall states can have this signature[3] it remains to discover the nature of these states. We have performed exact diagonalization of small systems for 3/8 and 4/11. We find that at 3/8 there are transitions as L_1 is varied whereas no transition is found at 4/11. In analogy with our other findings this suggests that 4/11 is a gapped QH state while the $\nu = 3/8$ state is metallic.

[2] In the thin limit, the $\nu = 1/3$ system maps onto an $s = 1$ spin chain that has quadratic and quartic terms in the Hamiltonian.

[3] To distinguish the two cases by looking at only R_{xx} one has to vary the temperature, T. The gapped states have $R_{xx} = 0$ as $T \to 0$ whereas R_{xx} remains finite for the gapless states in this limit. Data of this type in not available to date.

Chapter 7

Conclusions

In conclusion, we have presented a one-dimensional view of the FQHE that can account for the differences between various filling fractions and provides a natural framework for addressing numerous interesting questions regarding the physics in the lowest Landau level. A simple picture of the quantum Hall system has emerged; the gapped fractions originate from simple crystals in the electrostatic limit ($R \to 0$) while the gapless states may be studied as instabilities in the one-dimensional setting (as is the case at $\nu = 1/2$).

We obtained an exact analytical solution of the strongly interacting many-body problem in the half-filled lowest Landau level. The solution gives a description of the low energy sector of the electron gas on a thin torus in terms of free one-dimensional dipolar fermions. The ground state is homogenous and correspond to the filled Fermi sea while excitations are neutral particle-hole excitations out of this sea. Hence we have for the first time been able to explicitly derive how a system of strongly interacting electron in a magnetic field gives rise to a low energy sector of free particles (composite fermions) with a reduced coupling to the magnetic field. A renormalization group argument showed that the free fermion description evolves into an interacting Luttinger liquid in the neighborhood of the solvable point—this is also supported by extensive numerical calculations. By comparing with the Rezayi-Read state and by calculating overlaps, we conclude that the description on the thin torus is adiabatically connected to the bulk state.

The one-dimensional nature of the quantum Hall system may have experimental consequences—such as non-linear $I - V$ characteristics implied by the Luttinger liquid description of the half-filled Landau level.

Our theory is microscopic, in contrast to the conventional mean field theories used to describe the quantum Hall system. The scale of excitations is correct since the coupling becomes that of the interaction instead of that of the cyclotron frequency as in the Chern-Simons approach. Also, no reference to higher Landau levels is necessary and our theory is manifestly particle-hole symmetric. Furthermore, fractionally charged excitations are easily studied in the thin limit, and a simple way of predicting the nature of new states from small systems is proposed within our framework.

There are, of course, still many open ends and unsolved mysteries regarding the physics of the lowest Landau level. However, some of these might be resolved within our framework using the powerful techniques of one dimension.

Despite all advantages there is one obvious drawback with our approach. It is of course hard to prove what happens as $R \to \infty$. However, this theory is a complement to other theories and together with the existing ones we think that we are approaching a deeper understanding of the quantum Hall system.

Bibliography

[1] K.v. Klitzing, G. Dorda and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).

[2] R.B. Laughlin, Phys. Rev. B. **23**, 5632, (1981).

[3] B.I. Halperin, Phys. Rev. B. **25**, 2185, (1982).

[4] D.C. Tsui, H.L. Störmer and A.C. Gossard, Phys. Rev. Lett. **48**, 1599 (1982).

[5] R.B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[6] E.H. Rezayi and F.D.M. Haldane, Phys. Rev. B **50**, 17199 (1994).

[7] J.K. Jain, Phys. Rev. Lett. **63**, 199 (1989).

[8] S.R. White, Phys. Rev. Lett. **69**, 2863 (1992); Phys. Rev. B **48**, 10345 (1993).

[9] F.D.M. Haldane, J. Phys. C, **14**, 2585 (1981).

[10] E.J. Bergholtz and A. Karlhede, cond-mat/0304517, (2003).

[11] E.J. Bergholtz and A. Karlhede, Phys. Rev. Lett. **94**, 026802 (2005).

[12] E.J. Bergholtz and A. Karlhede, Unpublished (2005).

[13] M.I. Dyakonov, cond-mat/0209206, (2002).

[14] E.H. Hall, Journal of Mathematics **2**, 287 (1879).

