# Universal Critical Behavior of Noisy Coupled Oscillators

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We study the universal thermodynamic properties of systems consisting of many coupled oscillators operating in the vicinity of a homogeneous oscillating instability. In the thermodynamic limit, the Hopf bifurcation is a dynamic critical point far from equilibrium described by a statistical field theory. We perform a perturbative renormalization group study, and show that at the critical point a generic relation between correlation and response functions appears. At the same time, the fluctuation-dissipation relation is strongly violated.

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The collective behavior of many interacting elements generally leads to transitions and critical points in the large-scale and long-time properties of complex systems. This is well understood in the study of systems at thermodynamic equilibrium [1–3]. Nonequilibrium critical behaviors have been studied in a number of systems [4–6] but remain a serious challenge. An important example for criticality far from thermodynamic equilibrium is the behavior of coupled oscillators in the vicinity of a continuous homogeneous oscillatory instability or supercritical Hopf bifurcation. Such instabilities are important in many physical, chemical, and biological systems [7,8].

In this Letter, we apply concepts of the theory of dynamic critical points to study the generic properties of systems of coupled oscillators in the thermodynamic limit. In particular, we discuss linear response and two-point correlation functions defined for the oscillator ensemble. Since the system is far from a thermodynamic equilibrium, the fluctuation-dissipation (FD) relation between correlation and response functions in equilibrium systems is broken. We show that a Hopf bifurcation represents a nonequilibrium critical point and study the universal behaviors characterizing its approach from the nonoscillating state. We apply field theoretic renormalization group (RG) methods and develop an RG procedure which is appropriate for the case of a spontaneously oscillating system. This RG is performed in an oscillating reference frame with a scale-dependent oscillation frequency. The RG fixed points characterize the universal critical properties of locally coupled oscillators. We find that at the critical point of a Hopf bifurcation, an FD relation is formally satisfied if the system is described within the oscillating reference frame. In terms of physical variables, the FD relation is strongly broken but a relationship between correlation and response functions appears. Even though our calculations are performed in a $d = 4 - \epsilon$ dimensional space, we suggest that the main features of our results apply to Hopf bifurcations in general.

The generic behavior of an oscillator in the vicinity of a supercritical Hopf bifurcation can be described by a dynamic equation for a complex variable $Z$, which characterizes the phase and amplitude of the oscillations [7]. This variable can be chosen such that its real part is, to linear order, related to a physical observable, e.g., the displacement $X(t)$ generated by a mechanical oscillator: $X(t) = \mathrm{Re}[Z(t)] +$ nonlinear terms. In the presence of a periodic stimulus force $F(t) = \tilde{F}e^{-i\omega t}$ with a frequency $\omega$ close to the oscillation frequency at the bifurcation $\omega_0$, the generic dynamics obeys [9]

$$\partial_t Z = -(r + i\omega_0)Z - (u + iu_a)|Z|^2 Z + \Lambda^{-1} e^{i\theta} F(t). \quad (1)$$

For $F = 0$ and $r > 0$, the static state $Z = 0$ is stable. The system undergoes a Hopf bifurcation at $r = 0$ and exhibits spontaneous oscillations for $r < 0$. The nonlinear term characterized by the coefficients $u$ and $u_a$ stabilizes the oscillation amplitude for $u > 0$. The external stimulus appears linearly in this equation and couples in general with a phase shift $\theta$. In the case of a mechanical oscillator, the coefficient $\Lambda$ has units of a friction. From the point of view of statistical physics, the Hopf bifurcation is a critical point and Eq. (1) characterizes the corresponding mean field theory. Indeed, at $r = 0$ and $\omega = \omega_0$ and in terms of the amplitude $\tilde{X}$ of the limit cycle $Z(t) = \tilde{X}e^{-i\omega t}$, the system exhibits a power-law response $|\tilde{X}| \simeq |\tilde{F}|^{1/\delta}$ where $\delta = 3$ is a mean field critical exponent. For frequency differences $|\omega - \omega_0| \gg \Lambda^{-2/3}|\tilde{F}|^{2/3} \times |u + iu_a|^{1/3}$, the response becomes linear with $|\tilde{X}| \simeq \Lambda^{-1}|\tilde{F}|/|\omega - \omega_0|$.

