Erosion-deposition waves in shallow granular free-surface flows  

Nico Gray

- 15\textsuperscript{th} Oct 2000 an unintentional release of 150 000 m\textsuperscript{3} water led to a debris flow in Fully Switzerland that had regular surges
similar waves spontaneously develop on erodible beds in the lab

there are static regions between wave crests

Daerr & Douady (1999)
Borzsony et al. (2008)
Takagi et al (2011)
- waves are typically 5mm peak height and have 2mm stationary layer

- red line is the fixed base

- side-on "photo-finish" shows basal erosion and deposition
Granular solid-fluid phase transition in depth-averaged framework

- or ignore it ...
crude, BUT ....

- resolve erosion deposition interface
notoriously difficult

Use shallow water avalanche model ...

- Uses shallow water avalanche model (e.g. Grigorian et al. 1967, Gray et al. 1999, 2003) for the thickness $h$ and the depth-averaged velocity $\bar{u}$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) = 0$$

$$\frac{\partial}{\partial t}(h\bar{u}) + \frac{\partial}{\partial x}(\chi h\bar{u}^2) + \frac{\partial}{\partial x} \left( \frac{1}{2} h^2 g \cos \zeta \right) = hgS$$

- where $\chi = \frac{\bar{u}^2}{\bar{u}^2}$ is the shape factor, $g$ is the constant of gravitational acceleration and the source term

$$S = \sin \zeta - \mu \frac{\bar{u}}{|\bar{u}|} \cos \zeta$$

- consists of gravitational acceleration and basal friction $\mu$

Pouliquen & Forterre (2002)

- Measured basal friction by determining the thickness as which the grains came to rest
  - when they started moving again from a static state
- gave effective basal friction law

\[
\mu(h, \text{Fr}) = \begin{cases} 
\mu_1 + \frac{\mu_2 - \mu_1}{1 + h/\beta/(L \text{Fr})}, & \text{Fr} \geq \beta, \quad \text{dynamic} \\
\left(\frac{\text{Fr}}{\beta}\right)^\kappa (\mu_1 - \mu_3) + \mu_3 + \frac{\mu_2 - \mu_1}{1 + h/L}, & 0 < \text{Fr} < \beta, \quad \text{intermediate} \\
\mu_3 + \frac{\mu_2 - \mu_1}{1 + h/L}, & \text{Fr} = 0, \quad \text{static}
\end{cases}
\]

- where \( \text{Fr} \) is the Froude number, \( \kappa = 10^{-3} \) and \( \mu_1 = \tan \zeta_1, \mu_2 = \tan \zeta_2 \) and \( \mu_3 = \tan \zeta_3 \) are the tangents of the angles, \( \zeta_1, \zeta_2 \) and \( \zeta_3 \).

Travelling-wave solutions in the absence of viscosity

- In a frame travelling at speed $u_w$ with coordinates
  \[ \xi = x - u_w t, \quad \tau = t. \]

- Assuming $\partial/\partial \tau = 0$ and $\chi = 1$ the system is reduces to
  \[ \frac{d}{d\xi} (h(\bar{u} - u_w)) = 0, \]

  \[ h(\bar{u} - u_w) \frac{d\bar{u}}{d\xi} + hg \cos \zeta \frac{dh}{d\xi} = hg \cos \zeta (\tan \zeta - \mu) \]

- Since $\bar{u} = 0$ in a stationary layer of thickness $h = h_+$
  \[ h(\bar{u} - u_w) = -h_+ u_w \quad \Rightarrow \quad \bar{u} = u_w \left(1 - \frac{h_+}{h}\right). \]

- The flow thickness for which $Fr = \beta$ is now defined as $h = h_*$
  \[ \Rightarrow \quad u_w = \frac{\beta h_*^{3/2} \sqrt{g \cos \zeta}}{h_* - h_+}. \]

Integration of the first order ODE indicates a problem

- solution asymptotes to a critical thickness $h_* \gg h_{crit} > h_{stop}$
- To get through this point, ones needs a little bit of viscosity

