Lattice Fractional Quantum Hall States With Gapped Boundaries or Non-Abelian Defects – a microscopic investigation

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“Anyons in Quantum Many-Body Systems”
Dresden, Jan. 24, 2019
• All matters in nature are composed of bosons and fermions.

\[ R |\Psi(r_1, r_2)\rangle = |\Psi(r_2, r_1)\rangle = \pm |\Psi(r_1, r_2)\rangle \]
Anyons

- Particles constrained in 2D can have statistics interpolating between those of bosons and fermions! *Leinaa and Myrheim (1977)*

\[
|\Psi(\lambda_2)\rangle = |\Psi(\lambda_1)\rangle = |\Psi(0)\rangle
\]
\[
R^2 |\Psi(0)\rangle = |\Psi(0)\rangle
\]
\[
R^2 = 1, R = \pm 1
\]

either bosons or fermions

\[
|\Psi(\lambda_2)\rangle \neq |\Psi(\lambda_1)\rangle = |\Psi(0)\rangle
\]
\[
R^2 |\Psi(0)\rangle \neq |\Psi(0)\rangle, R^2 \neq 1
\]
\[
R = e^{i\theta} \quad \text{Abelian anyons}
\]
\[
R = \text{matrix} \quad \text{non-Abelian anyons}
\]

*Wilczek (1982)*
Anyons

- Braid non-Abelian anyons – the standard implementation of topological quantum computation.

It is challenging to realize non-Abelian anyons. Other routes to achieve and even facilitate the realization of TQC?
Non-Abelian defects (genons)

- Defects created in topologically ordered states can carry non-Abelian features, even when hosted by Abelian states.
- Enhance the GS degeneracy;
- Associated to a degenerate ground-state manifold;
- Obey projective non-Abelian braiding statistics.

Bombin (2010)

Linden, Berg, Refael, and Stern (2012)

Barkeshli, Jian, and Qi (2013)

Barkeshli and Qi (2012)
Gapped boundaries

- Gapless states on boundaries are often mentioned as a defining feature of topologically ordered states. However, we can gap out the counter propagating edge modes of nonchiral topological states.
- The GS degeneracy may be enhanced by gapped boundaries, depending on the boundary gapping condition on each boundary. Wang and Wen (2015); Hung and Wan (2015)

\[ \nu = \frac{1}{k} + \left( \frac{1}{k} \right)^* \]
\[ \text{GSD} = k^{N-1} \]
- Apply the same boundary gapping condition (tunneling or pairing) for all boundaries of an FQSH state → an FQH state on a higher-genus surface Barkeshli (2016); Ganeshan et al. (2017); Repellin et al. (2018)

- Braiding gapped boundaries + topological charge measurements + modular transformations in the mapping class group of the \( g>0 \) surface → universal TQC even though the intrinsic anyons of the underlying phase do not support it. Barkeshli and Freedman (2016); Cong, Cheng, and Wang (2017)
Our motivation

- The first step of implementing these beautiful ideas is to investigate the realizations of topological phases compatible with non-Abelian defects or gapped boundaries.

Effective field theory and exactly solvable models:

- Bombin (2010); You and Wen (2012); Lindner, Berg, Refael, and Stern (2012);
- Barkeshli and Qi (2012); Barkeshli, Jian, and Qi (2013); Barrett et al. (2013); Vaezi (2013, 2014);
- Kapustin (2014); Wang and Wen (2015); Hung and Wan (2015); Barkeshli (2016); Barkeshli and Freedman (2016);
- Ganeshan, Gorshkov, Gurarie, and Galitski (2017); Cong, Cheng, and Wang (2017); and more ...

Numerical simulations in microscopic models:

- Liu, Bergholtz, and Möller (2017); M.-S. Vaezi and A. Vaezi (2017);
- Repellin, Cook, Neupert, and Regnault (2018); Liu and Bergholtz (2019)

- Microscopic investigations are still relatively rare, but they are indispensable for guiding experiments and identifying problems obscured by the effective field theory.
This talk

- The building block: Kapit-Mueller model
  - an elegant lattice model mimicking the lowest Landau level

- Two layers of KM models with opposite chiralities
  + holes created by removing lattice sites + interlayer tunneling around holes + interactions
  - bosonic fractional quantum Hall states observed by extensive exact diagonalization

- Two layers of KM models with the same chirality
  + wormhole like branch cuts

- Outlook
The quantum Hall effect

- Cold 2DES in a strong magnetic field

Landau levels formed: exactly flat carrying Chern number $C=1$.

