Content

1. Scar states and ETH violation
2. Non-topological scar state in 1D
3. General construction
4. Topological scars in 1D, 2D, 3D

Collaboration

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Scars

**Rydberg atom experiments:** Emergent oscillations in many-body dynamics after sudden quench

Theoretical explanation: special “scar” states in the middle of the spectrum

[Lukin group, Nature 551, 579-584 (2017)]

[Turner et. al, Nature Physics (2018)]
Scar characteristics

Non-integrable model

Low (sub-volume law) entanglement entropy states

No disorder (distinct from many-body localization)

“Few” scar states distributed over spectrum

[Turner et. al, Nature Physics (2018)]
Analytical scars

Infinite series of low-entropy states in AKLT chain

MPS based construction

[S. Moudgalya, et al., PRB 2018, 2X]
Eigenstate thermalization hypothesis

**Classical thermalization:** equivalence of time average and ensemble average

**Quantum thermalization:**
- Assume system behaves thermally
- Unitary evolution cannot construct thermal state
- Thermal behavior in quantum systems occurs on the level of individual eigenstates
Eigenstate thermalization hypothesis

... and how it fails in many-body quantum systems

ETH obeying generic non-integrable system

nearby states have same expectation values of any observable

system with scar states

many-body localized

strong ETH

weak ETH

no ETH
What we do

Recipe for analytical construction of scar states

In systems of any dimension (1D, 2D, 3D)

In systems with topological character (SPT/topological order), inherited by the scar states

Area law entanglement scar states
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Simple (non-topological) example

1D spin-1/2 chain

\[ H(\beta) := \sum_i \alpha_i Q_i(\beta), \]

\[ \alpha_i := \alpha + (-1)^i, \quad Q_i(\beta) := e^{-\beta (Z_{i-1} Z_i + Z_i Z_{i+1})} - X_i \]

Transverse field Ising model
+ NNN coupling
+ sign-alternating coupling
Simple (non-topological) example

Proof of non-integrability: energy level statistics

\[ s_n := E_{n+1} - E_n \]
\[ r_n := \frac{\min(s_n, s_{n-1})}{\max(s_n, s_{n-1})} \]

in one momentum and inversion symmetry sector

\[ \langle r \rangle = 0.531 \]
\[ r_{\text{GOE}} = 0.5359 \]
\[ r_{\text{Poisson}} = 0.3863 \]
Simple (non-topological) example

Why the scar state?

Consider alternative Hamiltonian

\[ \tilde{H}(\beta) := \sum_i Q_i(\beta) \]

\[ Q_i(\beta) := e^{-\beta(Z_{i-1}Z_i + Z_iZ_{i+1})} - X_i \]

Each term is positive semidefinite

\[ Q_i(\beta)^2 = 2 \cosh (\beta (Z_{i-1}Z_i + Z_iZ_{i+1})) \cdot Q_i(\beta) \]

State annihilated by all terms

= ground state

area law entanglement

Exact eigenstate of

in the middle of the spectrum
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Recipe

Operators constituting a commuting projector model

\[ A_s \quad \quad A_s^2 = 1 \]

\[ [A_s, A_{s'}] = 0 \]

Extra set of operators we will use to deform the model

\[ M = \sum_i O_i \quad \quad M_s = \sum_{i \in s} O_i \]
\[ [O_i, O_j] = 0 \quad \quad \tilde{M}_s = \sum_{i \notin s} O_i \]

Such that

\[ \{M_s, A_s\} = 0 \]
\[ [\tilde{M}_s, A_s] = 0 \]
Recipe

Deformation:

\[ F_s = e^{\beta/2} M (1 - A_s) e^{-\beta/2} M \]

\[ = 1 - e^{\beta M_s} A_s \]

\[ = e^{\beta M_s} (e^{-\beta M_s} - A_s) =: e^{\beta M_s} Q_s \]

\[ Q_s = Q_s^\dagger \quad \text{is local operator!} \]