The $\mu(I)$-rheology for liquid-like granular flows

- GDR MIDI (2004) and Jop et al. (2006): proposed constitutive law
  \[ \tau = \mu(I)p \frac{D}{||D||} \]
- where 2nd invariant
  \[ ||D|| = \sqrt{\frac{1}{2} \text{tr} D^2} \]
- If $\mu = \text{const}$ this reduces to Mohr-Coulomb law
- BUT, friction $\mu$ is a function of the inertial number $I$
  \[ \mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{I_0/I + 1}, \quad I = \frac{2||D||d}{\sqrt{p/\rho^*}}, \]
- where $d$ is the particle diameter and $\rho^*$ is the intrinsic density.
The Bagnold solution

- For steady-uniform flow \( u = (u(z), 0, 0) \) the normal and downslope momentum balances imply that

\[
p = \rho g (h - z) \cos \zeta, \quad \tau_{xz} = \rho g (h - z) \sin \zeta
\]

- Rheology then implies \( \mu(I) = \tan \zeta \) and hence \( I \) is equal to a constant

\[
I_\zeta = I_0 \left( \frac{\tan \zeta - \tan \zeta_1}{\tan \zeta_2 - \tan \zeta} \right)
\]

- Solve \( I \) equation for the downslope velocity

\[
u = \frac{2I_\zeta}{3d} \sqrt{\Phi g \cos \zeta} \left( h^{3/2} - (h - z)^{3/2} \right).
\]

- The depth-averaged Bagnold velocity satisfies

\[
\bar{u} = \frac{2I_\zeta}{5d} \sqrt{\Phi g \cos \zeta} h^{3/2}
\]
The depth-averaged $\mu(I)$-rheology for granular flows

- To first order the inviscid avalanche equations emerge naturally with the dynamic basal friction law

$$\mu_b(h, Fr) = \mu_1 + \frac{\mu_2 - \mu_1}{\beta h/(L Fr) + 1}, \quad Fr > \beta,$$

- This is just Pouliquen & Forterre's (2002) law, where

$$Fr = \frac{|\bar{u}|}{\sqrt{gh \cos \zeta}}$$

- Now add in the in-plane deviatoric stress

$$\tau_{xx} = \mu(I)p \frac{D_{xx}}{\|D\|}$$

- Assume shallow and use Bagnold solution to evaluate

$$D_{xx} = \frac{\partial u}{\partial x}, \quad \|D\| = \frac{1}{2} \left| \frac{\partial u}{\partial z} \right|$$
• It follows that the in-plane deviatoric stress is

\[ \tau_{xx} = 2\rho g \sin \zeta \left( h^{1/2}(h - z)^{1/2} - (h - z) \right) \frac{\partial h}{\partial x}. \]

• formal depth-integration gives

\[ h \tilde{\tau}_{xx} = \frac{1}{3} \rho g \sin \zeta h^2 \frac{\partial h}{\partial x}. \]

• Use depth-averaged Bagnold velocity to reformulate

\[ h \tilde{\tau}_{xx} = \rho \nu h^{3/2} \frac{\partial \tilde{u}}{\partial x} \]

• where the angle dependent coefficient \( \nu \) is determined

\[ \nu = \frac{2L \sqrt{g}}{9 \beta} \frac{\sin \zeta}{\sqrt{\cos \zeta}} \left( \frac{\tan \zeta_2 - \tan \zeta}{\tan \zeta - \tan \zeta_1} \right). \]

Well-posed and ill-posed behaviour of the $\mu(I)$-rheology

- Coefficient $\nu$ is negative outside range of steady-uniform flow
- Depth-averaged $\mu(I)$-rheology is ill-posed in this region.

- Consistent with full $\mu(I)$-rheology, which is well-posed for angles in the grey region (above)
- 2D transient simulations of Bagnold flow blow-up via oblique pressure perturbations for angles in white region (above)

Application to granular roll-waves

\[
\frac{\partial}{\partial t}(h\tilde{u}) + \frac{\partial}{\partial x}(h\tilde{u}^2) + \frac{\partial}{\partial x}\left(\frac{1}{2}gh^2 \cos \zeta\right) = ghS_x + \frac{\partial}{\partial x}\left(\nu h^{3/2} \frac{\partial \tilde{u}}{\partial x}\right)
\]

- Adds a singular perturbation to the momentum equation
- This is the only form that is not singular in \(h\) or \(\tilde{u}\)

Measurements of the spatial growth rate of granular roll-waves

- Forterre & Pouliquen (2003) used loudspeaker to initiate roll waves of a given frequency
- Inviscid theory predicts critical Froude $Fr_c = 2/3$, but growth occurs at all frequencies $\omega$
- Depth-averaged rheology predicts the cut-off frequency $\omega_c$
- MATCHES WITHOUT ANY FITTING PARAMETERS
Exact travelling wave solutions for roll waves

- computed by numerically integrating 2nd order ODE with prescribed Fr and \( u_w \) until a limit cycle is formed

Exact travelling wave solutions for erosion-deposition waves

- For each solution $h_+$ and $h_*$ must be prescribed.
- viscosity allows solution to cross the critical line!