**Integer QHE:**
on non-interacting problem;
a result of Landau level quantization
Klitzing, Dorda, Pepper (1980)

**Fractional QHE:**
produced by strong interactions;
a notoriously difficult problem
Tsui, Stormer, Gossard (1982)
The fractional quantum Hall effect

- Single-particle states in the lowest LL:
  \[ A = \frac{B}{2}(-y, x) \]
  \[ \psi_m = z^m e^{-\frac{|z|^2}{4\ell^2}} \quad z = x + iy, \ell = \sqrt{\frac{\hbar c}{eB}} \quad m = 0, 1, 2, \ldots \]

- Ansatz wave functions in the lowest LL:
  \[ \nu = 1/3 : \Psi_{\text{Lau}} = \prod_{i<j} (z_i - z_j)^3 e^{-\sum_{i=1}^{N_e} \frac{|z_i|^2}{4\ell^2}} \quad \text{Laughlin (1983)} \]
  \[ \nu = 5/2 : \Psi_{\text{MR}} = \text{Pf} \left( \frac{1}{z_i - z_j} \right)^{N_e} \prod_{i<j} (z_i - z_j)^2 e^{-\sum_{i=1}^{N_e} \frac{|z_i|^2}{4\ell^2}} \quad \text{Moore and Read (1991)} \]

- Read-Rezayi series for \( \nu = k/(kM + 2), k \geq 1, M \geq 0 \)
  - odd \( M \) for fermions and even \( M \) for bosons
  - Laughlin and MR states as special examples
  - short-range parent Hamiltonians

\[(k+1)-\text{body contact interaction for the } \nu=k/2 \text{ bosonic RR state}\]
\[ \sum_{i_1<i_2<\cdots<i_{k+1}}^{N_b} \delta^2(z_{i_1} - z_{i_2})\delta^2(z_{i_2} - z_{i_3}) \cdots \delta^2(z_{i_k} - z_{i_{k+1}}) \]
The Kapit-Mueller model

- Modified Hofstadter model:
  - square lattice, flux quanta $\phi$ piercing each plaquette
  - infinite-range, but exponentially decayed hopping

$$t_{jk} = t_0 (-1)^{x+y+xy} e^{-\frac{\pi}{2} (1-|\phi|) (|x|^2 + |y|^2)} e^{-i\pi \phi (x_j + x_k) y}$$

$$x = x_j - x_k, \ y = y_j - y_k$$

Hofstadter (NN hopping)

Kapit-Mueller

The lowest band is exactly flat at any $\phi$. 

The Kapit-Mueller model

- The lowest band somehow mimics the LLL:
  - $\mathcal{C} = 1$, energetically exactly flat
    \[
    \text{Berry curvature still varying!} \quad \phi = 1/2
    \]
  - spanned by discretized lowest LL wave functions
    \[
    \psi_n(z_j) = z_j^n \exp \left( -\frac{\pi|\phi|}{2} |z_j|^2 \right)
    \]
    orthogonality lost on the lattice!
  - $(k+1)$-body onsite repulsion $\sum n_i(n_i - 1) \cdots (n_i - k)$
    gives discretized $\nu = k/2$ bosonic RR states as the ground states.
    excitations changed!

\[
\begin{align*}
k = 1 : & \Psi_{Lau} = \prod_{i<j}(z_i - z_j)^2 e^{-\frac{\pi|\phi|}{2} \sum_i |z_i|^2} \\
k = 2 : & \Psi_{MR} = \text{Pf} \left( \frac{1}{z_i - z_j} \right) \prod_{i<j}(z_i - z_j) e^{-\frac{\pi|\phi|}{2} \sum_i |z_i|^2}
\end{align*}
\]

E. Kapit and E. Mueller, PRL 105, 215303 (2010)
KM model + gapped boundaries

- Punch $M$ pairs of holes through two layers of KM model with opposite chiralities.
  - An extreme limit: each hole only contains a single removed lattice site, such that its edge contains eight sites.
  - Couple the two edges of each pair of holes with vertical interlayer tunneling, which gives an effective genus $g = M + 1$ surface.

