Understanding rainfall: the role of Turbulence and Large Deviation Theory

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Overview

- Understanding the origin of rain showers from ice-free clouds is a challenge. Raindrops grow by collisions, but the rate of collision is slow.
- It has been proposed that turbulence in shower clouds enhances the collision rate. There are two mechanisms: clustering of particles, and folding of the phase-space manifold. In typical clouds, neither effect is sufficient.
- Only a tiny fraction of droplets need to undergo runaway growth. Large deviation theory is the appropriate tool for rare events. In the case of rainfall, it shows that a shower can occur in a small fraction of the mean collision time.

The problem of rainfall

Raindrops (millimetre size) result from a ‘collector’ droplet sweeping up a vast number of microscopic droplets. The first few collisions happen very slowly (perhaps one collision per hour).

Droplet size: $a_0 = 10\,\mu\mathrm{m}$
Liquid water content: $\Phi_\mathrm{l} = 1\,\mathrm{g\,m^{-3}}$

One million collisions must occur in less than the mean time of a single collision. But this only a tiny fraction ($P \approx 10^{-7}$) of microscopic droplets need to undergo runaway growth.

Estimating collision rates

Collisions primarily due to differing settling rates. Stokes formula $F = 6\pi\eta a v$ gives

$$v = \tau_{\mathrm{p}} g = \kappa a^2 \ , \quad \kappa = \frac{2}{9}\frac{\rho_{\mathrm{l}}}{\rho_{\mathrm{g}}}\frac{g}{\nu} \qquad \kappa \approx 1.4 \times 10^8 \,\mathrm{m}^{-1}\mathrm{s}^{-1}$$

Simple kinetics (area X speed) gives the collision rate:

$$R_1 = \pi \varepsilon N_0 (a_0 + a_1)^2 \kappa (a_1^2 - a_0^2) \qquad N_0 = 2.5 \times 10^8 \,\mathrm{m}^{-3}$$

The collision efficiency is low for small droplets

$$a \leq 10\,\mu\mathrm{m} \to \varepsilon \leq 0.03$$

Collision rate estimate: for $a_0 = 10\,\mu\mathrm{m}\ , \quad a_1 - a_0 = 2.5\,\mu\mathrm{m}$

$$R_1 \approx 2 \times 10^{-5}\mathrm{s}^{-1}$$

Role of turbulence

Can turbulence initiate rainfall by increasing the rate of collisions between droplets? Several distinct proposals:

- Turbulence causes collisions by inducing shear (Saffman and Turner, 1956).
- Turbulence causes particles with significant inertia to cluster (Maxey, 1987). This should enhance the collision rate (Sundaram and Collins, 1997).
- The clustering of particles samples a fractal measure (Sommerer and Ott, 1993), implying large densities.
- Relative velocities may be increased by folding of phase-space manifolds (Falkovich, Fouxon and Stepanov, 2005), Wilkinson, Mehlig and Bezuggly, (2006).

Particles in turbulence

Droplets have small Reynolds number: their motion is dominated by viscosity. Equations of motion are

$$\dot{\boldsymbol{r}} = \boldsymbol{v} \ , \quad \dot{\boldsymbol{v}} = \frac{1}{\tau_{\rm p}}[\boldsymbol{u}(\boldsymbol{r},t) - \boldsymbol{v}] + \boldsymbol{g}$$

Damping time is determined by Stokes formula for drag

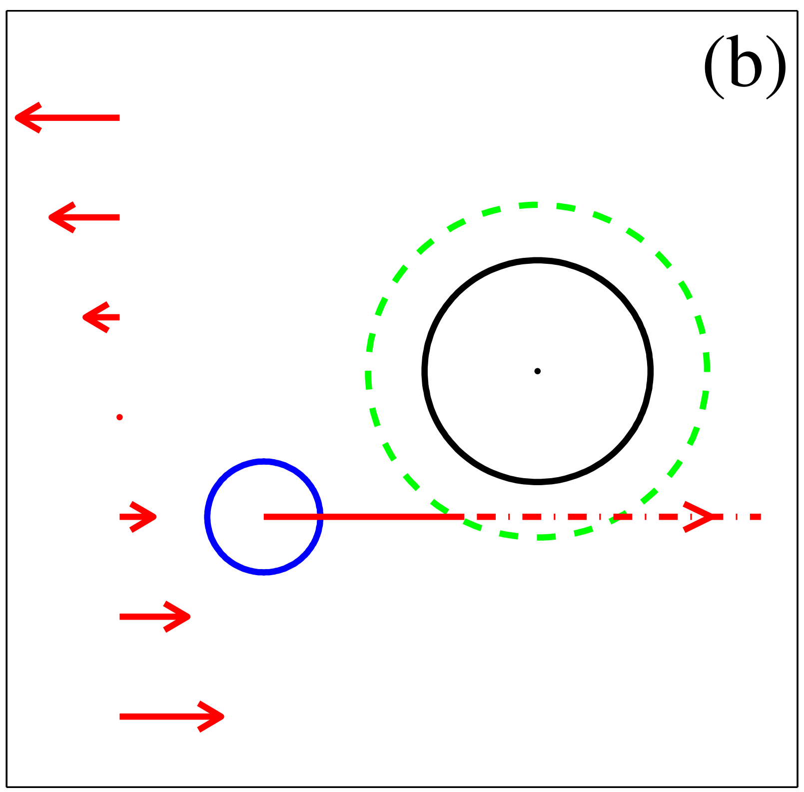
$$\tau_{\rm p} = \frac{2}{9}\frac{a^2}{\nu}\frac{\rho_{\rm p}}{\rho_{\rm f}}$$

