Vorticity-based Analytical Models for Internal Bores and Gravity Currents

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- *Motivation*
  - *Hydraulic jumps*
  - *Internal bores*
  - *Gravity currents*
- *Earlier modeling approaches*
- *Circulation-based modeling*
- *Summary and outlook*

Hydraulic jumps

Laminar circular hydraulic jump:

Hydraulic jumps

Hydraulic jump in a dam spillway:

Hydraulic jumps

Hydraulic jump in a dam spillway:

Hydraulic jumps

Tidal bore on the river Severn:

Internal bore

Undular bore in the atmosphere (Africa):

Internal bore

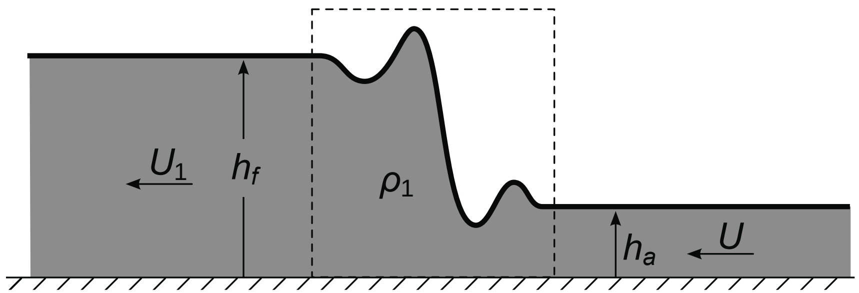
Atmospheric bore (Iowa):

Analytical models for stratified flows

Single-layer hydraulic jump (Rayleigh 1914):

Note: Simulation based on continuity + NS eqns. (mass, momentum)

In reference frame moving with the bore: steady flow



Task: Find U, U_1 as $f\,(h_f, h_a)$

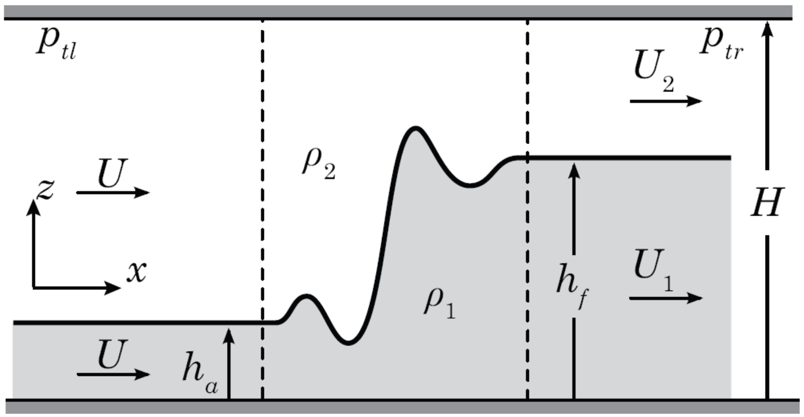
Mass conservation: $$Uh_a = U_1 h_f$$

Horiz. momentum conservation: $$U^2 h_a + \frac{1}{2} g h_a^2 = U_1^2 h_f + \frac{1}{2} g h_f^2$$

$$\rightarrow \quad u_f = \sqrt{\frac{1}{2} R\,(R+1)} \quad \text{where} \quad u_f = \frac{U}{\sqrt{g h_a}} \quad \text{and} \quad R = \frac{h_f}{h_a}$$

Analytical models for stratified flows (cont'd)

Two-layer internal bore for small density contrast (Boussinesq):



Find U, U_1, U_2 as $f\,(h_f\,, h_a\,, H, g')$

Have 3 conservation laws:

- mass in lower layer: $$U_1 h_f = U h_a$$
- mass in upper layer: $$U_2(H - h_f) = U(H - h_a)$$
- overall horiz. mom.: $$\int_0^H \left(p_l + \rho U^2\right)\,\mathrm{d}z = \int_0^H \left(p_r + \rho u_r^2\right)\mathrm{d}z$$

But: pressure difference $p_{tr} - p_{tl}$ appears as additional 4th unknown

→ closure assumption needed!

Two-layer internal bores (Boussinesq)

Closure assumption by Wood and Simpson (1984): no energy dissipation in the upper layer → apply Bernoulli eqn. along the top wall:

→ $$u_{ws} = \left\{ \frac{R\left(1+R\right)\left(1-Rr\right)^2}{R^2 r - 3Rr + 2} \right\}^{1/2}$$

where $u = U/(g'h_a)^{1/2}$ *,* $R = h_f/h_a$ *and* $r = h_a/H$

Alternative closure assumption by Klemp et al. (1997): no energy dissipation in lower layer → apply Bernoulli along lower wall:

→ $$u_{krs} = \left\{ \frac{R^2\left[2 - r\left(1+R\right)\right]\left(1-Rr\right)}{R^2 r - 3Rr + R + 1} \right\}^{1/2}$$

Two-layer internal bores (Boussinesq)

Why did Wood and Simpson (1984) and Klemp et al. (1997) need to invoke energy-based closure assumption, whereas DNS simulations require only conservation of mass and momentum?

Need to find: U, U_1, U_2, Δp across the bore

DNS simulation uses:

- *conservation of mass in each layer (2 eqns.)*
- *conservation of overall horizontal momentum*
- *conservation of overall vertical momentum*

Analytical models:

- *conservation of mass in each layer (2 eqns.)*
- *conservation of overall horizontal momentum*
- *DO NOT employ conservation of overall vertical momentum*

→ *existing analytical models do not satisfy conservation of vertical momentum. They use empirical energy closure assumption instead, to have enough equations to determine U, U_1, U_2, Δp*

Two-layer internal bores (Boussinesq)

Can we develop an analytical model that satisfies the conservation of vertical momentum, so that it does not require an empirical energy closure assumption?

Approach:

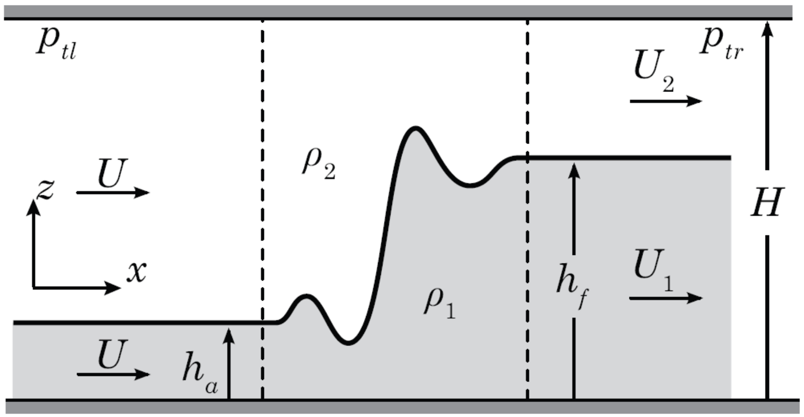
- *combine horizontal and vertical momentum eqns. → vorticity eqn.:*

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} \omega = -g' \frac{\partial \rho^*}{\partial x} + \nu \nabla^2 \omega$$

- *vorticity is generated at the interface between the two layers;*
- *it is then convected along by the fluid velocity*
- *it spreads diffusively as a result of viscosity*

Two-layer internal bores (Boussinesq)

Integrate over control volume containing the hydraulic jump:



$$\oint \omega \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S = \iint -g' \frac{\partial \rho^*}{\partial x} \, \mathrm{d}A + \oint \nu \boldsymbol{\nabla} \omega \cdot \boldsymbol{n} \, \mathrm{d}S$$

- for inviscid flow:

vorticity outflow = vorticity inflow + baroclinic vorticity production

- vorticity inflow = 0

- vorticity outflow = $(U_1 - U_2)(U_1 + U_2)/2 = (U_1^2 - U_2^2)/2$

- baroclinic vorticity production = $-g'(h_f - h_a)$

Two-layer internal bores (Boussinesq)

Conservation of vorticity yields:

$$\frac{1}{2}\left(U_2^2 - U_1^2\right) = g'\left(h_f - h_a\right)$$

combine with conservation of mass in both layers:

$$U_1 h_f = U h_a$$

$$U_2(H - h_f) = U(H - h_a)$$

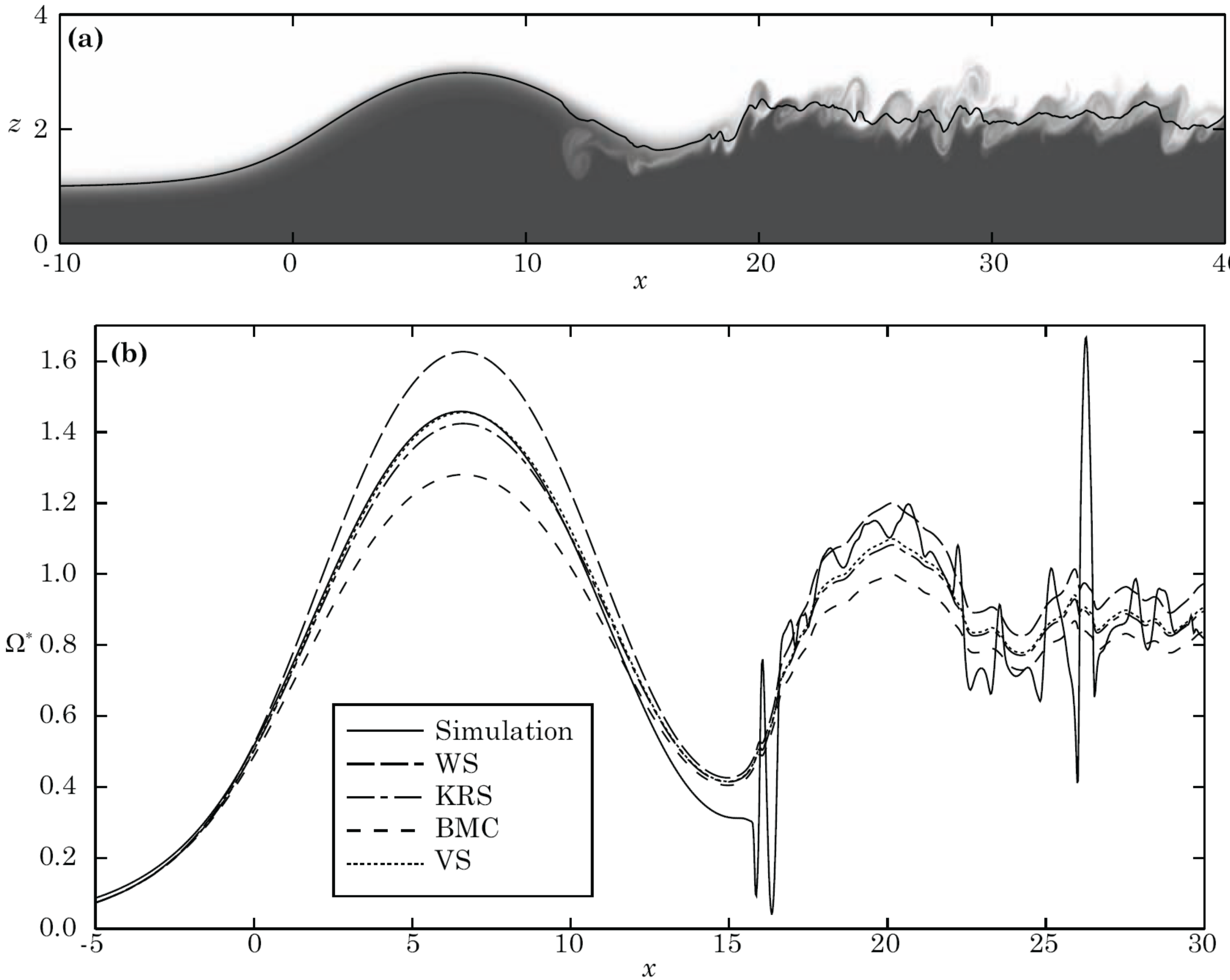
→ *have 3 equations for U, U_1 and U_2; note: p no longer shows up!*