$$H_0 = \sum_{j,k \notin R} \sum_{\sigma = \uparrow, \downarrow} t_{jk}^\sigma a_{jk}^\sigma a_{k \sigma} + \sum_{m=1}^{M} \sum_{e \in E_m} \left( t_e^\uparrow a_{e \uparrow}^\dagger a_{e \downarrow} + h.c. \right)$$

$$t_{jk}^\sigma = t_0 (-1)^{x_j + y_j + x_k} e^{-\frac{\pi}{2} (1-|\phi_{\sigma}|)(x_j^2 + y_j^2)} e^{i \pi \phi_{\sigma} (x_j + x_k) y_j}, \quad \phi_{\uparrow} = -\phi_{\downarrow} = \phi > 0$$

$M = 1 \quad g = M + 1 = 2$
Single-particle physics

- In the absence of holes: two decoupled KM models, the lowest band of each contains $\phi L_xL_y$ eigenstates $\rightarrow$ the lowest $2\phi L_xL_y$ eigenstates of $H_0$ are exactly degenerate.

$$\phi = 1/q, q > 2$$

- How do the holes change the band structure?

Single-particle physics

- $M$ pairs of holes distort the exactly flat lowest band: without interlayer tunneling, $2M$ states move into the band gap, but other $2\phi L_xL_y-2M$ states stay at original energies.

- The $2M$ ingap states are edge states – remnants of $M$ pairs of counter propagating continuum edge modes (not visible in the minimal hole limit).

$\epsilon_n = L_x \times L_y = 9 \times 9$, $\phi = 1/3$
Single-particle physics

- Interlayer tunneling splits the ingap edge states, such that a band gap is reopened (boundaries are gapped out). But the band structure can be further distorted.

- To make interactions dominant, can we design suitable interlayer tunneling to restore a nearly flat lowest band?

  - Two tunneling strengths for the eight edge sites of each hole.

  - Suitable tunneling phases to mimic a magnetic field consistent with that in each layer: each vertical plaquette between a pair of holes pierced inwardly by effective flux $\phi$. 

With suitable tunneling strength, our scheme of interlayer tunneling can indeed restore a flat lowest band containing $2\phi L_x L_y - M$ eigenstates of $H_0$: a higher-genus flat band.

With the decreasing of flux density $\phi$, we can get a flatter lowest band with weaker tunneling strength.

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\tilde{t}_1/t_0$</th>
<th>$\tilde{t}_2/t_0$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>0.52</td>
<td>0.36</td>
<td>4.40</td>
</tr>
<tr>
<td>1/4</td>
<td>0.42</td>
<td>0.24</td>
<td>5.81</td>
</tr>
<tr>
<td>1/5</td>
<td>0.36</td>
<td>0.19</td>
<td>6.94</td>
</tr>
<tr>
<td>1/6</td>
<td>0.33</td>
<td>0.15</td>
<td>7.96</td>
</tr>
</tbody>
</table>
The potential FQH states

- What topological states can we stabilize in this new flat band? Due to the relevance with the cold-atom implementation, we focus on the possibility of the \( \nu = k/2 \) bosonic RR state on a single \( g = M+1 \) surface.

- In the continuum, the \( \nu = k/2 \) RR state of \( N_b \) bosons on a genus-\( g \) surface resides in \( N_s = 2N_b/k-(1-g) \) exactly degenerate single-particle states in the lowest Landau level.

\[ N_s = 2\phi L_x L_y - M, \quad g = M + 1 \rightarrow N_b = k(\phi L_x L_y - M) \]

- The correct system size in our lattice model:

\[ H_{\text{int}} = U \sum_{\sigma = \uparrow, \downarrow} \sum_{i \notin \mathcal{R}} : n_{i,\sigma} n_{i,\sigma} \cdots : \]

---

Wen and Zee (1992)

Topological degeneracy

- We use exact diagonalization to identify the nature of the ground state.

- Can we observe ground-state topological degeneracies consistent with the $\nu = k/2$ RR state?