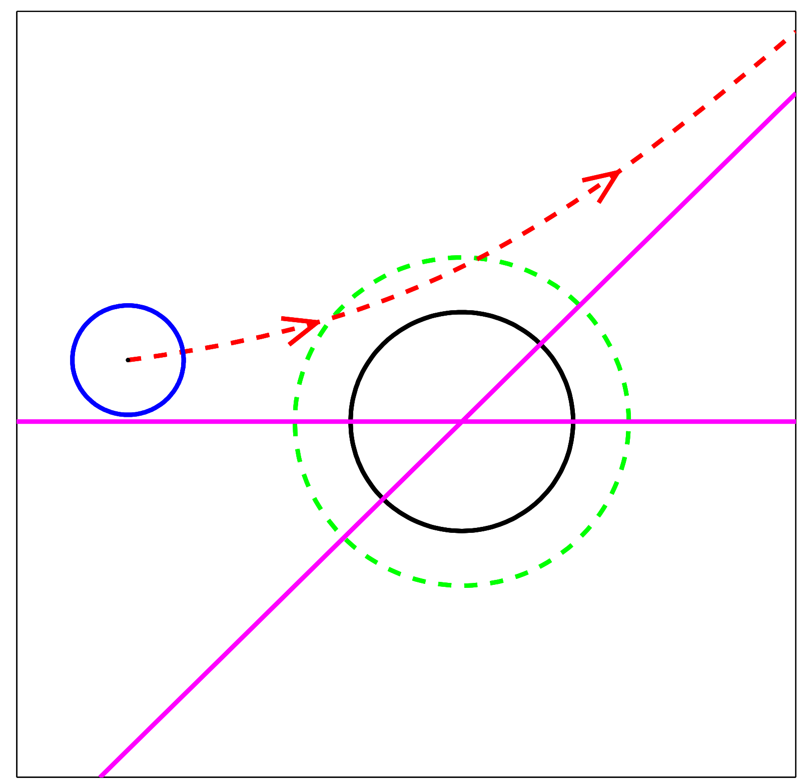
Turbulent motion of fluid is described by Kolmogorov theory

$$\langle[\boldsymbol{u}(\boldsymbol{R},t) - \boldsymbol{u}(\boldsymbol{0},t)]^2\rangle = C\left(\epsilon|\boldsymbol{R}|\right)^{2/3} \qquad \tau_{\rm K} = \sqrt{\nu/\epsilon}$$

Saffman-Turner mechanism

Saffman and Turner: turbulence has shearing motions, which induce collisions:



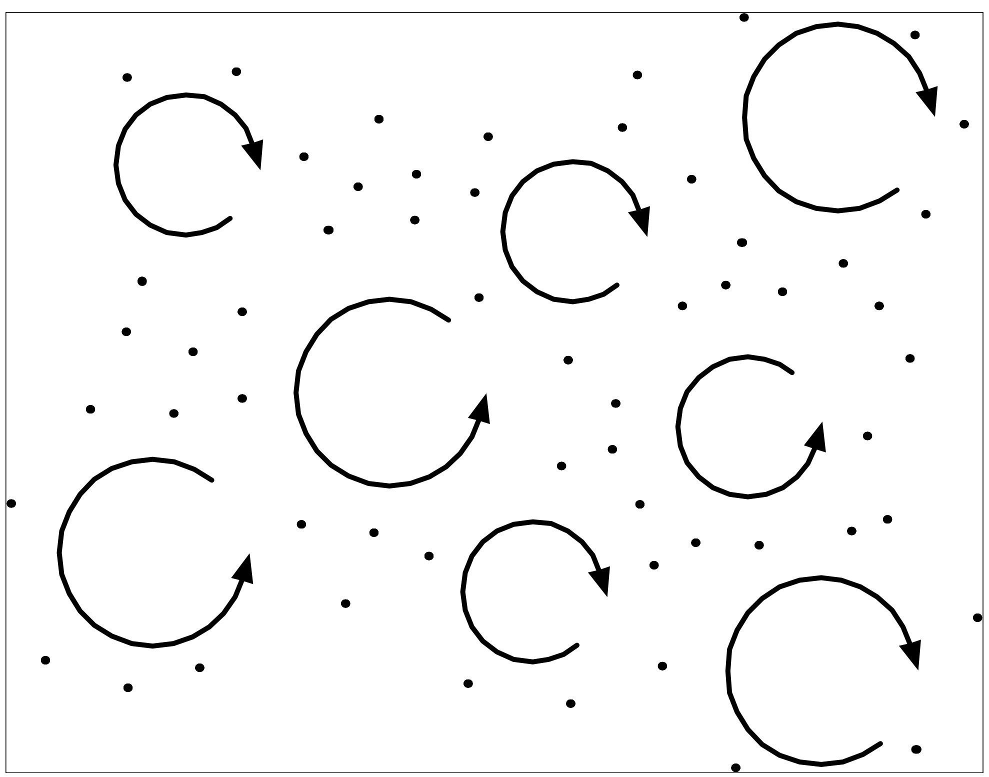


$$R_{\mathrm{turb}} = \sqrt{\frac{8\pi}{15}\frac{N_0 \varepsilon (2a)^3}{\tau_{\mathrm{K}}}}$$

$$\tau_{\mathrm{K}} = \sqrt{\nu / \epsilon}$$

Clustering

Droplets density becomes inhomogeneous due to inertial effects. Maxey (1987) proposed that dense particles are ejected from vortices by centrifugal effects. The effect can only work at moderate Stokes number.



