Periodic Trajectories in Saltation Transport

Alexandre Valance\textsuperscript{1}, Jim Jenkins\textsuperscript{2} and Diego Berzi\textsuperscript{3}

\textsuperscript{1} Institut de Physique de Rennes, Université de Rennes 1, France
\textsuperscript{2} Cornell University, Ithaca, USA
\textsuperscript{3} Politecnico Milano, Milano, Italia

\textit{Dresden, Max Planck Institut, 14-19 March, 2016}
1. Introduction

2. Saltation over a Rigid bed

3. Saltation over an Erodible Bed

4. Aquatic transport and Extra-terrestrial atmospheres
Saltation Transport

- Aeolian sand transport (air)
- Snow drift (air)
- Bed load transport (water)
- Saltation transport on extra-terrestrial planets
Aeolian saltation transport (Bagnold, 1941)

- Sand density: \( \sigma = \frac{\rho_p}{\rho_{air}} = 2200 \)
- Mode of transport:
  - Saltation: \( d \approx 0.1 - 0.6 \text{ mm} \)
  - Suspension: \( d < 0.1 \text{ mm} \)
Mass flow rate : Bagnold law

- Hypothesis : Trajectories replaced by an ’averaged’ trajectory of length $l_s$

- Particle shear stress : $s_{grain}(y = 0) = \phi_{eq}(u_0\downarrow - u_0\uparrow)$

- Momentum conservation :
  
  $s_{grain}(y) + S_{air}(y) = S_{\infty} = \rho_{air} u^*^2$

- Mass flow rate :
  
  $Q_{eq} = l_s \phi_{eq} = \frac{l_s}{(u_0\downarrow - u_0\uparrow)}(S_{\infty} - S_{air}(0))$
Mass flow rate: Bagnold law

- **Bagnold assumptions:**
  - $S_{air}(y = 0) \approx 0$
  - $(u_{0\downarrow} - u_{0\uparrow}) \approx u_{0\downarrow} \propto u^*$
  - $l_s \propto (u^*^2/g)$

- **Bagnold cubic scaling law:**
  \[
  Q_{eq} = \frac{l_s}{(u_{0\downarrow} - u_{0\uparrow})} S_\infty \propto \frac{\rho_{air} u^*^3}{g}
  \]

- **Strong resemblance with the Meyer-Peter and Müller law for bed-load transport in water:**
  \[
  Q \propto (S^* - S_c^*)^{3/2} \quad \text{where} \quad S^* = \frac{\rho_p u^*^2}{(\rho_p - \rho_f)gd} \quad \text{(Shields)}
  \]
Experimental data: Aeolian Sand transport

Wind-tunnel experiments: Mass flow rate measurements

(Iversen & Rasmussen 1990, Ho et al. 2011)

Erodible bed

Rigid bed

\[ Q \propto (S^* - S_{c}^*) \]

\[ Q_{\text{max}} \propto (S^* - S_{c}^*)^{3/2} \]
Experimental data: Aeolian Sand transport

- Mean saltation hop length: Erodible vs Rigid bed

\[(Ho \ et \ al. \ 2011)\]

- Erodible bed: \( l_s \approx \text{cst} \)
- Rigid bed: \( l_s \propto (S^* - S_c^*) \)
Different modeling approaches

- Eulerian/Eulerian Approach:
  

- Eulerian/Lagrangian Approach:
  
  Anderson and Haff 1988, Werner 1990  
Motivation

- Develop the simplest Eulerian/Lagrangian approach
- Idea: describe the Aeolian saltation transport in terms of a single periodic trajectory (instead of a distribution of trajectories)

Consider the single trajectory to be the ensemble average of the trajectories that participate in a steady motion.
Strategy and System parameters

- Phrase the motion equations as a two-point boundary value problem (solving simultaneously for ascending and descending motion)

- Characteristic scales:
  - unit length: particle diameter $d$, unit velocity: $\sqrt{gd}$
  - unit stress: $\rho_p gd$

- Dimensionless numbers:
  - Particle Reynolds number: $Re = d \sqrt{gd} / \nu (\approx 1 - 10)$
  - Density ratio: $\sigma = \rho_p / \rho_{air} (\approx 2000)$
  - Stokes number: $St = \sigma Re (\approx 10^4)$
  - Shields number: $S^* = \rho_f u^*^2 / (\rho_p - \rho_f)gd (\approx 0.01 - 1)$
1 Introduction
2 Saltation over a Rigid bed
3 Saltation over an Erodible Bed
4 Aquatic transport and Extra-terrestrial atmospheres
Particle Motion

Ascending Motion:

\[ a_x^+ = \xi_y^+ \frac{d\xi_x^+}{dy} = D^+(U - \xi_x^+) , \]  
\[ a_y^+ = \xi_y^+ \frac{d\xi_y^+}{dy} = -D^+ \xi_y^+ - 1 , \]  

Descending Motion:

\[ a_x^- = \xi_y^- \frac{d\xi_x^-}{dy} = D^-(U - \xi_x^-) , \]  
\[ a_y^- = \xi_y^- \frac{d\xi_y^-}{dy} = -D^- \xi_y^- - 1 , \]  