Positive semidefinite

\[ Q_s^2 = (e^{-\beta M_s} - A_s)(e^{-\beta M_s} - A_s) \]

\[ = e^{-2\beta M_s} - (e^{-\beta M_s} + e^{-\beta M_s}) A_s + 1 \]

\[ = 2 \cosh(\beta M_s) Q_s \]
Recipe

Ground state of commuting projector model

\[ |\Psi_0\rangle = \prod_{s'} (1 + A_{s'}) |\Omega\rangle \]

\[(1 - A_s) |\Psi_0\rangle = (1 - A_s) \prod_{s'} (1 + A_{s'}) |\Omega\rangle \]

\[= (1 - A_s)(1 + A_s) \prod_{s' \neq s} (1 + A_{s'}) |\Omega\rangle \]

\[= 0. \]

Deformed state is annihilated by all \(Q_s\)

\[|\Psi_\beta\rangle = e^{-\beta / 2 M} |\Psi_0\rangle \]

\[F_s |\Psi_\beta\rangle = e^{-\beta / 2 M} (1 - A_s) e^{\beta / 2 M} e^{-\beta / 2 M} \prod_{s'} (1 + A_{s'}) |\Omega\rangle \]

\[= 0 \]

\[\Rightarrow Q_s |\Psi_\beta\rangle = 0 \]
\[ |\Psi_\beta\rangle = e^{-\beta/2M} |\Psi_0\rangle \]

is ground state of \( H = \sum_s Q_s \)

And excited eigenstate of \( H' = \sum_s \alpha_s Q_s \) if coefficients of different sign

No need for deformed Hamiltonians to be integrable \([Q_s, Q_{s'}] \neq 0\) since \( O_i \) may belong to different \( s \)
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SPT example

Nontrivial gapped paramagnet

\[ \mathcal{H}_{\text{SPT}} = \sum_i Z_{i-1} X_i Z_{i+1} \]

spin-1/2 chain

Pauli algebra of chain end operators

- commute with Hamiltonian
- double degeneracy of each state (including the ground state)
- anticommute with protecting symmetries (cannot be added as perturbations)

\[ \prod_{i \in \text{even}} X_i \prod_{i \in \text{odd}} X_i \]

\[ Z_1, \quad X_1 Z_2, \quad Y_1 Z_2 \]

or

\[ T = K \prod_i X_i \]

\[ \mathbb{Z}_2 \times \mathbb{Z}_2 \]

\[ \mathbb{Z}_2^T \]

Protect topological degeneracy at each end, stable to symmetry-preserving deformations (unitary spectral evolution)
SPT example

Deform SPT paramagnet

\[ H = H_1 + H_2 \]

\[ H_{\text{SPT}} = \sum_i Z_{i-1}X_i Z_{i+1} \]

\[ H_a = \sum_{i \in \text{SL}_a} \alpha_{a,i} Q_{a,i} \]

\[ \alpha_{a,i} = \alpha + (-1)^{(i-a)/2} \]

\[ Q_{a,i} = e^{-\beta_a (X_{i-1} + X_{i+1})} - Z_{i-1}X_i Z_{i+1} \]

Has scar state:

\[ |\text{scar}\rangle = \exp \left( \frac{\beta_1}{2} \sum_{i \in \text{SL}_1} X_{i-1} \right) \exp \left( \frac{\beta_2}{2} \sum_{i \in \text{SL}_2} X_{i-1} \right) |+, \cdots, +\rangle \]

But: integral of motion \( H_1 \)
SPT example

\[ H_a = \sum_{i \in SL_a} \alpha_{a,i} Q_{a,i} \]

Unitary transformation:
\[ Q_{a,i} = e^{-\beta_a (X_{i-1} + X_{i+1})} - Z_{i-1} X_i Z_{i+1} \]

\[ W := \exp \left( i \frac{\pi}{4} \sum_{j \in SL_1} Z_j Z_{j+1} - i \frac{\pi}{4} \sum_{j \in SL_2} Z_j Z_{j+1} \right) \]

\[ \tilde{Q}_{a,i} = e^{-\beta_a (Z_{i-2} X_{i-1} Z_i + Z_i X_{i+1} Z_{i+2})} - X_i \]

For diagonalization of \( H_1 \) alone, fix X state of every other spin (integral of motion)
\[ Q_i(\beta) := e^{-\beta (Z_{i-1} Z_i + Z_i Z_{i+1})} - X_i \]