The present vortex sheet model for two-layer internal bores yields:

$$u_{vs} = \left\{\frac{2R^2\left(Rr-1\right)^2}{R-2Rr+1}\right\}^{1/2}$$

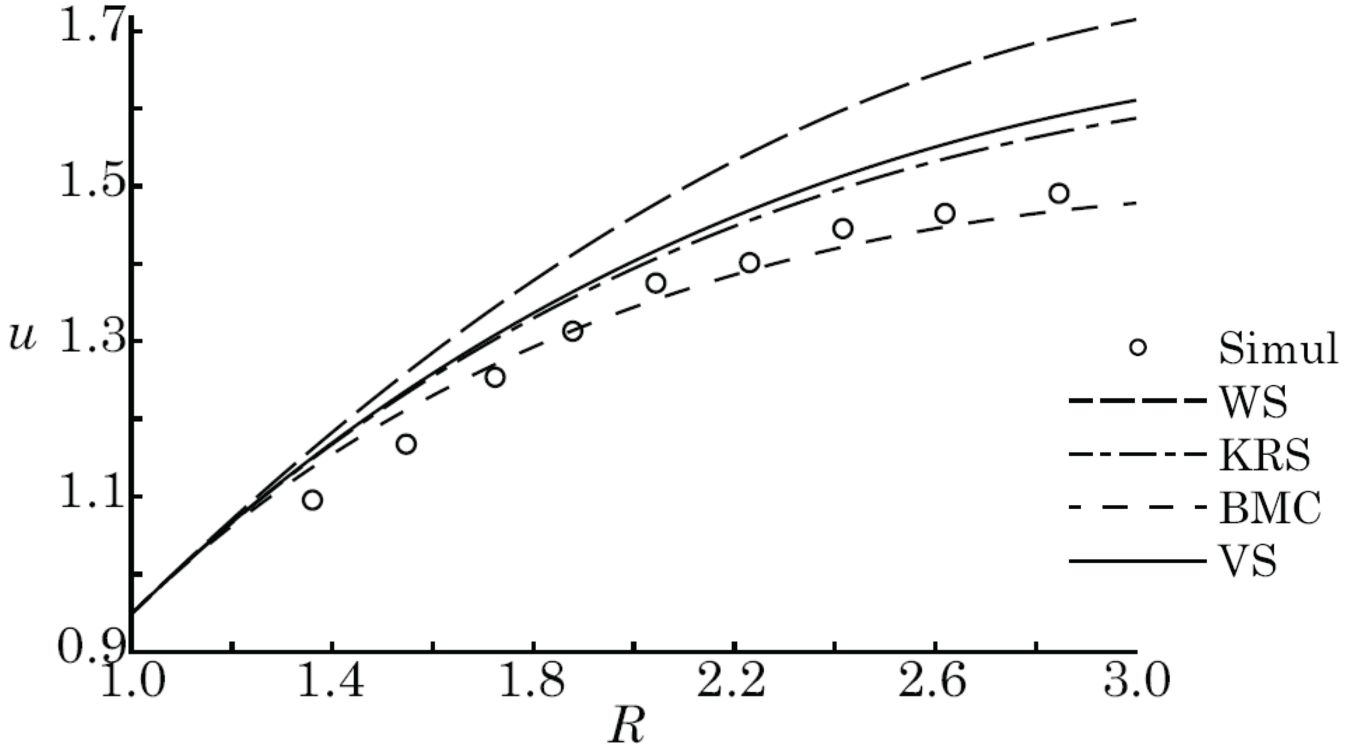
Note: We used only linear combination of horizontal and vertical momentum conserv. eqns. → could still use horizontal momentum eqn. by itself to determine Δp, but p-information is not needed to get bore velocity → consistent with NS simulations in (ψ,ω)-form

Comparison of different bore models: vorticity flux



Where the flow is approximately steady-state, the new circulation model yields the closest agreement with the DNS simulations

Comparison of different bore models: bore velocity

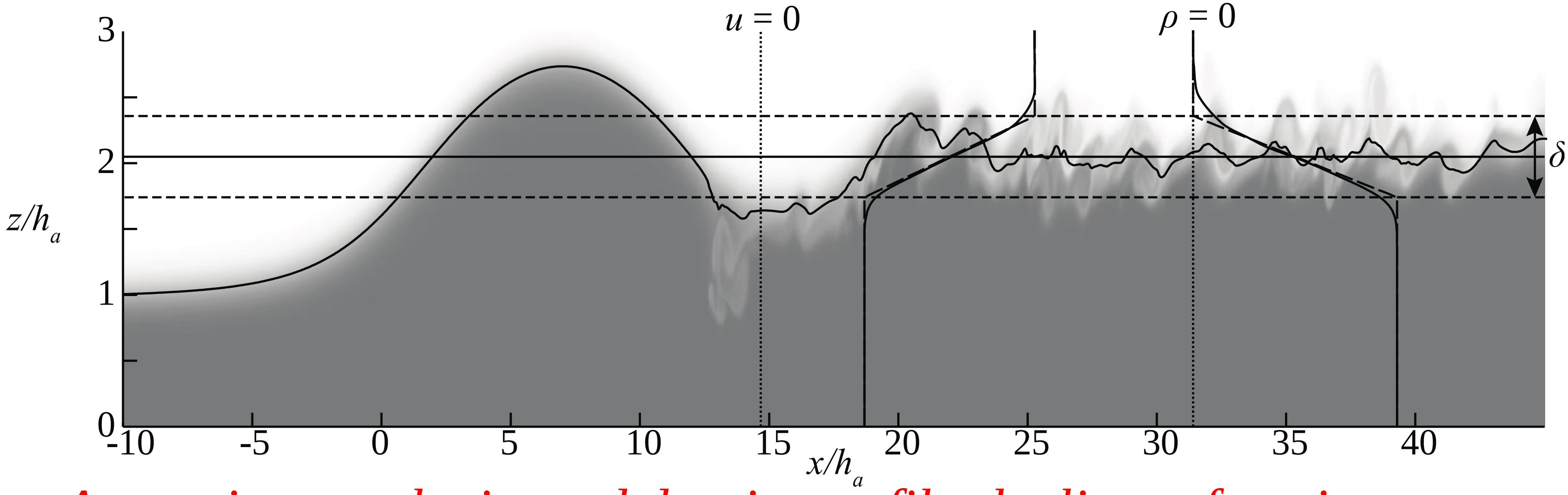


Circulation model does not agree very closely with NS data

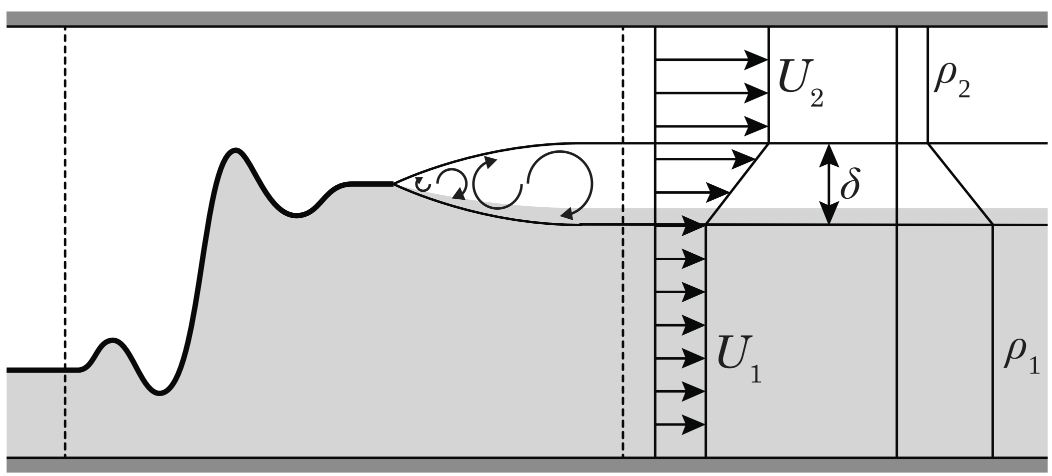
Why?