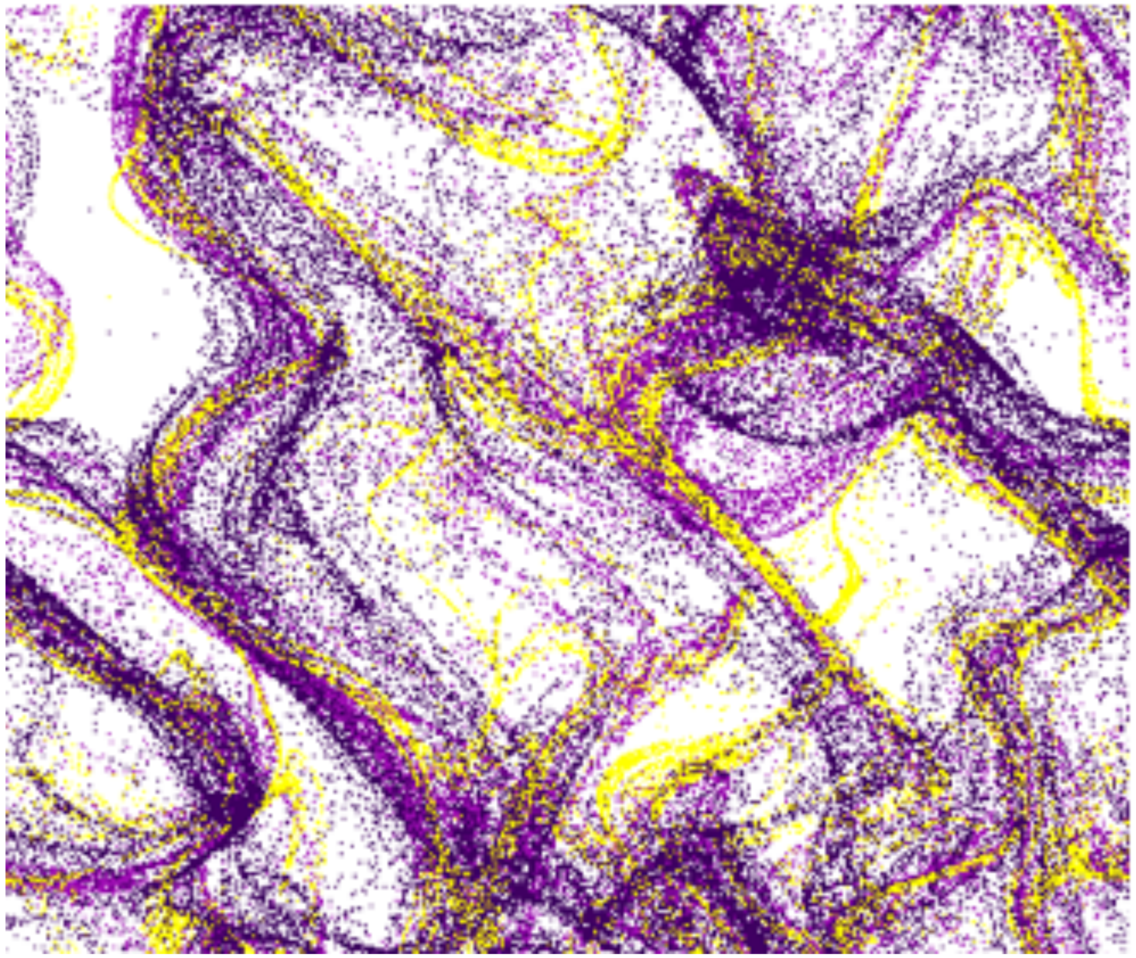
$$\mathrm{St} = \frac{\tau_\mathrm{p}}{\tau_\mathrm{K}}$$

$$\boldsymbol{v} = \boldsymbol{u}(\boldsymbol{x}(t), t) - \tau_\mathrm{p} \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}(\boldsymbol{x}(t), t)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = -\tau_\mathrm{p} \left[\mathrm{tr}(\mathbf{E}^2) - \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{\omega} \right]$$

Fractal clustering

The clustering effect is much stronger than Maxey's centrifuge argument implies. Ott and Sommerer (1993) used dynamical systems theory to show that the clustering is fractal. Characterised by the correlation dimension:

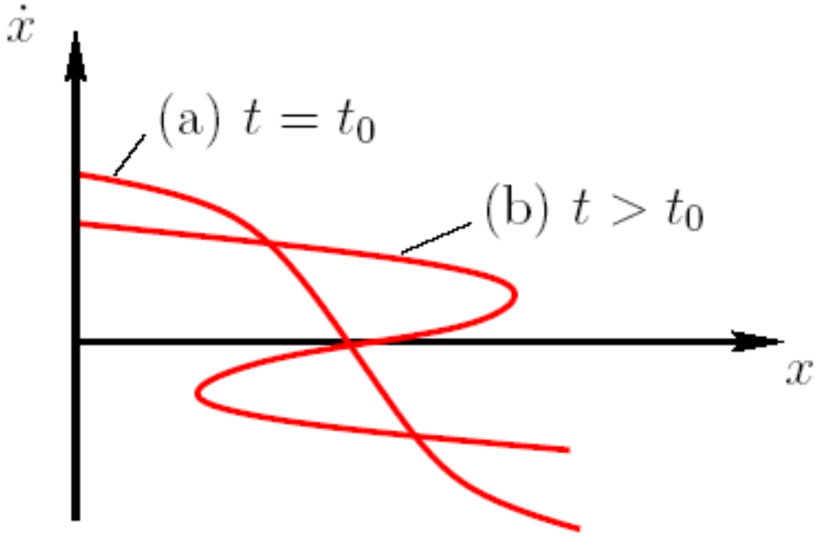


$$\mathcal{C}(\epsilon) = \frac{1}{4\pi n \epsilon^2} \frac{\mathrm{d}\langle \mathcal{N} \rangle}{\mathrm{d}\epsilon}\bigg|_{\epsilon}$$

$$\langle \mathcal{N}(\epsilon) \rangle \sim n\eta^3 \left(\frac{\epsilon}{\eta}\right)^{D_2}$$

Caustics

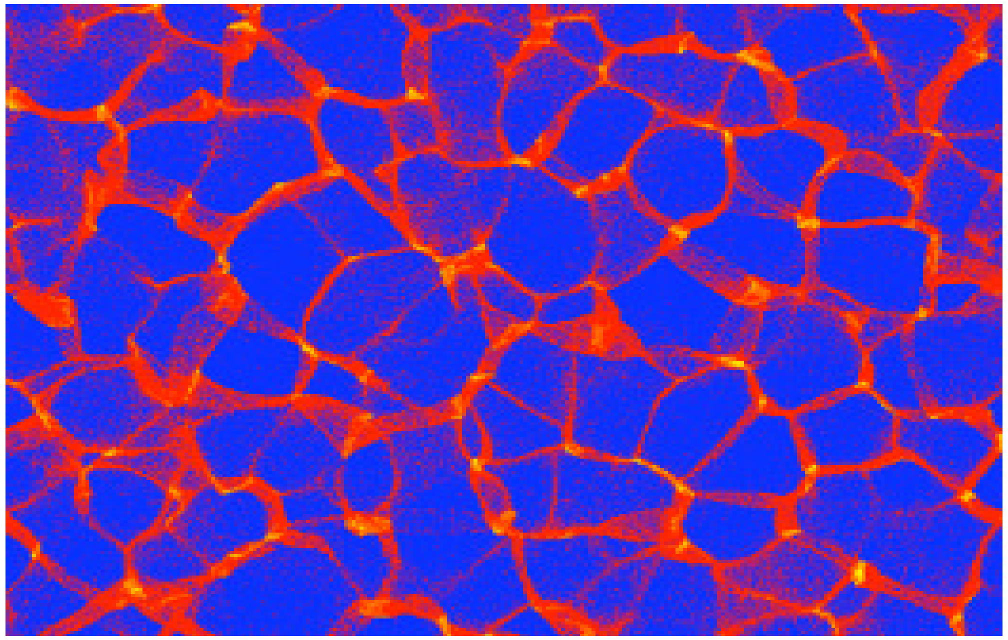
A competing effect is that collisions may occur if faster particles overtake slower ones: think about phase space.



$$R = \frac{4\pi a^2 n\eta}{\tau_{\mathrm{K}}} F(\mathrm{St}, \mathrm{Re})$$

$$F(\mathrm{St}, \mathrm{Re}) \propto \exp(-S/\mathrm{St})$$

The effect is most clearly understood in terms of caustics:



High Stokes number

The ‘overtaking’ mechanism was termed the *sling effect* by Falkovich, Fouxon and Stepanov (2005), and discussed in terms of caustics by Wilkinson, Mehlig & Bezuglyy (2006).

When the Stokes number is very large, the particle trajectories are uncorrelated with the flow (Abrahamson, 1975), also termed ‘random uncorrelated motion’ (Reeks *et al*, 2011). The collision rate is inferred from the Kolmogorov theory (Mehlig & Wilkinson, 2007):

$$\langle|\Delta v|\rangle = \mathcal{K}\sqrt{\epsilon\tau_{\mathrm{p}}} \qquad\qquad R \approx K\frac{na^2\eta}{\tau_{\mathrm{K}}}\sqrt{\mathrm{St}}$$

Collision rate formula

Collision rates due to shearing effects and caustics are additive:

$$R = R_{\mathrm{adv}} + R_{\mathrm{caust}}$$

$$R_{\mathrm{adv}} = \sqrt{\frac{8\pi}{15}}\,\frac{n(2a)^3}{\tau_{\mathrm{K}}}\mathcal{C}(2a)$$

$$R_{\mathrm{caust}} = K\frac{na^2\eta}{\tau_{\mathrm{K}}}\sqrt{\mathrm{St}}\,\exp(-S/\mathrm{St})$$