  Ardonne, Bergholtz, Kailasvuori, and Wikberg (2008)

<table>
<thead>
<tr>
<th>State</th>
<th>GS degeneracy</th>
<th>$g=2$  ((M=1))</th>
<th>$g=3$  ((M=2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=1$ Laughlin</td>
<td>$2^g$</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>$k=2$ MR</td>
<td>$2^{g-1}(2^g + 1)$</td>
<td>10</td>
<td>36</td>
</tr>
<tr>
<td>$k=3$ Z$_3$ RR</td>
<td>$2[(5 + \sqrt{5})^{g-1} + (5 - \sqrt{5})^{g-1}]$</td>
<td>20</td>
<td>120</td>
</tr>
</tbody>
</table>

- For large numerical efficiency, we project the interaction to the restored flat band, and neglect its dispersion.
Nice (approximate) ground-state degeneracies exist!
For a fixed $\phi$, the ground-state splitting is reduced relative to the gap as the system size is increased.
**$k=2$: Moore-Read state?**

<table>
<thead>
<tr>
<th>State</th>
<th>GS degeneracy</th>
<th>$g=2$ ($M=1$)</th>
<th>$g=3$ ($M=2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=2$ MR</td>
<td>$2^{g-1}(2^g + 1)$</td>
<td>10</td>
<td>36</td>
</tr>
</tbody>
</table>

- $N_b = 8, L_x \times L_y = 3 \times 5, \phi = 1/3$
- $N_b = 6, L_x \times L_y = 4 \times 4, \phi = 1/4$
- $N_b = 8, L_x \times L_y = 4 \times 5, \phi = 1/4$
- $N_b = 10, L_x \times L_y = 4 \times 6, \phi = 1/4$

$D = 10$

- $M = 1$

$D = 36$

- $M = 2$

---

\(k=3: \mathbb{Z}_3\) Read-Rezayi state?

<table>
<thead>
<tr>
<th>State</th>
<th>GS degeneracy</th>
<th>(g=2) (\text{ (}M=1\text{) )})</th>
<th>(g=3) (\text{ (}M=2\text{) )})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k=3) (\mathbb{Z}_3) RR</td>
<td>(2[(5 + \sqrt{5})^{g-1} + (5 - \sqrt{5})^{g-1}])</td>
<td>20</td>
<td>120</td>
</tr>
</tbody>
</table>

- Compared with the Laughlin case, non-Abelian states require multibody interactions, and lower flux densities.
Evidence beyond degeneracy?

- The modular $S$ matrix contains the information of anyonic statistics of underlying quasiparticles.

- For the simplest case of Abelian states on a $g=2$ surface, the direct product of two $S$ matrices gives the transformation between two special bases of the ground-state manifold.

$$|a'c'b'\rangle_{\beta_1\gamma\beta_2} = \sum_{a,b} S_{aa'} S_{bb'} |acb\rangle_{\alpha_1\gamma\alpha_2}$$

$|acb\rangle_{\alpha_1\gamma\alpha_2}$
- quasiparticles $a$, $c$, and $b$ threading the nonintersecting, noncontractible circles $\alpha_1$, $\gamma$, and $\alpha_2$, respectively.
- $c$ must be identity for Abelian states.

Quasiparticle statistics

- The basis states are minimally entangled states with respect to a specific bipartition of the whole system.

\[ |a \uparrow b \rangle_{\alpha_1 \gamma \alpha_2} \]

\[ |a \uparrow b \rangle_{\beta_1 \gamma \beta_2} \]

Zhang, Grover, Turner, Oshikawa, and Vishwanath (2012)

Cuts go through both layers. A and B are bilayer subsystems.

- We must diagonalize the full Hamiltonian in real space.
  - limited to smaller systems compared with band projection
  - hardcore condition imposed to increase numerical efficiency

Quasiparticle statistics

- We focus on the $\frac{1}{2}$ filling with $M=1$ pair of holes (i.e., $g=2$). We do observe four-fold degeneracies by real-space ED!

- We minimize the Renyi-2 entropy $S_2 = -\ln \text{Tr} \rho_A^2$ in this ground-state subspace for two cuts. For each cut, we indeed find four (almost orthogonal) minimally entangled states with similar $S_2$.