Have to analyze the effects of turbulent mixing in the bore

Two-layer internal bores: effects of turbulent mixing



Approximate velocity and density profiles by linear functions over mixing layer of thickness δ :



$$u(z^*) = U_1 + \frac{z^*}{\delta}(U_2 - U_1)$$

$$\rho^*(z^*) = 1 - \frac{z^*}{\delta}$$

Two-layer internal bores: effects of turbulent mixing

Modified conservation equations:

- *mass in lower layer:*

$$U h_a = U_1 \left(h_f - \frac{\delta}{2} \right) + \int_0^\delta u(z^*) \rho^*(z^*) \, \mathrm{d}z^*$$
$$= U_1 h_f + \frac{1}{6} \delta \left(U_2 - U_1 \right)$$

- *mass in upper layer:*

$$U \left(H - h_a \right) = U_2 \left(H - h_f - \frac{\delta}{2} \right) + \int_0^\delta u(z^*) \left[1 - \rho^*(z^*) \right] \mathrm{d}z^*$$
$$= U_2 \left(H - h_f \right) - \frac{1}{6} \delta \left(U_2 - U_1 \right)$$

- *vorticity:*

$$\frac{1}{2} \left(U_2^2 - U_1^2 \right) = g' \left(h_f - h_a \right)$$

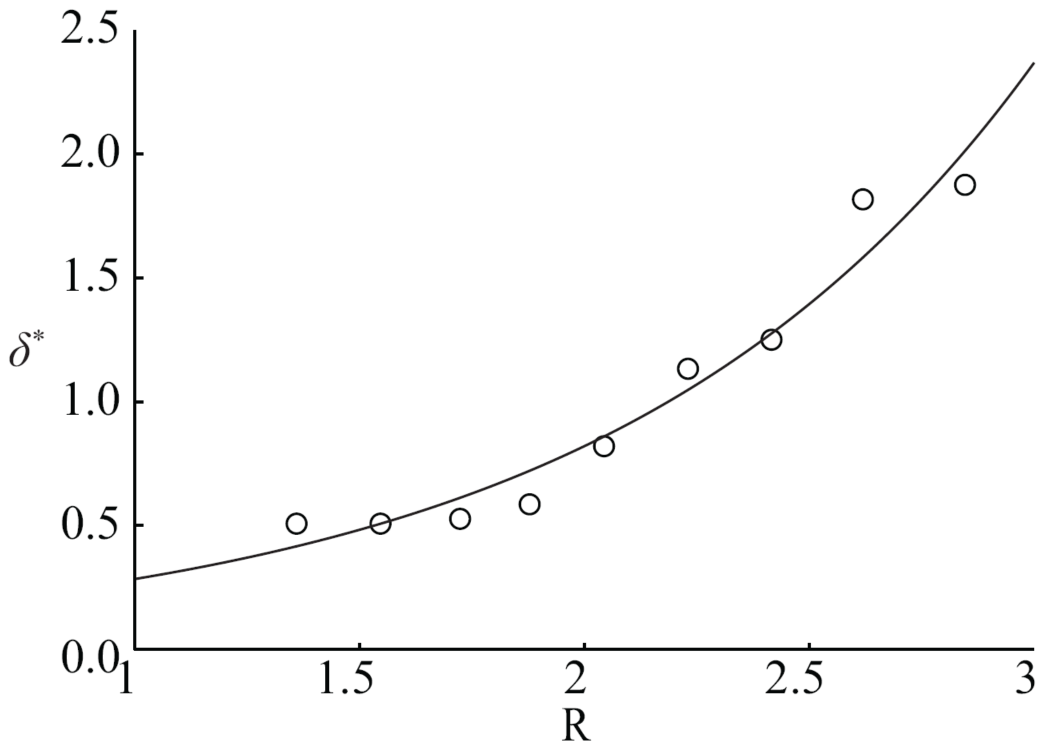
Vorticity conservation equation remains the same as before
→ mixing affects bore only via mass conservation, not via vorticity conservation

Bore velocity with mixing:

$$u_{dvs} = \frac{\left(R^2 r - R + \delta^*/6 \right) \left[-6 \left(6Rr - 3R - 3 + \delta^* \right) \right]^{1/2}}{6Rr - 3R - 3 + \delta^*}$$

Two-layer internal bores: effects of turbulent mixing

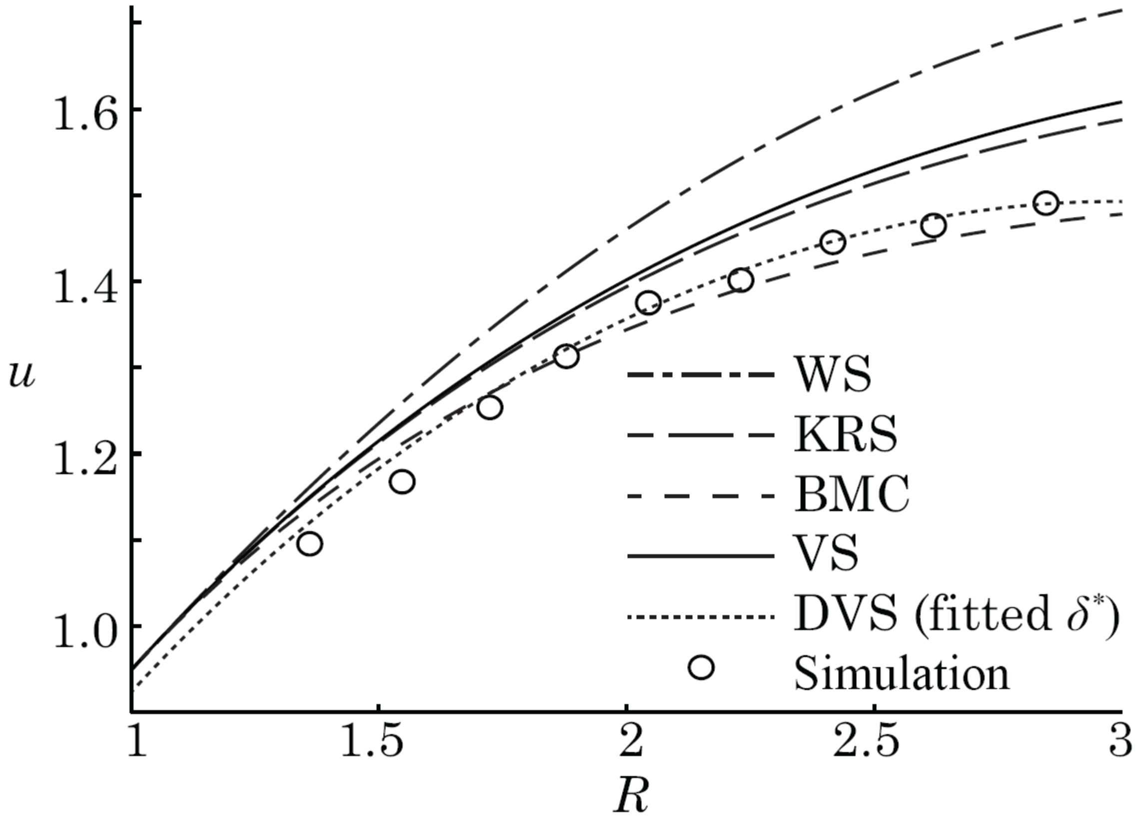
Determine interface thickness δ from DNS simulations:



Can fit smooth function through the DNS data to obtain δ(R), then substitute δ(R) into the finite interface thickness model

Two-layer internal bores: effects of turbulent mixing

Comparison with DNS simulations:



Diffuse vortex sheet model closely agrees with DNS data

Summary

- *by employing the vertical momentum equation, in addition to the conservation of mass and horizontal momentum, we avoid the need for an empirical closure condition based on energy considerations*
- *pressure equation becomes decoupled, so that information on the pressure is not required for predicting the bore velocity*
- *new circulation-based model yields very close agreement with DNS simulation data with regard to the vorticity flux*
- *in order to obtain good agreement regarding the bore velocity, we need to account for turbulent mixing effects*

Related problem: gravity currents

Haboob (atmospheric gravity current):

Driven by hydrostatic pressure gradient due to density difference

Sandstorms

Pyroclastic flow

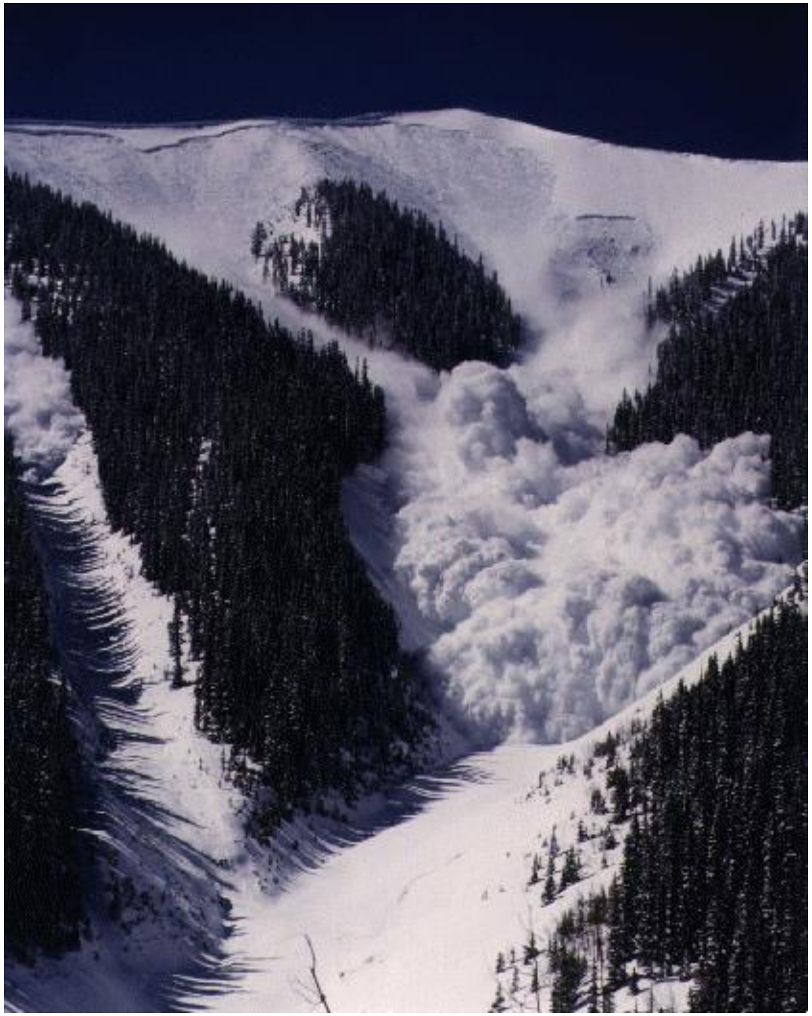
- *Small particles rise with buoyant ambient gas*
- *Large particles form pyroclastic avalanche*

Mt. St. Helens (USGS)