In the presence of fluctuations, the critical point of an individual oscillator is concealed in the same way as finite-size effects destroy a phase transition in equilibrium thermodynamics. However, a true critical point can exist in a thermodynamic limit where many oscillators, distributed on a lattice in a $d$-dimensional space, are coupled by nearest-neighbor interactions. The combined system undergoes a dynamic phase transition at which all oscillators synchronize and an order parameter, which characterizes the global phase and amplitude of oscillators, becomes nonzero. Since at the critical point the correlation length diverges and oscillators become synchronized over large distances, a discrete model can, on large scales, be described by a continuum field theory which characterizes the universal features of the critical point [1,10]. In the case of locally coupled oscillators, this field theory is given by the complex Ginzburg-Landau equation [11] with fluctuations

$$\partial_t Z = -(r + i\omega_0)Z + (c + ic_a)\Delta Z - (u + iu_a)|Z|^2 Z + \Lambda^{-1} e^{i\theta} F + \eta. \tag{2}$$

Here, $Z(\mathbf{x}, t)$ becomes a complex field defined at positions $\mathbf{x}$ in a $d$-dimensional space and $\Delta$ denotes the Laplace operator. The coefficients $c$ and $c_a$ characterize the local coupling of oscillators and the effects of fluctuations are described via a complex random forcing term $\eta(\mathbf{x}, t)$. For a vanishing external field $F(\mathbf{x}, t)$ and in the absence of fluctuations, Eq. (2) is invariant with respect to phase changes of the oscillations $Z \to Ze^{i\phi}$. As far as long-time and long wavelength properties are concerned, $\eta$ can be chosen Gaussian and white with the correlations $\langle \eta(\mathbf{x}, t)\eta(\mathbf{x}', t')\rangle = 0$ and $\langle \eta(\mathbf{x}, t)\eta^*(\mathbf{x}', t')\rangle = 4D\delta^d(\mathbf{x} - \mathbf{x}')\delta(t - t')$, which respect phase invariance.

The linear response function $\chi_{\alpha\beta}$ and the two-point autocorrelation function $C_{\alpha\beta}$ of this field theory are defined by $\langle \psi_\alpha(\mathbf{x}, t)\rangle = \int d^d x' dt' \chi_{\alpha\beta}(\mathbf{x} - \mathbf{x}', t - t') \times F_\beta(\mathbf{x}', t') + O(|F|^2)$ and $C_{\alpha\beta}(\mathbf{x} - \mathbf{x}', t - t') = \langle \psi_\alpha(\mathbf{x}, t) \times \psi_\beta(\mathbf{x}', t')\rangle_c$. Here we have expressed $Z = \psi_1 + i\psi_2$ and $F = F_1 + iF_2$ by their real and imaginary parts, and $\langle \ldots \rangle_c$ denotes a connected correlation function. Because of phase invariance, these functions obey symmetry relations, e.g., $C_{11} = C_{22}$, $C_{21} = -C_{12}$. In the following, we focus for simplicity on the elements $C \equiv C_{11}$ and $\chi \equiv \chi_{11}$, which characterize the correlation and response of the observable $X$.

It is convenient to eliminate the frequency $\omega_0$ from Eq. (2) by a time-dependent variable transformation $Y \equiv e^{i\omega_0 t} Z$, $H \equiv e^{i\omega_0 t}\Lambda^{-1}e^{i\theta}F$, and $\zeta \equiv e^{i\omega_0 t}\eta$. This leads to the amplitude equation

$$\partial_t Y = -rY + (c + ic_a)\Delta Y - (u + iu_a)|Y|^2 Y + H + \zeta, \tag{3}$$

where the noise $\zeta$ has the same correlators as $\eta$. For the particular case $c_a = 0$ and $u_a = 0$, Eq. (3) becomes identical to the model A dynamics of a real Ginzburg-Landau field theory with an $O(2)$ symmetry of the order parameter [3]. The critical behavior of this theory at thermodynamic equilibrium has been extensively studied [12]. This leads, in this particular case, to a formal analogy between an equilibrium phase transition and a Hopf bifurcation. The correlation and response functions $C_{\alpha\beta}$ and $\chi_{\alpha\beta}$ can here be obtained from those of the equilibrium field theory by using the time-dependent variable transformation given above. Since the theory at thermodynamic equilibrium obeys an FD relation, a generic relation between the correlation and response functions $C_{\alpha\beta}$ and $\chi_{\alpha\beta}$ appears. This special case provides a further example of an equilibrium universality class found in a nonequilibrium dynamics with nonconserved order parameter [6]. It is the case, e.g., for the model A dynamics of the real Ginzburg-Landau theory with a $Z_2$ symmetry [13], even when the symmetry is broken by the nonequilibrium perturbations [14], and for some of its generalizations to the $O(n)$ symmetry [15].