Drag coefficient: \[ D = \frac{0.3 \sqrt{(U - \xi_x)^2 + \xi_y^2 + 18/Re}}{\sigma} \]
Particle Motion

Downstream position $x^+$ and $x^-$ are functions of $y$:

\[ \xi_y^+ \frac{dx^+}{dy} = \xi_x^+ , \]  \hfill (5)

\[ \xi_y^- \frac{dx^-}{dy} = \xi_x^- . \]  \hfill (6)
Fluid Motion

Steady and fully developed turbulent boundary layer

- Momentum conservation:
  \[ S_{\text{air}}(y) + s_{\text{particle}}(y) = S^* = \text{constant} \]

- Mixing length turbulence model:
  \[ \sigma S_{\text{air}} = \kappa^2 l^2 \left| \frac{dU}{dy} \right| \left( \frac{dU}{dy} \right) \text{ with } l = (y + y_0) \]

\[ \Rightarrow \frac{dU}{dy} = \frac{\left[ \sigma (S^* - s_{\text{particle}}) \right]^{1/2}}{\kappa(y + y_0)} \]
Eulerian Velocity and Particle Shear Stress

We consider a steady state consisting in an ensemble of particles with periodic trajectories.

We introduce the concentration of ascending and descending particles: \( c^+ \) and \( c^- \)

- Eulerian particle velocity:
  \[
  u(y) = \frac{(c^+ \xi_x^+ + c^- \xi_x^-)}{(c^+ + c^-)}
  \]

- Particle shear stress:
  \[
  s(y) = - \left( c^+ \xi_y^+ \xi_x^+ + c^- \xi_y^- \xi_x^- \right)
  \]
Two-point boundary value problem

- Counting equations and unknown parameters
  - Seven first order differential equations for: 
    \( \xi^+_x, \xi^-_x, \xi^+_y, \xi^-_y, x^+, x^- \) and \( U \)
  - Three additional unknown parameters:
    - the height \( H \) and the length \( L \) of the trajectory
    - the mass hold-up \( M = \int_0^H c(y)dy \)

- Continuity relations provide only 7 conditions
  - \( \xi^+_x(H) = \xi^-_x(H), \xi^+_y(H) = 0 \) and \( \xi^-_y(H) = 0 \)
  - \( x^+(0) = 0, x^- (0) = L \) and \( x^+(H) = x^- (H) \)
  - \( U(0) = 0 \)

- Need for additional conditions
Rebound on a rigid and bumpy surface

Rebound law

\[ e = \frac{\xi^+(0)}{\xi^-(0)} = (A - B \sin \theta) \]
\[ e_y = \frac{\xi^+_y(0)}{\xi^-_y(0)} = -(A_y / \sin \theta - B_y) \]

\[ A = 0.9, \quad B = 0.4, \quad A_y = 0.5 \quad \text{and} \quad B_y = 0 \]

(values obtained from D.E.M simulation with \( e_n = 0.8 \))
Two additional conditions provided by the rebound law:
\[ \xi^+(0) = e(\theta) \xi^-(0) \text{ and } \xi_y^+(0) = -e_y(\theta) \xi_y^-(0) \]

We have now \((7 + 2)\) boundary conditions for 7 first order differential equations + 3 unknown parameters

For a given flow strength, there is therefore one free parameter (the mass hold-up \(M\))
Model predictions: Particle trajectory

System parameters:
$Re = 0.73 \ (d = 0.23 \ mm), \ \sigma = 2200 \ and \ S^* = 0.06$

- Varying the mass holdup at a fixed Shields number

Trajectories decrease in size with increasing mass holdup
Model predictions: Air and particle velocity

System parameters:

\[ Re = 0.73 \ (d = 0.23 \ mm), \ \sigma = 2200 \ \text{and} \ S^* = 0.06 \]

- Varying the mass holdup at a fixed Shields number

Air flow profile

Particle velocity profile

Air and particle velocity decrease with increasing mass holdup
Model Predictions: Mass flow rate

System parameters:
\[ Re = 0.73 \ (d = 0.23 \ mm), \ \sigma = 2200 \ \text{and} \ S^* = 0.06 \]

- Varying the mass holdup at a fixed Shields number

The mass flow rate exhibits a maximum
Model Predictions : Mass flow rate

System parameters :
\( Re = 0.73 \ (d = 0.23 \ mm) \) and \( \sigma = 2200 \)

- Mass flow rate vs mass holdup for different Shields numbers

The mass flow rate and the mass holdup at maximum capacity both increase with increasing Shields number
Model Predictions: Flow at maximum capacity

- Trajectory height and length at maximum capacity:
  \[ L_{\text{max}} \propto H_{\text{max}} \propto (S^* - S^*_c) \]

- Mass hold-up at maximum capacity:
  \[ M_{\text{max}} \propto (S^* - S^*_c) \]