Thunderstorm outflow

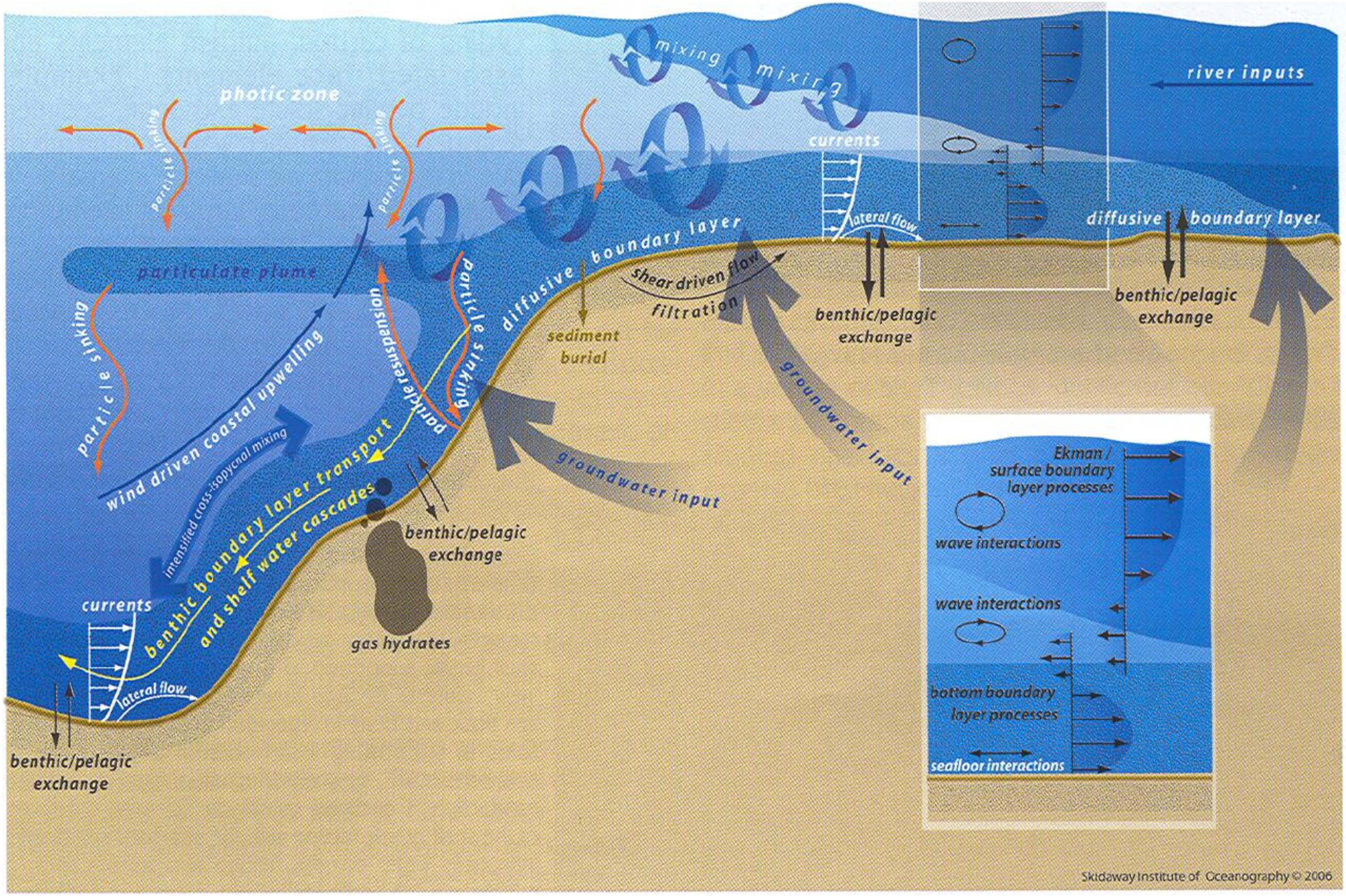
Borden and Meiburg (2010)

Avalanche



- *Non-Boussinesq*
- *Formation*
- *Growth / Amplification*
- *Front velocity*
- *Particle-particle interaction*
- *Erosion / Resuspension*
- *Deposition*
- *Influence of bottom topography*
- *Runout length*

Coastal margin processes



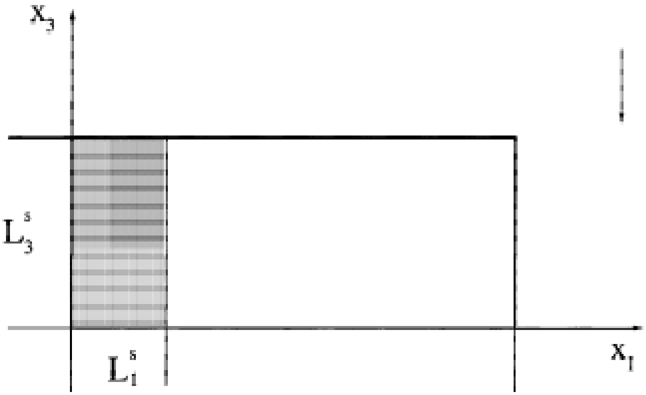
Turbidity current

- *Underwater sediment flow down the continental slope*
- *Can transport many km^3 of sediment*
- *Can flow O(1,000)km or more*
- *Often triggered by storms or earthquakes*
- *Repeated turbidity currents in the same region can lead to the formation of hydrocarbon reservoirs*

Turbidity current.
http://www.clas.ufl.edu/

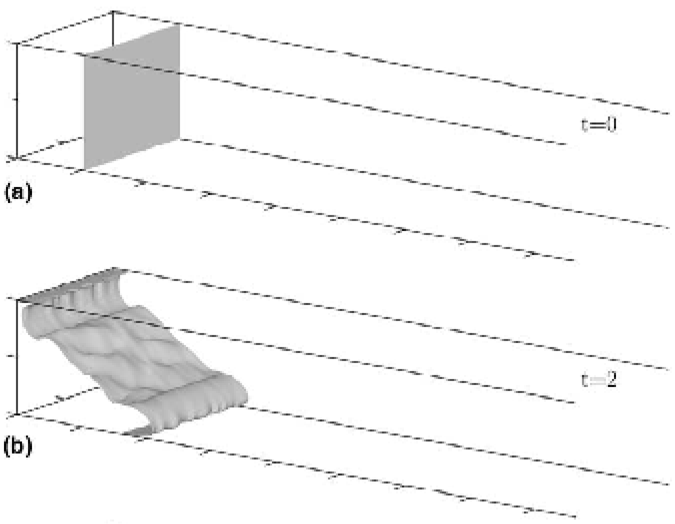
Model problem: Lock-exchange gravity current

Lock exchange configuration



Dense front propagates along bottom wall

Light front propagates along top wall



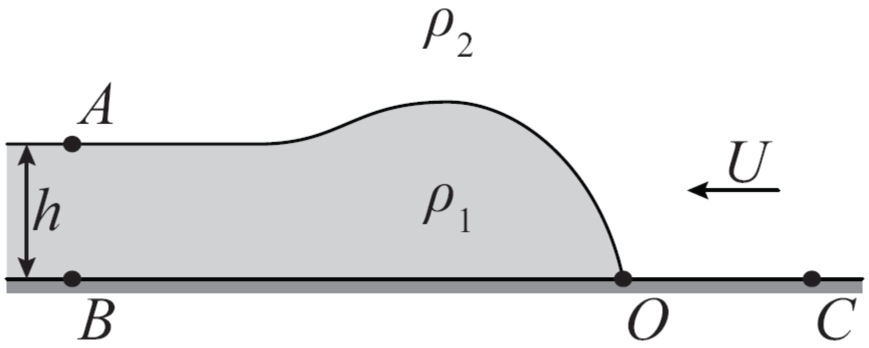
Model problem: Lock-exchange gravity current

3D DNS simulation (M. Nasr-Azadani 2012):

Can we develop simplified analytical model for predicting the front velocity?

Analytical models for gravity currents

von Karman (1940):



Goal: determine $F_h = U/\sqrt{g'h}$ where $g' = g(\rho_1 - \rho_2)/\rho_1$

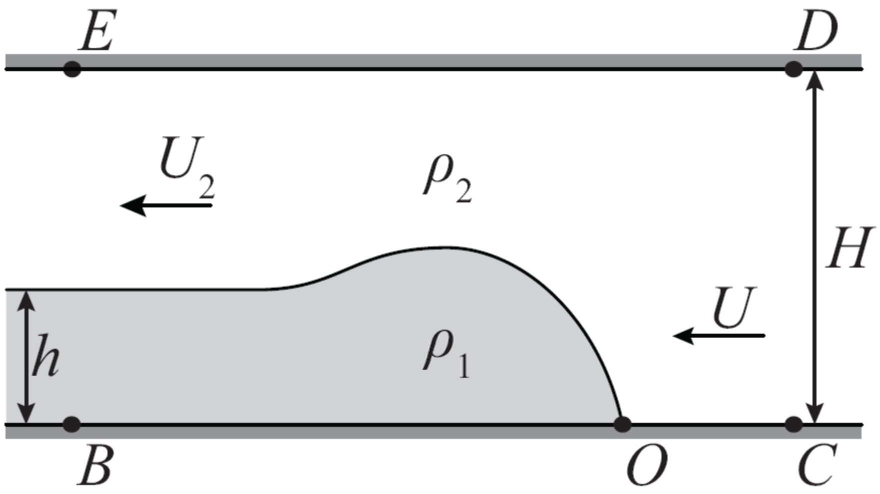
Assumptions:
- infinitely deep ambient
- apply Bernoulli along C-O and O-A

$\rightarrow \quad F_h = \sqrt{\frac{2}{\sigma}}$ where $\sigma = \rho_2/\rho_1$

Boussinesq: $\sigma \approx 1 \quad \rightarrow \quad F_h = \sqrt{2}$

Analytical models for gravity currents

Benjamin (1968): Bernoulli should not be applied along the interface, where turbulent mixing and dissipation occurs
→ alternative model:



Goal: determine U, U_2 as f(h, H, g')

- mass conservation in ambient $UH = U_2(H-h)$

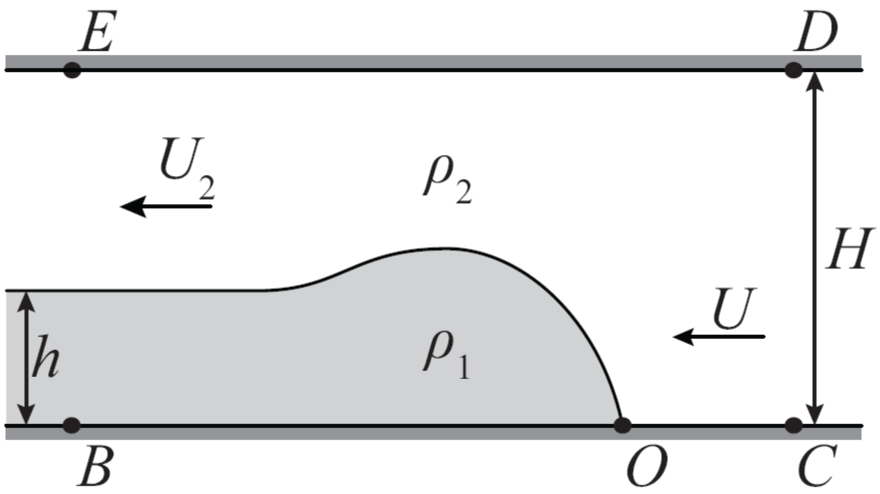
- horizontal momentum conservation

$$p_C H + \rho_2 U^2 H = p_B H + \frac{1}{2} g\left(\rho_1 - \rho_2\right) h^2 - g\left(\rho_1 - \rho_2\right) H h + \rho_2 U_2^2 \left(H - h\right)$$

But: pressure difference $p_B - p_C$ appears as additional 3rd unknown
→ closure assumption needed!