<table>
<thead>
<tr>
<th>Cut</th>
<th>$S_2$ Values</th>
<th>$S_2$ Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>cut I</td>
<td>$S_2 = 1.37908, 1.36319, 1.36319, 1.37908$</td>
<td>$S_2 = 1.76580, 1.71694, 1.71694, 1.76580$</td>
</tr>
<tr>
<td>cut II</td>
<td>$S_2 = 2.86280, 2.82103, 2.91412, 2.86280$</td>
<td>$S_2 = 3.12519, 3.27780, 3.27780, 3.42259$</td>
</tr>
<tr>
<td>Cut between two layers</td>
<td>$S_2 = 0.357869, 0.357887, 0.530498, 0.536709$</td>
<td>$S_2 = 0.327539, 0.327342, 0.350278, 0.356425$</td>
</tr>
</tbody>
</table>
Quasiparticle statistics

- What is the overlap matrix between MESs? \( O_{mn} = \langle \Sigma^I_m | \Sigma^II_n \rangle \)

\[
O \approx \begin{pmatrix}
0.523 & 0.525 & 0.517 & 0.523 \\
0.477 & -0.472 & 0.483 & -0.477 \\
0.477 & 0.472 & -0.483 & -0.477 \\
0.523 & -0.525 & -0.517 & 0.523 \\
\end{pmatrix}
\]

\[
O \approx \begin{pmatrix}
0.493 & 0.494 & 0.494 & 0.496 \\
0.507 & -0.505 & 0.505 & -0.503 \\
0.507 & 0.505 & -0.505 & -0.503 \\
0.493 & -0.494 & -0.494 & 0.496 \\
\end{pmatrix}
\]

\[
|a'c b'\rangle_{\beta_1 \gamma_2} = \sum_{a,b} S_{aa'} S_{bb'} |acb\rangle_{\alpha_1 \gamma_2}
\]

- \( O \) is very close to the direct product of two modular \( S \) matrices of the Laughlin state!

\[
S \otimes S = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 \\
\end{pmatrix}
\]

- Together with the nonzero interlayer entropy, we confirm that the ground state is the Laughlin state on a single \( g=2 \) surface.
KM model + gapped boundaries

- The key message from our results: the idea of gapped boundaries works even in the most extreme lattice limit with minimal holes + relatively high flux densities, even though it would appear doubtful that the insights from low-energy field theory would apply in this limit even qualitatively.

- This limit is the most attractive regime from a practical point of view since the involved energy scales are much larger than in the dilute limit. Our results are thus encouraging in the context of experimental realizations.
KM model + defects

- Introduce defects in two layers of KM model with the same chirality.
  - A pair of defects is connected by a straight branch cut. Hopping across a branch cut is switched from intralayer to interlayer.
  - With wormhole-like branch cuts, we again have an effective high-genus surface.

\[ g = M + 1 \]

\[ g = 1 + 1 \]  \[ g = 2 \]  \[ g = 3 \]
What we find for defects

- Single-particle states localized at defects exist in the band gap.

- The lowest flat band can be restored by a local potential around defects.

\[
V = - \sum_{n=1}^{2\phi L_x L_y + M} \epsilon_n T_R(\langle \psi_n | \psi_n \rangle)
\]

- Switching on interactions in this new flat band also gives bosonic RR states.

Relevance to experiments

Good news

– Long-range hopping is NOT necessary! GS degeneracies still exist for the Hofstadter model (but with larger finite-size effects).

– Key ingredients available in experiments:
  the Hofstadter model;
  lattice shaking / pairs of beams → bilayer;
  beam shaping → holes, branch cuts → high-genus surface

Challenges

– More realistic schemes to restore the lowest flat band, which may be very important for making interactions dominant.

– Multibody interactions needed for non-Abelian states.

– Realistic planar geometry works?

Kim, Zhu, Porto, and Hafezi (2018)
Outlook

- More complicated states if we use a higher Chern number model as the building block?
- Microscopic lattice models of dislocations, pairing?
- Microscopic investigation of anyons in lattice FQH systems: the quasiparticle tunneling, the interplay between intrinsic anyons and defects/gapped boundaries, ...

Microscopic characterization of Abelian quasiholes on lattices: density profile, quasihole size, braiding, effective lattice magnetic length:

$$\ell_{B}^{\text{lat}} = \sqrt{A/(2\pi)}$$

Zhao Liu, R. N. Bhatt, and Nicolas Regnault (2015)
Błażej Jaworowski, Nicolas Regnault, and Zhao Liu (2019)
Thank you!
Welcome to visit Hangzhou in the future!

a sister city of Dresden