This raises the question of whether the equilibrium universality class also characterizes the general case where $u_a$ and $c_a$ are finite. Dimensional analysis reveals that for $d > 4$, mean field theory applies. In this case,

$$\chi^{\rm mf}(\mathbf{q}, \omega) = \frac{1}{2\Lambda}\left[\frac{e^{i\theta}}{R - i(\omega - \Omega_0)} + \frac{e^{-i\theta}}{R - i(\omega + \Omega_0)}\right]$$

$$C^{\rm mf}(\mathbf{q}, \omega) = \frac{D}{R^2 + (\omega - \Omega_0)^2} + \frac{D}{R^2 + (\omega + \Omega_0)^2}, \tag{4}$$

where $R = r + c\mathbf{q}^2$, $\Omega_0 = \omega_0 + c_a\mathbf{q}^2$ and where $\mathbf{q}$ and $\omega$ are wave vector and angular frequency, respectively.

For $d < 4$, mean field theory breaks down. We apply RG methods using an $\epsilon$ expansion near the upper critical dimension ($d = 4 - \epsilon$) [16]. Defining two real fields $\phi_\alpha$ by $Y = \phi_1 + i\phi_2$, Eq. (3) reads

$$\partial_t \phi_\alpha = -R_{\alpha\beta}\phi_\beta - U_{\alpha\beta}\phi_\beta\phi_\gamma\phi_\gamma + H_\alpha + \zeta_\alpha, \tag{5}$$

where $H = H_1 + iH_2$, $R_{\alpha\beta} = (r - c\Delta)\delta_{\alpha\beta} - c_a\Delta\varepsilon_{\alpha\beta}$, and $U_{\alpha\beta} = u\delta_{\alpha\beta} + u_a\varepsilon_{\alpha\beta}$, with $\epsilon_{21} = -\epsilon_{12} = 1$ and $\epsilon_{ij} = 0$ for $i = j$. We introduce the Martin-Siggia-Rose response field $\tilde{\phi}_\alpha$ [17] and apply the Janssen–De Dominicis formalism [18] to write a generating functional with action

$$S[\tilde{\phi}_\alpha, \phi_\alpha] = \int d^d x dt \{D\tilde{\phi}_\alpha\tilde{\phi}_\alpha - \tilde{\phi}_\alpha[\partial_t\phi_\alpha + R_{\alpha\beta}\phi_\beta] - U_{\alpha\beta}\tilde{\phi}_\alpha\phi_\beta\phi_\gamma\phi_\gamma\}. \tag{6}$$

Using a Callan-Symanzik RG scheme [1,2], we define the renormalized theory such that its effective action is of the form (6). This requires to introduce a phase shift $\delta\theta$ and a frequency shift $\delta\omega_0$ between the bare fields $(\phi_\alpha^0, \tilde{\phi}_\alpha^0)$ and the renormalized fields $(\phi_\alpha, \tilde{\phi}_\alpha)$:

$$\phi_\alpha^0(\mathbf{x}, t^0) = \Omega_{\alpha\beta}(-\delta\omega_0 t) Z_\phi^{1/2} Z_\omega \phi_\beta(\mathbf{x}, t)$$

$$\tilde{\phi}_\alpha^0(\mathbf{x}, t^0) = \Omega_{\alpha\beta}(-\delta\theta - \delta\omega_0 t) Z_{\tilde{\phi}}^{1/2} Z_\omega \tilde{\phi}_\beta(\mathbf{x}, t). \tag{7}$$

Here we have introduced $Z$ factors for the renormalization of the fields and the time ($t^0 = Z_\omega^{-1} t$), and $\Omega_{\alpha\beta}(\theta)$ denotes the rotation matrix by an angle $\theta$ in two dimensions. We furthermore introduce dimensionless coupling constants $g$ and $g_a$ and a scale factor $\mu$ by $u = \mu^\epsilon(4\pi)^{-\epsilon/2} g$ and $u_a = \mu^\epsilon(4\pi)^{-\epsilon/2} g_a$. The bare and renormalized quantities are now related depending on $\mu$ [19]. We define the correlation and response functions $G_{\alpha\beta} = \langle \phi_\alpha\phi_\beta\rangle_c$ and

$\gamma_{\alpha\beta} = \langle \phi_\alpha \tilde{\phi}_\beta \rangle_c$. They are related to the physical observables $C_{\alpha\beta}$ and $\chi_{\alpha\beta}$ via [20]

$$\chi_{\alpha\beta}(\mathbf{x}, t^0) = \Lambda^{-1}\Omega_{\alpha\sigma}(\theta - \omega_0 t)(Z_\phi Z_{\tilde{\phi}})^{1/2} Z_\omega^2 \gamma_{\sigma\beta}(\mathbf{x}, t)$$
$$C_{\alpha\beta}(\mathbf{x}, t^0) = \Omega_{\alpha\sigma}(-\omega_0 t) Z_\phi Z_\omega^2 G_{\sigma\beta}(\mathbf{x}, t). \quad (8)$$

The dependence of the renormalized parameters $g$, $g_a$, and $c_a$ on $\mu$ defines three $\beta$ functions. Writing $\vec{g} = (g, g_a, c_a)$, we have $\vec{\beta}(\vec{g}, \epsilon) = \mu(\partial_\mu \vec{g})_0$, where $\vec{\beta} = (\beta, \beta_a, \beta_c)$ and $(\partial_\mu)_0$ denotes differentiation with fixed $u^0$, $u_a^0$, and $c_a^0$.