[15] R.L. Willett, J.P. Eisenstein, H.L. Störmer, D.C. Tsui, A.C. Gossard and J.H. English, Phys. Rev. Lett. **59**, 1776 (1987) (1987).

[16] W. Pan, H.L. Störmer, D.C. Tsui, L.N. Pfeiffer, K.W. Baldwin, and K.W. West, Phys. Rev. Lett. **90**, 016801 (2003).

[17] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics* (Pergamon Press, New York, 1994).

[18] R. Tao and D.J. Thouless, Phys. Rev. B **28**, 1142 (1983).

[19] S.A. Trugman and S. Kivelson, Phys. Rev. B **31**, 5280 (1985)

[20] F.D.M. Haldane in the *The Quantum Hall Effect*, eds. R.E. Prange and S.M. Girvin, (Springer-Verlag, New York, 1990).

[21] F.D.M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).

[22] B.I. Halperin, Phys. Rev. Lett. **52**, 1583, 2390 (E) (1984).

[23] For a review, see, *e.g.* J.K. Jain and R.K. Kamilla, in *Composite Fermions*, ed. O. Heinonen, (World Scientific, Singapore, 1998).

[24] S.M. Girvin, cond-mat/9907002, (1999).

[25] S.C. Zhang, T.H. Hansson and S.A. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).

[26] A. Lopez, and E. Fradkin, Phys. Rev. B. **44**, 5246 (1991).

[27] V. Kalmeyer and S.C. Zhang, Phys. Rev. B. **46**, 9889 (1992).

[28] N. Read, Phys. Rev. Lett. **62**, 86 (1989).

[29] See *e.g.* S.H. Simon,cond-mat/9812186 (1998).

[30] Illustration by Emma Wikberg (2005). Inspiered by Kwon Park.

[31] H.W. Jiang, H.L. Störmer, D.C. Tsui, L.N. Pfeiffer, and K.W. West, Phys. Rev. B **40**, R12013 (1989).

[32] B.I. Halperin, P.A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).

[33] R.L. Willett, and L.N. Pfeiffer, Surf. Sci. **362**, 38 (1996).

[34] W. Kang, H.L. Störmer, L.N. Pfeiffer, K.W. Baldwin, and K.W. West, Phys. Rev. Lett. **71**, 3850 (1993).

[35] V.J. Goldman, B. Su, and J.K. Jain, Phys. Rev. Lett. **72**, 2065 (1994).

[36] J.H. Smet, D. Weiss, R.H. Blick, G. Lütjering, K. von Klitzing, R. Fleischmann, R. Ketzmerick, T. Geisel, and G. Weimann, Phys. Rev. Lett. **77**, 2272 (1996).

[37] G. Murthy and R. Shankar, in *Composite Fermions*, ed. O. Heinonen, (World Scientific, Singapore, 1998) and Rev. Mod. Phys. **75**, 1101 (2003).

[38] D.-H. Lee, Phys. Rev. Lett. **80**, 4745 (1998), Erratum **82**, 2416 (1999).

[39] N. Read, Phys. Rev. B **58**, 16262 (1998).

[40] V. Pasquier and F.D.M. Haldane, Nuclear Physics B **516**, 719 (1998).

[41] E.H. Rezayi and N.Read, Phys. Rev. Lett. **72**, 900 (1994).

[42] See *e.g.* J.J. Sakurai, *Modern Quantum Mechanics*(Revised Edition) (Addison-Wesley Publishing company, New York, 1994).

[43] E. Westerberg and T.H. Hansson, Phys. Rev. B **47** 16554 (1993).

[44] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. **16**, 407 (1961).

[45] F.D.M. Haldane and E.H. Rezayi, Phys. Rev. B **31**, R2529 (1985).

[46] W.P. Su, and J.R. Schrieffer, Phys. Rev. Lett. **46**, 738 (1981).

[47] H.J. Schulz, G. Cuniberti and P. Pieri, in *Field Theories for Low-Dimensional Condensed Matter Systems*, eds. G. Morandi et al., (Springer, 2000).

[48] F.D.M. Haldane, Phys. Lett. 93A, 464 (1983).

[49] F.D.M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983).

[50] C.-C Chang and J.K. Jain, Phys. Rev. Lett. **92**, 196806 (2004).

[51] M.O. Goerbig, P. Lederer, and C.M. Smith, Phys. Rev. Lett. **93**, 216802 (2004).