Same as trivial scar Hamiltonian:
- non-integrable (as argued before)
- scar state is not ground state of integral of motion
Toric code example

\[ H = H_1 + H_2 \]

\[ H_1 = - \sum_s A_s \]

\[ H_2 = - \sum_p B_p \]

\[ B_p = \prod_{i \in p} Z_i \]

\[ A_s = \prod_{i \in s} X_i \]

4-fold degenerate ground state due to Wilson loops (and every excited state)

\[ H_1 = \sum_s \left[ \exp \left( -\beta_1 \sum_{i \in s \cap P_1} Z_i \right) - A_s \right] \]

\[ H_2 = \sum_p \left[ \exp \left( -\beta_2 \sum_{i \in p \cap P_2} X_i \right) - B_p \right] \]
Toric code example

\[ H = H_1 + H_2 \]

\[ H_1 = \sum_s \alpha_s \left[ \exp \left( -\beta_1 \sum_{i \in s \cap \mathcal{P}_1} Z_i \right) - A_s \right] \]

\[ H_2 = \sum_p \alpha_p \left[ \exp \left( -\beta_2 \sum_{i \in p \cap \mathcal{P}_2} X_i \right) - B_p \right] \]

Choice of paths:
- non-intersecting
- cover all sites
- connected

Guarantees no extra integrals of motion

Consider only \( H_1 \): Z eigenvalues for spins not on path are integrals of motion.

In each sector, \( H_1 \) reduces to 1D scar paramagnet

Non-integrable, supports scar state in the middle of spectrum

+ 4-fold “topological” degeneracy of scar state
Let's go 3D
Fracton topological order

Topological order beyond the gauge theory paradigm

- 3D (or higher D)
- (partially) immobile excitations
- **exponential in L** (ground) state degeneracy: **great topological quantum memory**

Motivation: T-stable topological order (did not work)

**First model**

[C. Castelnovo, C. Chamon, and D. Sherrington, PRB 81, 184303 (2010)]

**Haah code**

[J. Haah, PRA 83, 042330 (2011)]
X-cube model

spin 1/2 on cubic lattice bonds

\[ H = - \sum_c B_c - \sum_s A_s \]

\[ B_c = \prod_{i \in c} Z_i \]

\[ A_s = \prod_{i \in s} X_i \]

cubes

degeneracy on LxLxL system with pbc: \(6L-3\)

Excitations:

corners of membrane operators, mobile only in pairs

1D confined particles

eigenvalues of independent stars/cubes

\[ |+, \cdots, +; \zeta \rangle \]

\[ \zeta \in \{-, +\}^{6L-3} \]
X-cube model

Deformed version

\[ H = H_1 + H_2 \]

\[ H_1 = \sum_s \alpha_s \left[ \exp \left( -\beta_1 \sum_{i \in s \cap \mathcal{P}_1} Z_i \right) - A_s \right] \]

\[ H_2 = \sum_c \alpha_c \left[ \exp \left( -\beta_2 \sum_{i \in c \cap \mathcal{P}_2} X_i \right) - B_c \right] \]

- alternating sign coefficients
- same conditions on paths as in 2D

AGAIN:

Consider only \( H_1 \): Z eigenvalues for spins not on path are integrals of motion.

In each sector, \( H_1 \) reduces to 1D scar paramagnet.

Non-integrable, supports scar state in the middle of spectrum

\[ |\text{scar}; \zeta\rangle = \exp \left( \frac{\beta_1}{2} \sum_{i \in \mathcal{P}_1} Z_i \right) \exp \left( \frac{\beta_2}{2} \sum_{i \in \mathcal{P}_2} X_i \right) |+, \cdots, +; \zeta\rangle \]
Conclusions

- new models with scar states in all dimensions [only single scar state]
- strong numerical evidence for their non-integrability
- topological features of the scar states, in particular exp(L) degeneracy for fracton models in 3D

Open questions

- Is there any sense of stability of the “topological” scar state degeneracies?
- Stability of scar states in general?
- Observables/physical consequences of single scar states?

arxiv:1901.01260
Open questions

2D model: **phase transition** (in ground state)

\[ H_1 = \sum_s \alpha_s \left[ \exp \left( -\beta_1 \sum_{i \in s \cap \mathcal{P}_1} Z_i \right) - A_s \right] \]

related to Ising model phase transition

Turn into scar state: **phase transition** in excited states?