Numerical studies

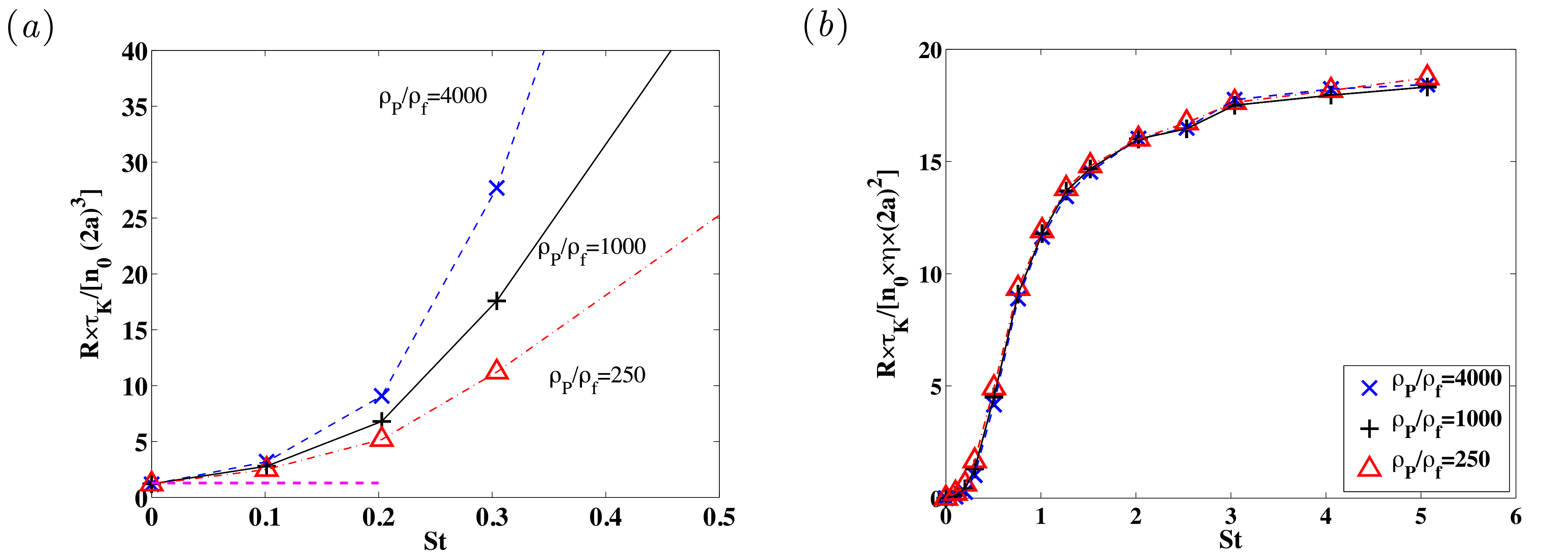


FIGURE 1. The collision rate R as a function of the Stokes number St and for the ratios of density $\rho_\mathrm{p}/\rho_\mathrm{f} = 250$, 10^3 and 4×10^3. The collision rate R is normalized by $n_0(2a)^3/\tau_\mathrm{K}$ (a), and $n_0(2a)^2\eta/\tau_\mathrm{K}$ (b). The horizontal dashed line in (a) corresponds to the Saffman-Turner prediction.