- Particle velocity and Mass flow rate at maximum capacity:
  \[ u_{0\text{max}} \propto \sigma^{1/2} (S^* - S^*_c)^{1/2} \]
  \[ Q_{\text{max}} \propto M_{\text{max}} u_{0\text{max}} \propto \sigma^{1/2} (S^* - S^*_c)^{3/2} \]
Comparison with Experiments

- Wind-tunnel experiments on rigid and bumpy bed:
  (Ho, Phd Thesis 2012, Rennes)
  Parameters: $d = 0.230\, mm$ ($Re = 0.73$) and $\sigma = 2200$

Maximum capacity of transport vs Shields number

![Graph showing the comparison between periodic simulation and experimental data from Ho et al.](image-url)
Introduction

Saltation over a Rigid bed

Saltation over an Erodible Bed

Aquatic transport and Extra-terrestrial atmospheres
Main differences between Rigid and Erodible bed:

Finite supply & Inexhaustible source of particles

No mass exchange between the bed and the flow & Mass exchange via the splash process

Main issue: How to account for the splash process?
Splash process

- Model Collision Experiment (Beladjine et al, PRE 2007)

Particle parameters: \( d = 6 \text{ mm} \) and \( \rho_p = 2300 \text{ kg/m}^3 \)
Splash process : Experimental outcomes

- Rebound: law similar to that on a rigid bed
- Ejections of particles

\[ N_{tot} = N_{rebound} + N_{ej} \]

\[ \theta_{impacting} \rightarrow \text{rebound} \]
\[ \text{Ejecta} \]

\[ N_{tot}(\xi) = \begin{cases} 
1 + N_0(1 - e^2) \left( \frac{\xi}{\xi_c} - 1 \right) & \text{if } \xi > \xi_c \\
1 & \text{if } 1 \leq \xi \leq \xi_c \\
0 & \text{if } \xi \leq 1 
\end{cases} \]

- Steady State:

\[ N_{tot} = 1 \Rightarrow 1 < \xi_0 < \xi_c \Rightarrow \xi_0 \approx \frac{1 + \xi_c}{2} \approx 20 \]

Additional condition leaving the system without any free parameter
Model predictions: Air velocity

System parameters:
\( \text{Re} = 0.73 \) (\( d = 0.23 \) mm) and \( \sigma = 2200 \)

- Air velocity profiles: \( 0.01 < S^* < 0.1 \)

Log profiles with a focus point (Bagnold 1941)
Model predictions: Periodic trajectories

System parameters:
Re = 0.73 (d = 0.23 mm) and \( \sigma = 2200 \)

- Saltation trajectory

- Saltation Height and Length

Trajectory invariant with Shields

\( H \) and \( L \) invariant with Shields
Model predictions: concentration and mass flux rate

System parameters:
\( Re = 0.73 \) \((d = 0.23 \text{ mm})\) and \( \sigma = 2200 \)

- **Concentration at the bed**
  \[ c_0 \propto (S^* - S_{c}^*) \]
  and \( M \approx c_0 H \approx (S^* - S_{c}^*) \)

- **Mass flow rate**
  \[ Q \propto (S^* - S_{c}) \]
Conclusion and Perspectives

- Conclusion
  - Simple predictive model for saltation transport over erodible and rigid bed
  - Two different saltation regimes:
    - Unlimited saltation: \( Q \propto (S^* - S^*_c)^{3/2} \)
    - Splash-limited Saltation: \( Q \propto (S^* - S^*_c) \)

- Perspectives
  - Application to bed load-transport in water
  - Application to saltation transport on other planetary aeolian environments
1. Introduction

2. Saltation over a Rigid bed

3. Saltation over an Erodible Bed

4. Aquatic transport and Extra-terrestrial atmospheres
Sediment transport in water

- Differences with transport in Air:
  - $\sigma = \rho_p/\rho_{fluid} \approx 2.6$
  - $1 < St = \sigma Re < 1000$

- Model modifications:
  - Particle motion equations:
    - Replace $g \rightarrow g' = (1 - 1/\sigma)g$ (Buoyancy)
  - Collision:
    - $e \rightarrow e' = e - 6.9(1 + e)/St$ (Lubrication force)
Aquatic transport over Rigid bed

System parameters:
$Re = 450 \ (d = 0.23 \ mm), \ \sigma = 2.65 \ and \ St = 1200 \ (no \ lubrication)$

- Particle velocity at the maximum capacity

\[ \begin{align*}
\nu_0 & \propto (S^* - S^*_c)^{1/2} \\
\nu_0 & \ll \xi_c \approx 40 \Rightarrow \text{Splash not triggered}
\end{align*} \]

- Mass flux at the maximum capacity $Q$

\[ Q \propto (S^* - S^*_c)^{3/2} \]

Aquatic bed-load transport Mass: Unlimited saltation transport
Saltation transport on extra-terrestrial atmospheres

- General phase diagram for saltation transport
  (Berzi et al. JFM 2016)

\[ St = 1000 \]

- Venus : \( \sigma = 80 \) and Titan : \( \sigma = 200 \)
  \( \Rightarrow \) Expected transition from unlimited to splash-limited saltation