Analytical models for gravity currents

Benjamin assumes Bernoulli along C-O and O-B:



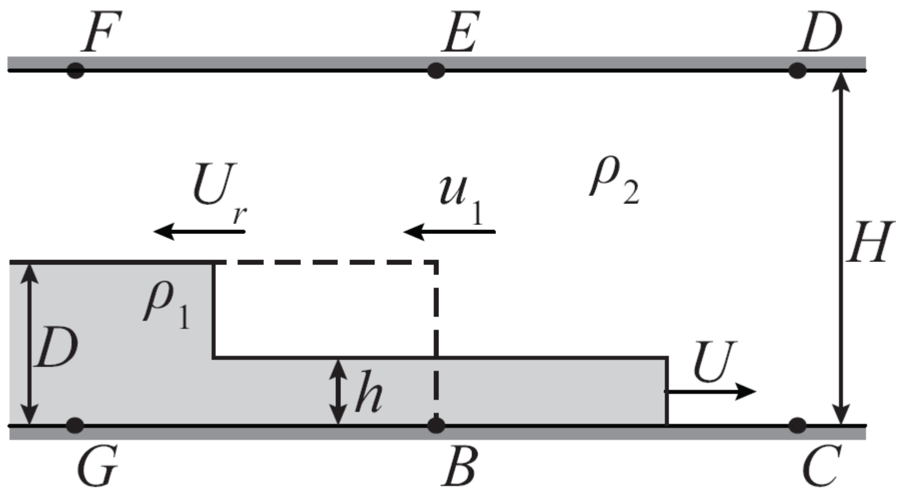
Obtains:

$$F_{H,B} = \frac{U}{\sqrt{g'H}} = \left[\frac{\alpha(1-\alpha)(2-\alpha)}{\sigma(1+\alpha)}\right]^{1/2} \quad \textit{where} \quad \alpha = h/H$$

By applying Bernoulli along D-E, Benjamin shows that an energy-conserving current requires α=1/2. Currents with α<1/2 lose energy, currents with α>1/2 require external energy input.

Analytical models for gravity currents

Shin et al. (2004) consider entire current, not just one front:



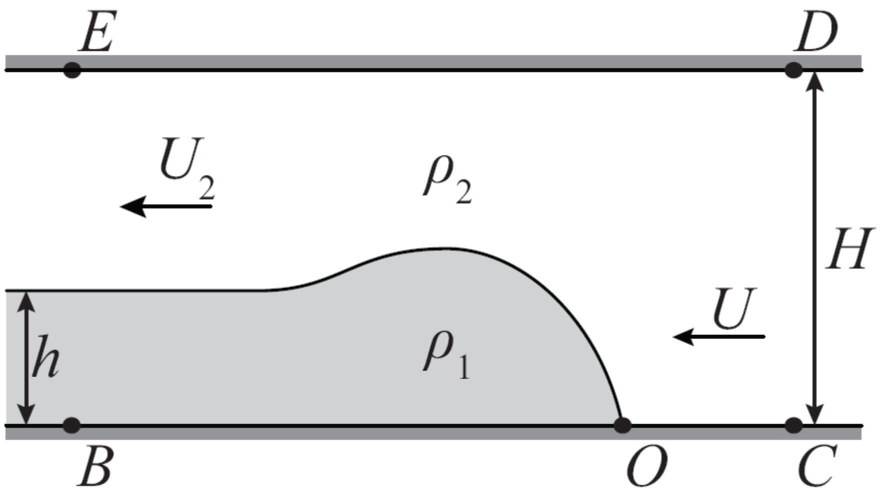
Apply Bernoulli along D-F, obtain:

$$F_{H,S} = \left\{ \frac{\beta(\beta - \alpha)(1-\alpha)}{2\alpha\left[1 - \alpha\left(1-\sigma\right)\right]} \right\}^{1/2}$$ *where* $\beta = D/H$

Above models do not employ vertical momentum eqn. As a result, they require additional energy-related closure assumption. By contrast, NS simulations reproduce gravity currents based on mass and momentum conservation only → develop new model that satisfies vertical momentum eqn., doesn't require empirical energy closure

Analytical models for gravity currents

Consider same set-up as Benjamin (1968):



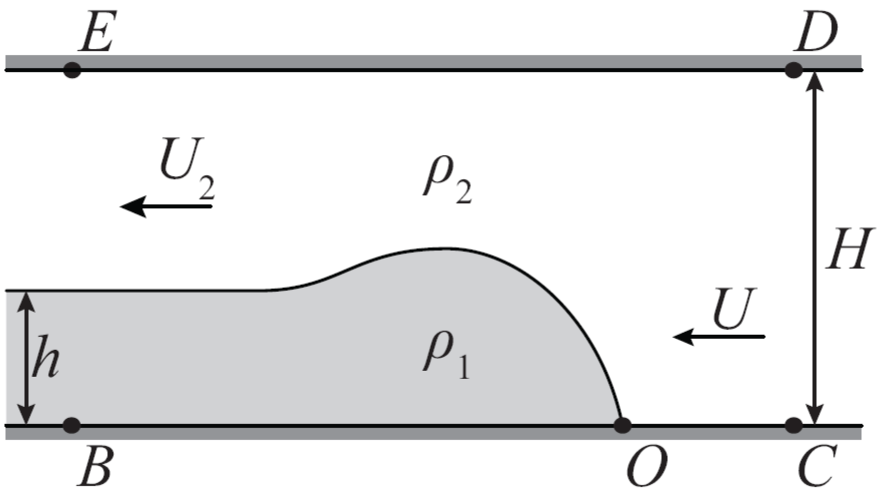
Task: determine U, U_2, Δp_{ED} as $f(h, H, g')$

Available equations:

- *mass conservation in ambient flow*
- *conservation of overall horizontal momentum*
- *conservation of overall vertical momentum*

Analytical models for gravity currents

Combine two momentum eqns to get vorticity equation (Boussinesq):



$$\oint \omega \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S = \iint g' \frac{\partial \rho^*}{\partial x} \, \mathrm{d}A + \oint \frac{1}{Re} \nabla \omega \cdot \boldsymbol{n} \, \mathrm{d}S$$

for inviscid flow:

vorticity outflow = vorticity inflow + baroclinic vorticity production

- *vorticity inflow = 0*
- *vorticity outflow =* $U_2^2/2$
- *baroclinic vorticity production = g' h*

Analytical models for gravity currents

Conservation of vorticity yields:

$$\frac{1}{2}U_2^2 = g'h$$

combine with conservation of mass in ambient stream:

$$UH = U_2(H - h)$$

→ have 2 equations for U and U_2; pressure problem is decoupled

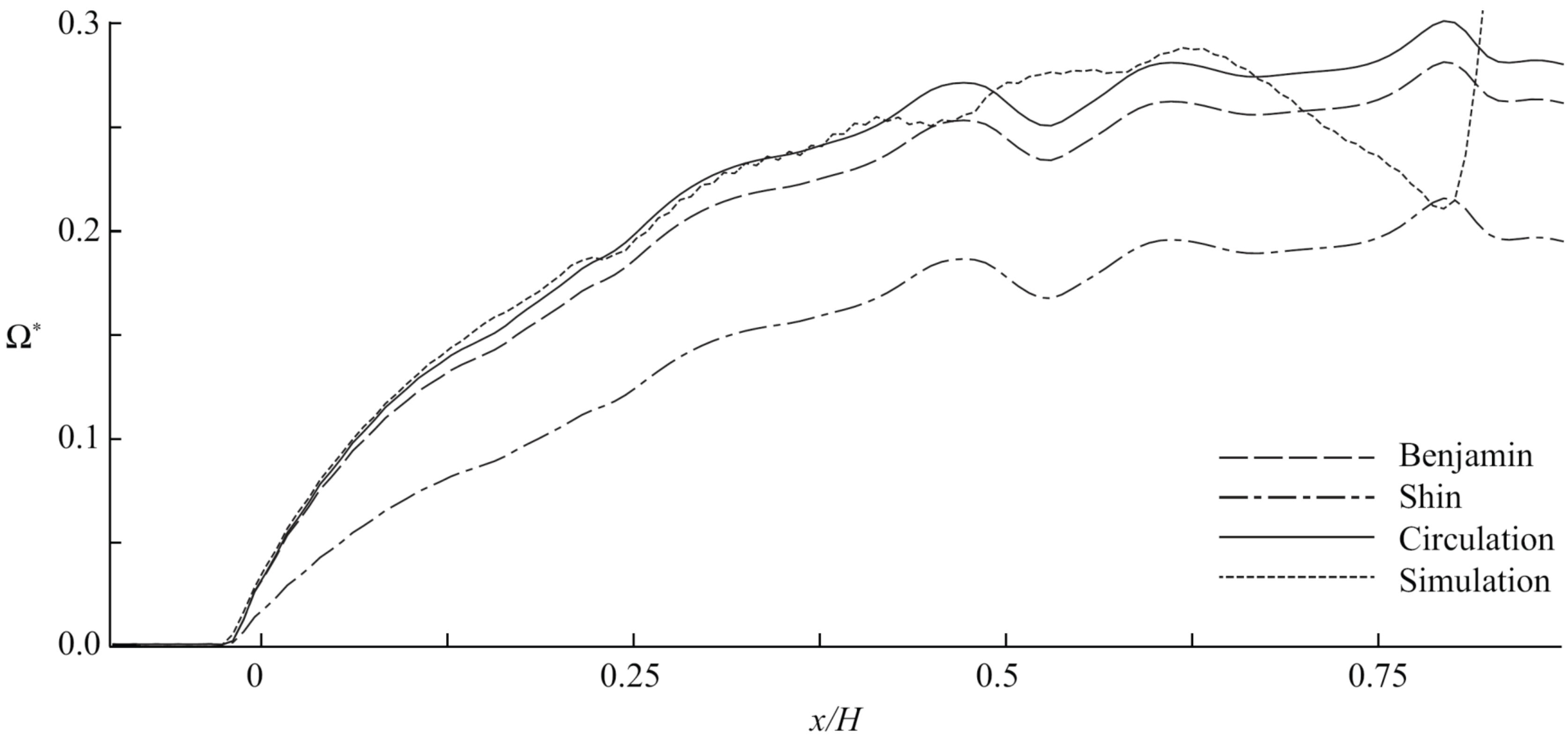
The circulation model for Boussinesq gravity currents yields:

$$F_{H,C} = \sqrt{2\alpha}(1 - \alpha)$$

Note: We used only linear combination of horizontal and vertical momentum conserv. eqns. → can still use horizontal momentum eqn. by itself to determine Δp_{ED}, but p-information not needed to get current velocity → consistent with NS simulations in (ψ,ω)-form

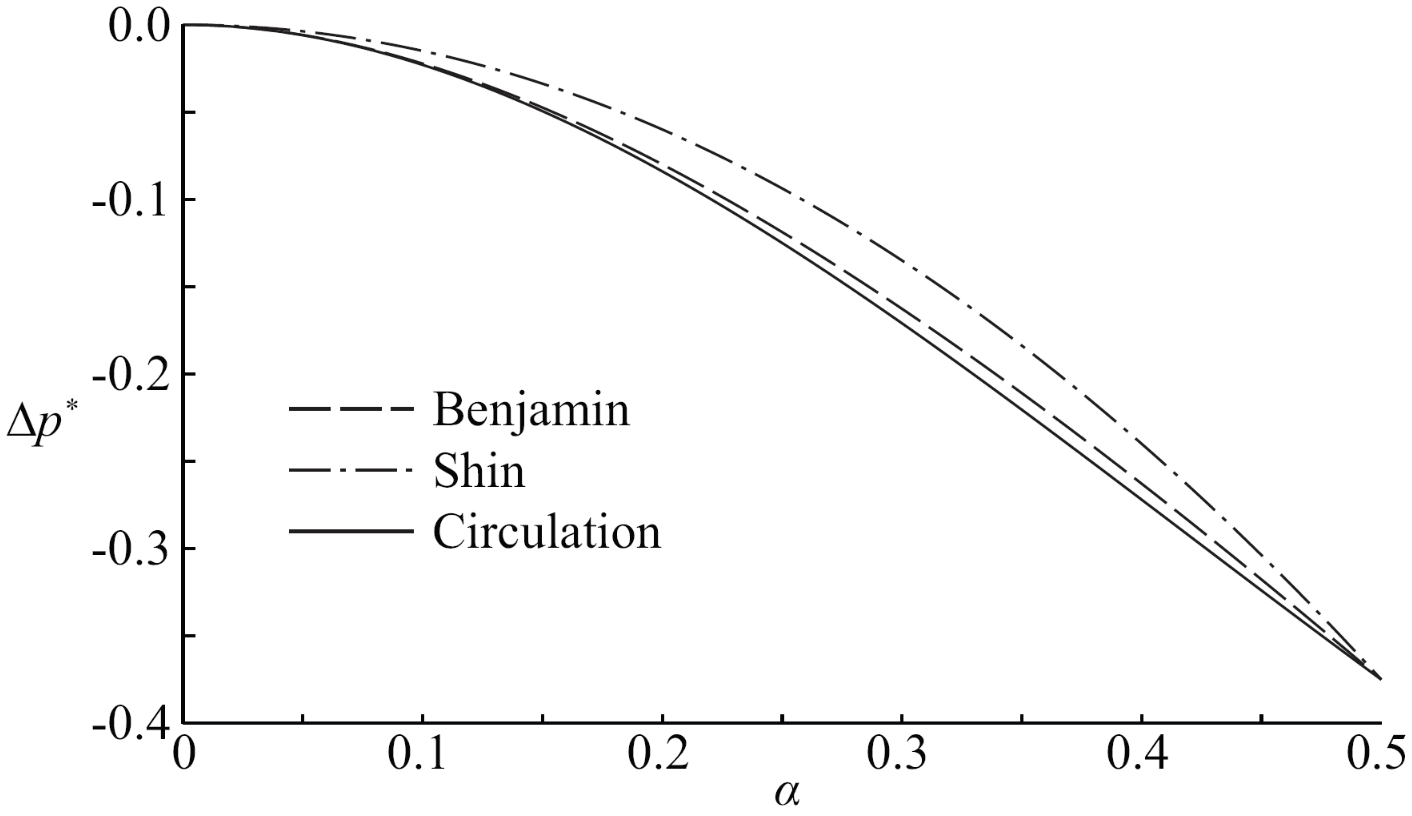
$$\Delta p^* = \frac{p_E - p_D}{\rho_1 g' H} = -\frac{1}{2}\alpha^2\left(5 - 4\alpha\right)$$

Comparison of gravity current models: vorticity flux



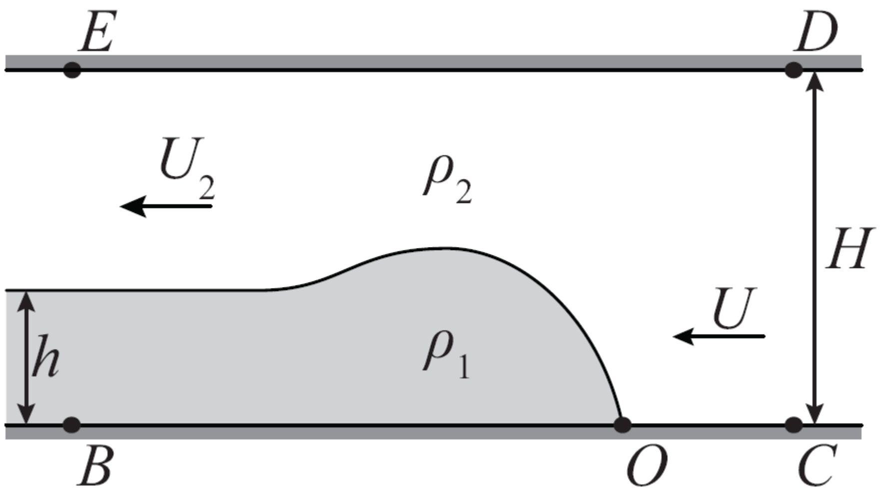
Where the flow is approximately steady-state (near the current front), the new circulation model yields the closest agreement with the DNS simulations

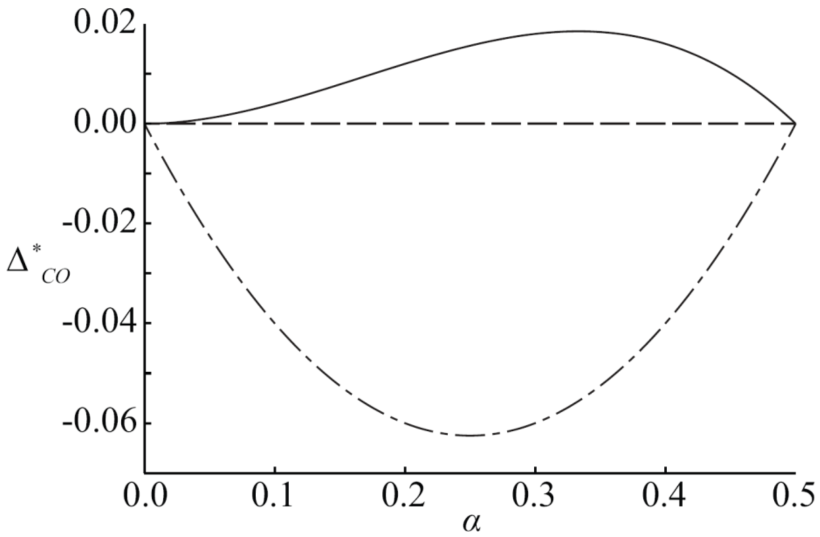
Comparison of gravity current models: Δp_{ED}