Large Stokes number limit

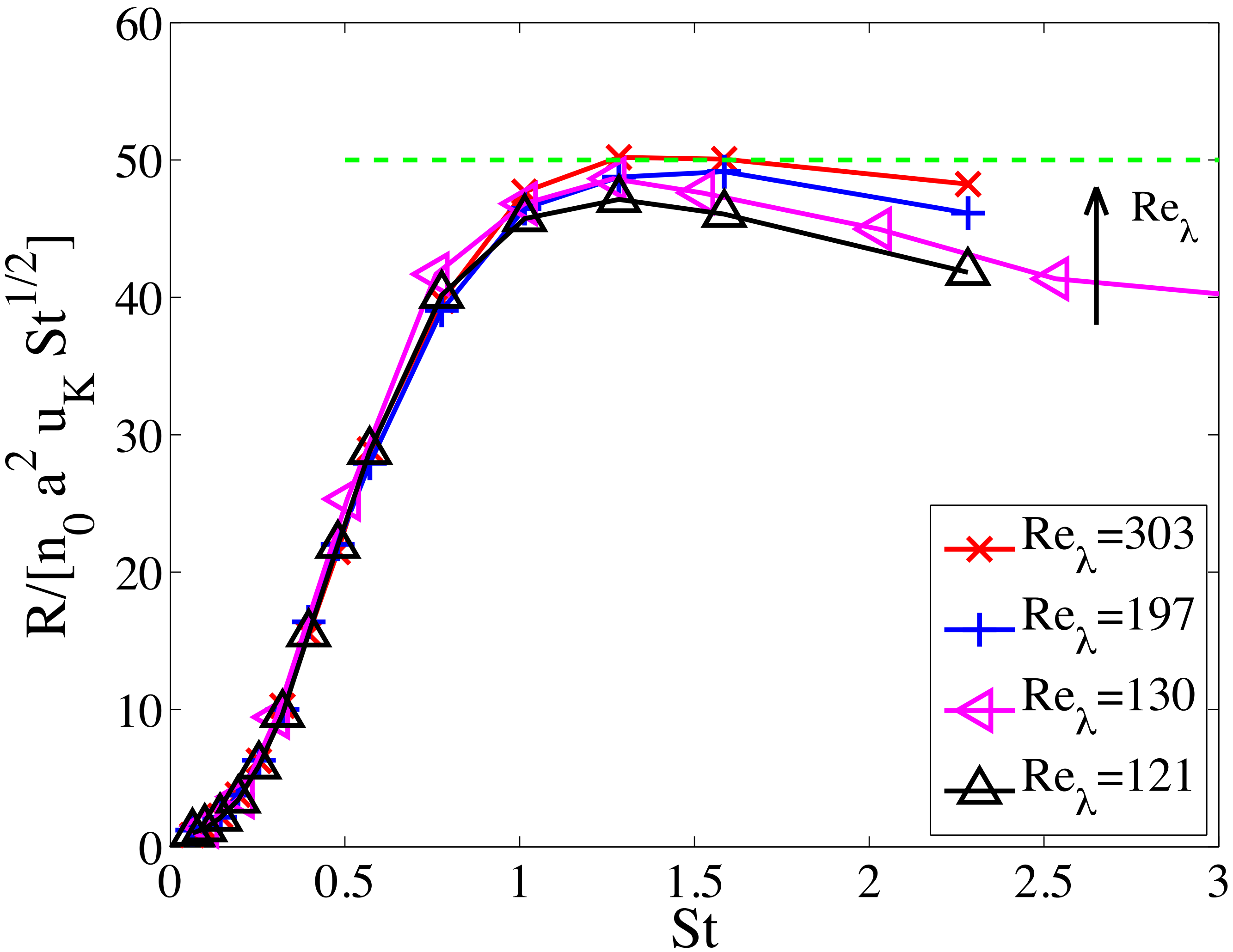


Figure 4. (Colour online). Plot of the collision rate R divided by $n_0 a^2 u_{\rm K}\sqrt{\rm St}$, as a function of St. The curves appear to approach a plateau at large St as the Reynolds number increases, consistent with equation (12). The data for $\mathrm{Re}_\lambda = 130$ is from [24], the other data is re-plotted from [10].

Does turbulence explain rain?

Estimate intensity of turbulence (rate of dissipation):

$$\epsilon \sim \frac{U^3}{L} \approx 10^{-3}\,\mathrm{m^2 s^{-1}} \qquad \tau_\mathrm{K} \approx 10^{-2}\,\mathrm{s}$$

Parameters of typical droplets:

$$a_0 = 10\,\mu\mathrm{m}\ , \quad a_1 - a_0 = 2.5\,\mu\mathrm{m} \qquad\qquad N_0 = 2.5 \times 10^8\,\mathrm{m}^{-3}$$

The Saffman-Turner collision rate is low:

$$R_\mathrm{turb} \approx 10^{-6}\,\mathrm{s}^{-1}$$

And so is the Stokes number:

$$\mathrm{St} = \frac{\tau_\mathrm{p}}{\tau_\mathrm{K}} \approx 10^{-2}$$

So turbulence does not have much effect.

Model for runaway growth

Kostinski and Shaw (BAMS, 2005) gave a model for runaway droplet growth. Time for growth of raindrop is:

$$T = \sum_{i=1}^{\mathcal{N}} t_n \qquad\qquad \mathcal{N} \approx 10^6$$

Individual collision times are Poisson distributed, with a rate that increases as the droplet grows:

$$P_n(t_n) = R_n \exp(-R_n t_n) \qquad\qquad R_n = R_1 n^{\gamma}$$

Ignoring collision efficiency gives $\gamma = 4/3$. Including collision efficiency, $\gamma = 2$ may be a better model. What is

$$P(\tau)\ , \quad \tau = \frac{T}{\langle T \rangle}$$

Large deviation form

We are interested in the very small probability that the droplet grows explosively in a very short time. What is the (dimensionless) time τ^* at which the probability to grow to size $\mathcal{N}$ is

$$P(\tau^*) = \frac{1}{\mathcal{N}} \ , \quad \tau = \frac{T}{\langle T \rangle}$$

This is a large-deviation problem: introduce an 'entropy'

$$P(T) = \frac{1}{\langle T \rangle} \exp[-J(\tau)] \ , \qquad \tau = \frac{T}{\langle T \rangle}$$

Dimensionless time for onset of rainfall is solution of:

$$J(\tau^*) = \ln \mathcal{N}$$

Bromwich integral solution

Mean time converges when $\gamma > 1$: raindrops result from cases where the first few collisions where unusually rapid.

$$\lim_{\mathcal{N}\to\infty} \langle T \rangle = \lim_{\mathcal{N}\to\infty} \frac{1}{R_1} \sum_{n=1}^{\mathcal{N}} \frac{1}{f(n)} = \frac{1}{R_1}\zeta(\gamma)$$

Laplace transform in terms of explicit cumulent:

$$\exp[-\lambda(k)] = \langle \exp(-kT) \rangle = \int_0^\infty \mathrm{d}T\ P(T)\ \exp(-kT)$$

$$\lambda(k) = -\sum_{n=1}^{\mathcal{N}} \ln\langle \exp(-kt_n)\rangle = \sum_{n=1}^{\mathcal{N}} \ln\left(1 + \frac{k}{R_n}\right)$$