Exact travelling wave solutions for erosion-deposition waves

- Exact solution picks off the correct amplitude and wavelength
- ALTHOUGH its shape is a little different
- MAJOR STEP FORWARD in modelling erosion-deposition problems with shallow erodible layers

• Numerical solutions with random noise rapidly coarsen into large amplitude waves
• Close to stopping very destructive waves are formed!

Complex coarsening dynamics is qualitatively reproduced

- Experimental space-time plot shows:
  - regions of stationary material as horizontal straight lines
  - the wave-fronts as white lines
- very similar in computations (right)

The depth-averaged $\mu(I)$-rheology also plays critical role in fingering instabilities.
Schematic diagram for the levee formation process

- larger particles are shouldered to the sides to create levees
- this is an example of a segregation-mobility feedback effect
Inviscid avalanche model for segregation-mobility induced fingers

- For avalanche thickness $h$, small particle thickness $\eta$ and depth-averaged velocity $\overline{u}$ the 2D coupled model is

$$
\frac{\partial h}{\partial t} + \text{div}(h\overline{u}) = 0,
$$

$$
\frac{\partial \eta}{\partial t} + \text{div}\left(\eta\overline{u} - (1-\alpha)\eta \left(1 - \frac{\eta}{h}\right) \overline{u}\right) = 0,
$$

$$
\frac{\partial}{\partial t}(h\overline{u}) + \text{div}(h\overline{u} \otimes \overline{u}) + \text{grad}\left(\frac{1}{2}gh^2\cos\zeta\right) = hgS,
$$

- source terms composed of gravity and basal friction

$$
S = \left(\begin{array}{c}
\sin\zeta - \mu(\overline{u}/|\overline{u}|)\cos\zeta, \\
- \mu(\overline{v}/|\overline{u}|)\cos\zeta,
\end{array}\right)
$$

- coupling through $\overline{\phi} = \eta/h$ dependent friction coefficient

$$
\mu = \left(1 - \overline{\phi}\right)\mu^L + \overline{\phi}\mu^S, \quad \mu^L > \mu^S
$$


- The model is hyperbolic
- captures the instability mechanism
- and forms large rich lateral levees, BUT ....
Numerical solutions are grid dependent ...

- Such ill-posed behaviour is an indication that some important physics is missing – in this case viscosity.

A two-dimensional fully coupled model including rheology

- When the depth-averaged $\mu(I)$-rheology is generalized to 2D it suggests a system of conservation laws of the form

$$\frac{\partial h}{\partial t} + \text{div}(h\bar{u}) = 0,$$

$$\frac{\partial \eta}{\partial t} + \text{div} \left( \eta \bar{u} - (1 - \alpha) \eta \left(1 - \frac{\eta}{h}\right) \bar{u} \right) = 0,$$

$$\frac{\partial}{\partial t} (h\bar{u}) + \text{div}(h\bar{u} \otimes \bar{u}) + \text{grad} \left(\frac{1}{2} h^2 \cos \zeta\right) = hS + \text{div} \left(\nu h^{3/2} \overline{D}\right),$$

- where the two-dimensional strain-rate tensor is

$$\overline{D} = \frac{1}{2} \left( L + L^T \right)$$

- and $L = \text{grad}(\bar{u})$ is the depth-averaged velocity gradient

- Numerics converges ... (Baker, Johnson & Gray in prep)
characteristics coincide when

\[ Fr = Fr_c = \frac{1}{(1 - \alpha)|2\eta_0 - 1|} \]

produces unbounded growth in inviscid case \( \nu = 0 \).

The depth-averaged \( \mu(I) \)-rheology regularizes the equations
Important for geophysical mass flows which often form levees which enhance run-out!