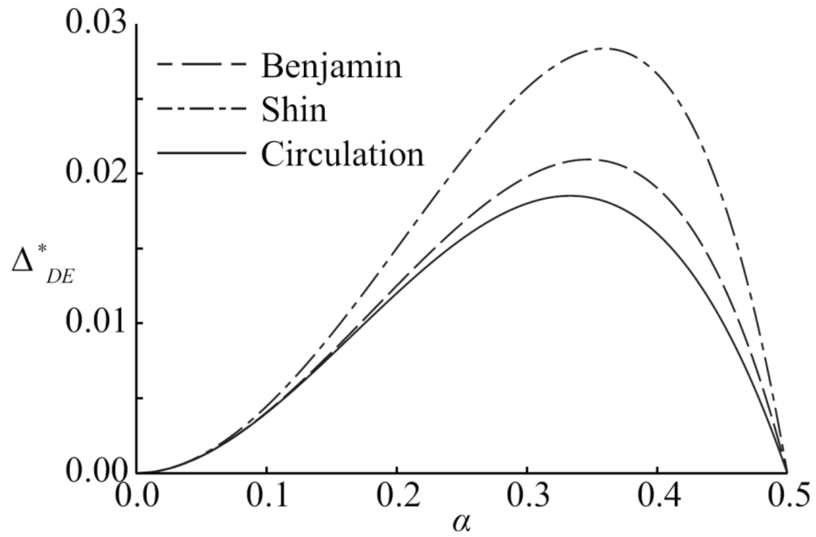


All models predict similar pressure drops across the current front

Comparison of gravity current models: head loss



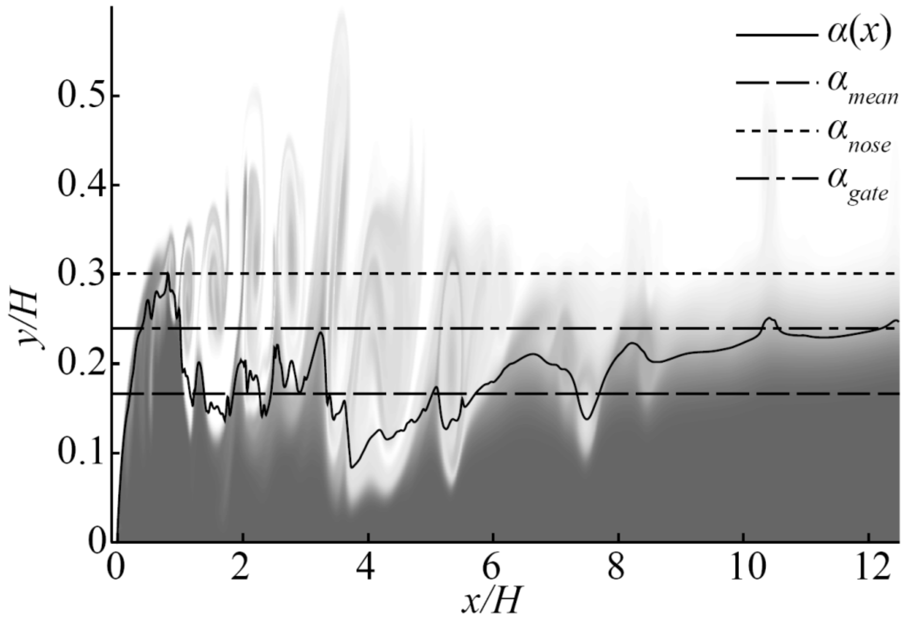


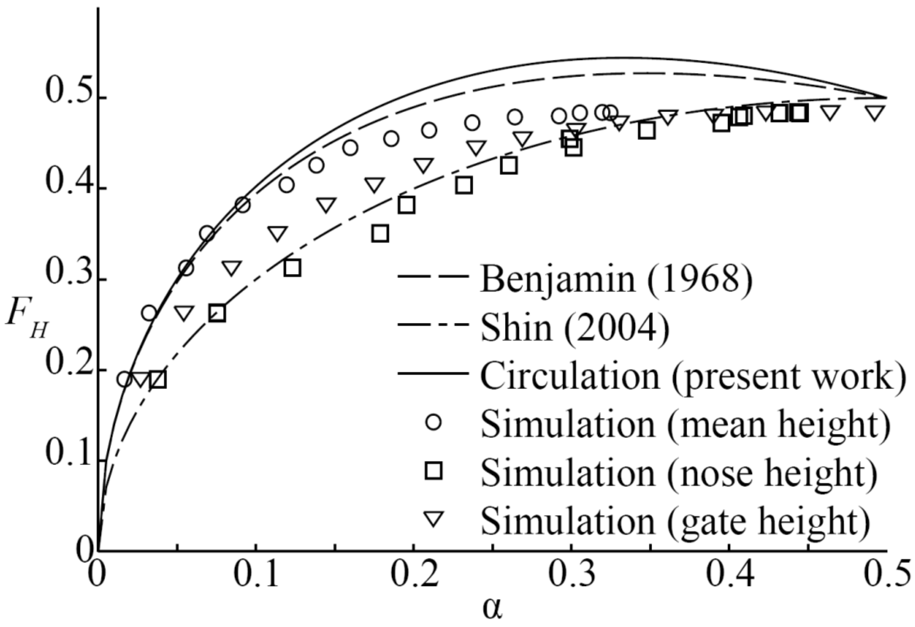


Shin et al. (2004) model predicts a head gain along C-O

Comparison of gravity current models: current velocity

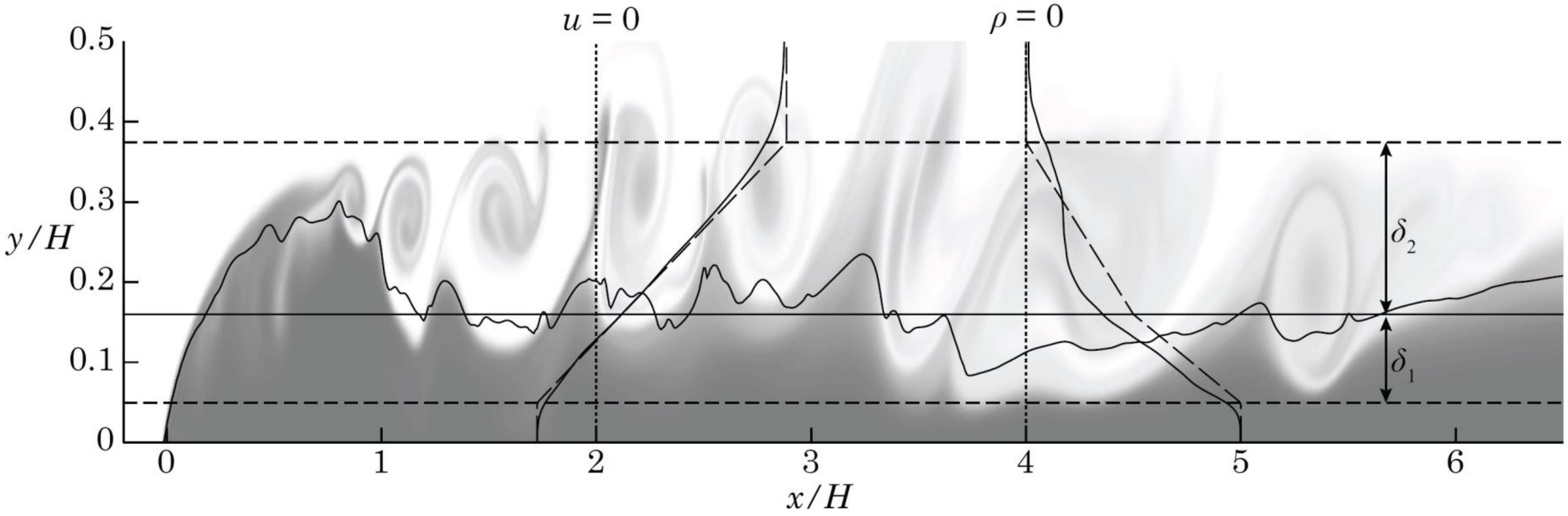
Need to determine current height:



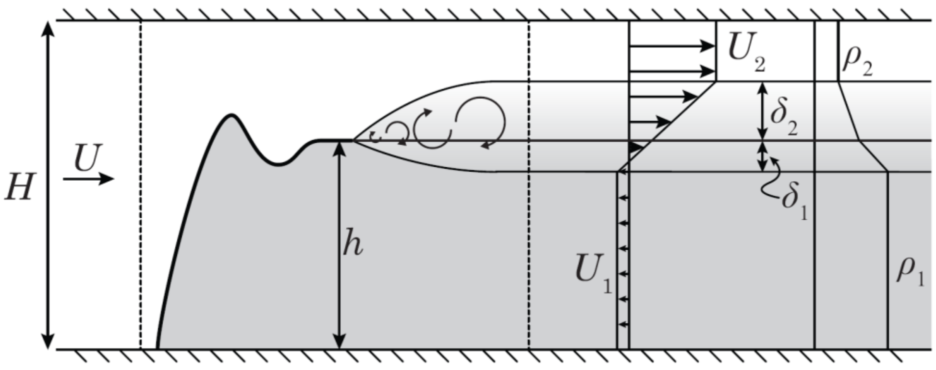


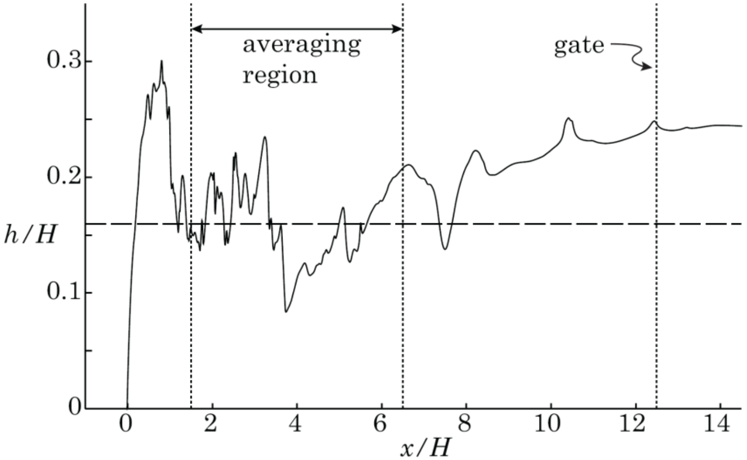
Given the uncertainty associated with determining the current velocity, no model appears to yield better agreement than others

Gravity currents: effects of turbulent mixing



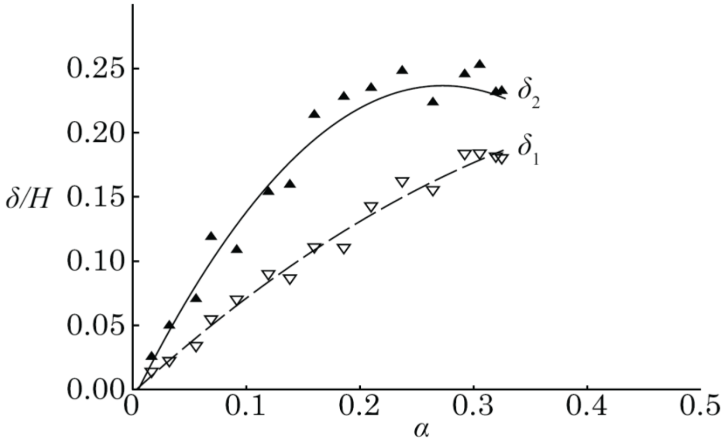
Approximate velocity and density profiles by linear functions over mixing layer of thickness δ_1 and δ_2 :