Laplace inverted numerically by Bromwich integral:

$$P(T) = \frac{1}{2\pi \mathrm{i}} \int_{C-\mathrm{i}\infty}^{C+\mathrm{i}\infty} \mathrm{d}z\ \exp[zT - \lambda(z)]$$

Saddle-point approximation

Seek a stationary point of the exponent in Bromwich integral: relates time to saddle point:

$$T = \sum_{n=1}^{\mathcal{N}} \frac{1}{R_n + k^*}$$

Saddle point approximation:

$$P(T) = \frac{1}{R_1} \frac{1}{\sqrt{2\pi \mathcal{S}(k^*)}} \exp[-J(\tau)] \qquad \tau = T/\langle T \rangle$$

Dimensionless parametric equations for entropy:

$$\tau(\kappa) = \frac{1}{\zeta(\gamma)} \sum_{n=1}^{\infty} \frac{1}{\kappa + n^\gamma} \qquad J(\kappa) = \sum_{n=1}^{\mathcal{N}} \ln\left(1 + \kappa n^{-\gamma}\right) - \kappa\tau$$

Asymptotics of asymptotics

Small τ asymptotics of saddle requires large κ form of:

$$S(\kappa) = \sum_{n=1}^{\infty} \ln\left(1 + \kappa n^{-\gamma}\right)$$

Some analysis gives:

$$S \sim S_0 - \frac{1}{2}\ln(\kappa) - \gamma C + O(\kappa^{-1}) \qquad S_0 = \int_0^{\infty} \mathrm{d}n \;\ln(1 + \kappa n^{-\gamma})$$

$$C = \lim_{n\to\infty}\left[(n-1)\ln\left(\frac{n-1}{n}\right) + 1\right] - \frac{\ln(n)}{2} \approx 0.91896611$$

Other asymptotics follow from this, e.g.:

$$T(\kappa) \sim A(\gamma)\kappa^{-\frac{\gamma-1}{\gamma}} + \frac{1}{2\kappa} \qquad A(\gamma) = \int_0^{\infty} \mathrm{d}x \;\frac{x^{\frac{\gamma-1}{\gamma}}}{1+x} = \frac{1}{\gamma}\beta(\frac{1}{\gamma}, 1 - \frac{1}{\gamma})$$

Explicit asymptotic theory

Seek a stationary point of the exponent in Bromwich integral: relates time to saddle point:

$$T = \sum_{n=1}^{\mathcal{N}} \frac{1}{R_n + k^*}$$

Saddle point approximation:

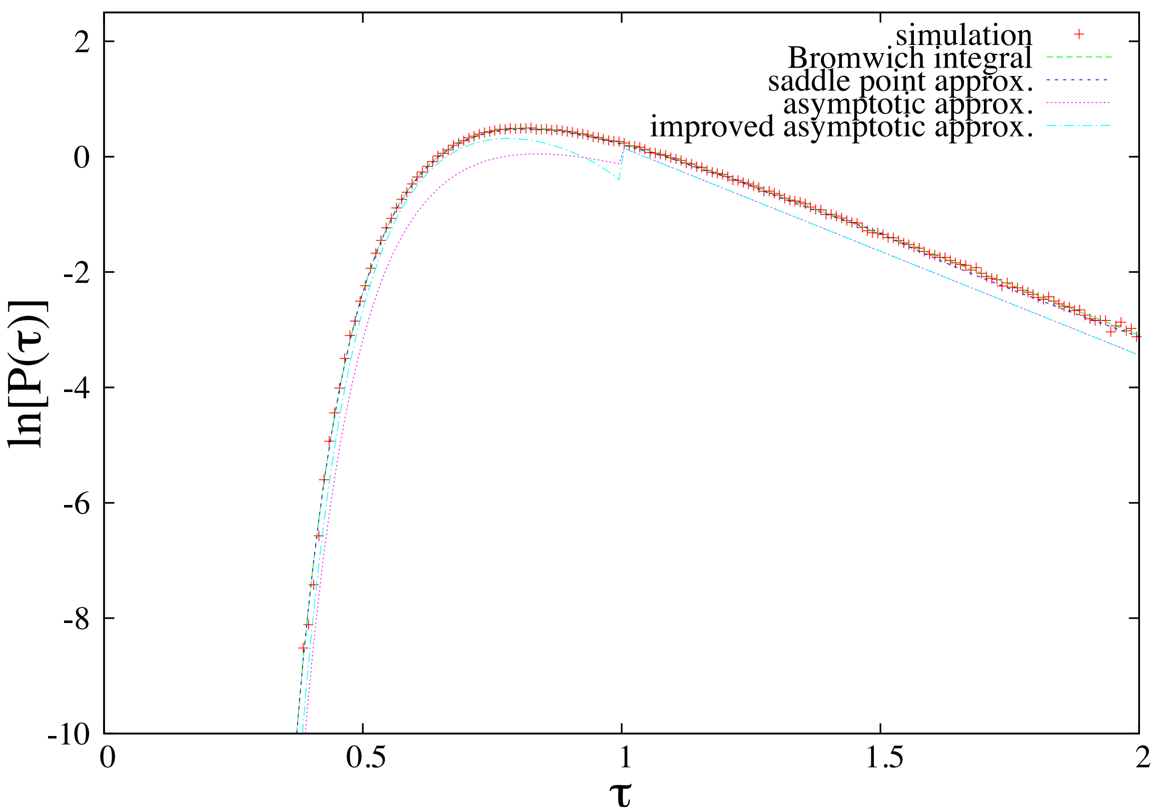
$$P(T) = \frac{1}{R_1} \frac{1}{\sqrt{2\pi \mathcal{S}(k^*)}} \exp[-J(\tau)] \qquad \tau = T/\langle T \rangle$$

Analysis of sums leads to an asymptotic formula

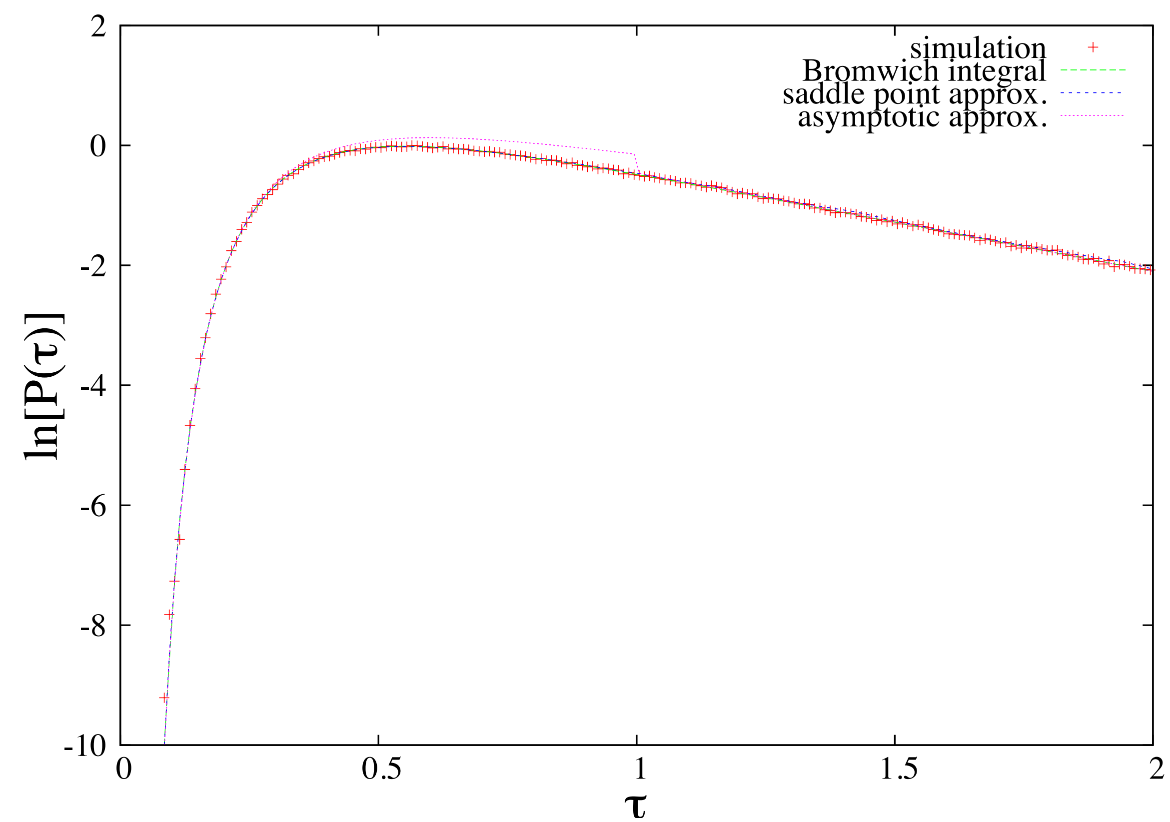
$$P(\tau) = K \tau^{-\frac{3\gamma-1}{2(\gamma-1)}} \exp\left(-\mathcal{C}/\tau^{\frac{1}{\gamma-1}}\right)$$

Numerical results

- Monte-Carlo simulation
- Exact probability density from Bromwich integral
- Saddle-point approximation
- Explicit asymptote for very small time



$\gamma = 4/3$



$\gamma = 2$

Implications for rainfall

Consider $\gamma = 2$. The probability of undergoing $\mathcal{N} = 10^6$ collisions is approximately $P = 10^{-6}$ when $\tau^* = 0.12$, that is a shower is triggered after a small fraction of the timescale for the first collision:

$$T_{\text{shower}} = \frac{\zeta(\gamma)}{R_1}\tau^* \approx 0.19 T_1$$

A small fraction of particles undergoes an enormous number of collisions in a fraction of the mean time for the first collision.

Surprisingly the time to make a shower decreases as the number of collisions increases:

$$\tau \sim (\ln \mathcal{N})^{-(\gamma-1)}$$

Summary

Since Saffman and Turner’s classic paper in 1956, a vast literature has developed on whether turbulence enhances droplet collision rates in clouds. Recent numerical evidence shows that ‘caustics’ are the dominant mechanism, but in most clouds their effect is insignificant.

An alternative view is that raindrops are the result of rare cases where droplets undergo a succession of rapid collisions. Large deviation theory is the appropriate tool. Surprisingly, the more droplets have to collide, the shorter the time for the onset of rainfall.