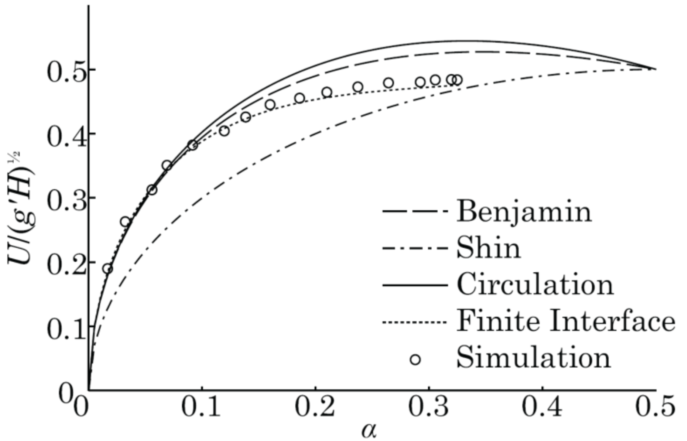


Gravity currents: effects of turbulent mixing

Determine δ_1 and δ_2 from DNS simulations:



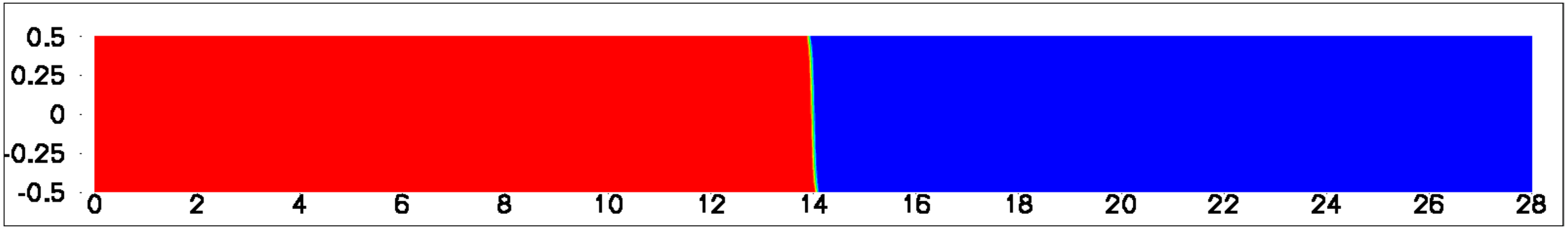
Substitute into the conservation equations for mass and vorticity:



→ obtain good agreement between model predictions and DNS data

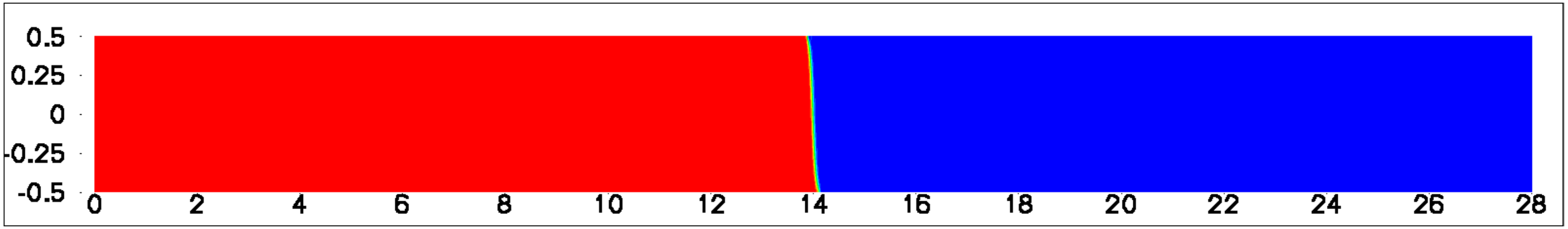
Extensions: Strong density difference (non-Boussinesq)

$\gamma = \frac{\rho_2}{\rho_1} \approx 1$:



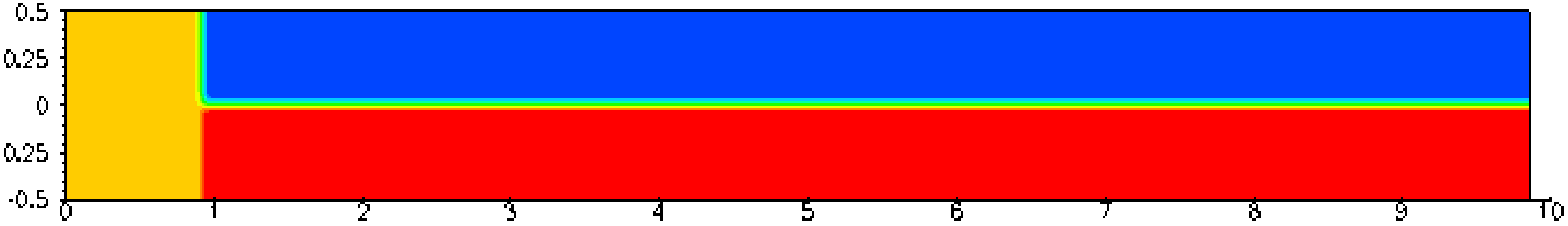
- *small density contrast (Boussinesq case): fronts are symmetric*

$\gamma = 0.4$:



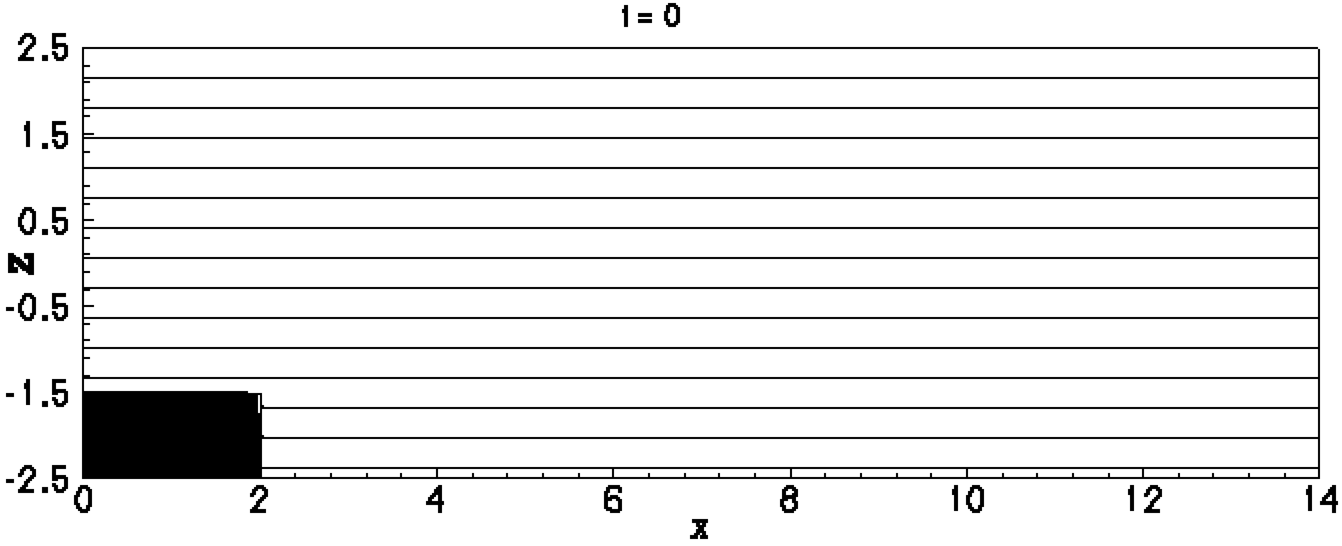
- *large density contrast (non-Boussinesq): asymmetric fronts*

Gravity currents in stratified ambients: Intrusions



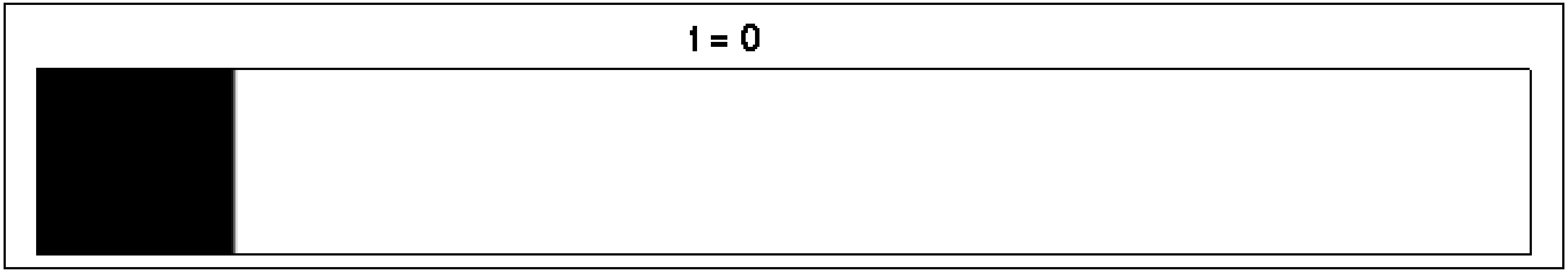
- *generation of internal waves*
- *complex interaction of the current with the stratified ambient*

Stratification: Internal wave generation



- *Excitation of internal waves in the ambient fluid*

Reversing buoyancy currents



- *propagates along bottom over finite distance, then lifts off*
- *subsequently propagates along top*

Summary

- *it is possible to develop simplified models for gravity-driven interfacial flows without invoking empirical energy arguments, by employing the vertical momentum eqn., in addition to the conservation equations for mass and horizontal momentum*
- *pressure information is not required for determining the velocity of bores and gravity currents, consistent with NS simulations based on vorticity-streamfunction formulation*
- *circulation-based models yield very close agreement with DNS simulation data regarding the vorticity flux*
- *by accounting for turbulent mixing, we can also obtain good agreement regarding the velocities of bores and gravity currents*
- *current extensions to non-Boussinesq flows and intrusions*

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- *National Science Foundation*

University of California at Santa Barbara

- *Founded 1944*
- *~ 20,000 students*
- *5 Nobel Prizes since 1997*
- *Reputation for outstanding scientific research and interdisciplinary collaboration*

Mechanical and Environmental Engineering

- *~ 500 undergrads*
- *~ 85 graduate students, 50 of them international*
- *~ 30 faculty members, 10 members of the NAE*

Research Areas

- *Computational Science and Engineering*
- *Dynamics, Control, and Robotics*
- *Fluids and Thermal Transport*
- *Microscale and Nanoscale Engineering*
- *Solid Mechanics, Materials, and Structures*