To one-loop order in perturbation theory (see Fig. 1 for examples of Feynman diagrams of the theory), only $\omega_0$, $r$, $g$, and $g_a$ are renormalized. The two nontrivial $\beta$ functions are given by:

$$\beta_a \simeq -\epsilon g_a - \mathcal{D}\left[\frac{c_a}{1 + c_a^2}(g^2 - g_a^2 + 2 g g_a c_a) - 6 g g_a\right]$$
$$\beta \simeq -\epsilon g - \mathcal{D}\left[\frac{g_a^2 - g^2 - 2 g g_a c_a}{1 + c_a^2} - 4 g^2\right], \quad (9)$$

where $\mathcal{D} = 4D/(4\pi)^2$. The RG fixed point corresponds to the values $\vec{g}^*$ of the parameters $\vec{g}$ for which the three $\beta$ functions are simultaneously zero. Since $\beta_c = 0$ to one-loop order, one condition is lacking to fully determine $\vec{g}^*$. Choosing $c_a$ as a parameter, we obtain $g^* \simeq \epsilon/5\mathcal{D}$ and $g_a^* \simeq c_a \epsilon/5\mathcal{D}$. In order to determine completely the fixed point $\vec{g}^*$, we need to go to two-loop order in perturbation theory. To this order, all parameters (apart from $c$ and $D$), the fields and the time are renormalized and explicit expressions for all the Wilson's functions and $Z$ factors of the theory can be obtained.

Only one fixed point exists that describes the universality class of Hopf bifurcations. It obeys $c_a^* = 0$, $g_a^* = 0$ and is infrared-stable. This fixed point is formally equivalent to the one of the real Ginzburg-Landau theory with $O(2)$ symmetry. As a consequence, we find $\nu \simeq 1/2 + \epsilon/10$, $\eta \simeq \epsilon^2/50$, and $z \simeq 2 + \epsilon^2[6\ln(4/3) - 1]/50$, which are the corresponding equilibrium critical exponents. Here, $\nu$ denotes the exponent characterizing the divergence of the correlation length $\xi$, $z$ is the dynamic exponent, and $\eta$ denotes the exponent characterizing the field renormalization [1,3].

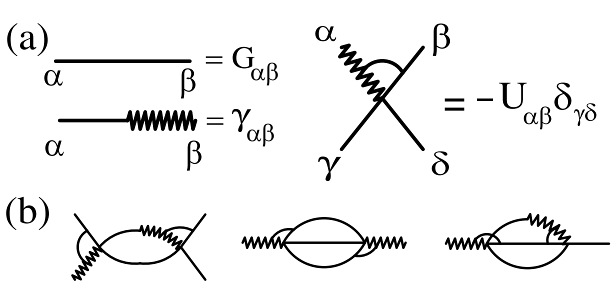


FIG. 1. (a) Graphic representation of the propagators $G_{\alpha\beta}$ and $\gamma_{\alpha\beta}$ and of the vertex $U_{\alpha\beta}\delta_{\gamma\delta}$. (b) Examples of Feynman diagrams of the theory to one and two-loop order.

The RG flow in the vicinity of the fixed point, however, is defined here in a larger parameter space as the one corresponding to the $O(2)$ dynamic model. Furthermore, the effective theory discussed here is expressed in an oscillating reference frame with scale-dependent frequency and phase. Therefore the correlation and response functions $C_{\alpha\beta}$ and $\chi_{\alpha\beta}$ differ from those of the equilibrium model and additional universal exponents appear. We can derive generic expressions for these functions using the RG flow of all parameters in the vicinity of the critical point and employing a matching procedure [21]. For example, we find for $q\xi \gg 1$ and for stimulation at the effective oscillation frequency $\omega_0^{\mathrm{eff}}$:

$$\chi(q, \omega = \omega_0^{\mathrm{eff}}) \simeq \frac{1}{q^{2-\eta}} \frac{1}{2\Lambda_{\mathrm{eff}}}\left[\frac{e^{i\theta(q)}}{1 + i\gamma(q)}\right], \quad (10)$$

where we have introduced the functions $\theta(q) \simeq \theta_{\mathrm{eff}} + \alpha_{\mathrm{eff}} q^{\omega_1} + \beta_{\mathrm{eff}} q^{\omega_2}$ and $\gamma(q) \simeq \gamma_{\mathrm{eff}} q^{\omega_2}$, and nonuniversal effective quantities denoted by the index "eff." The universal exponents $\omega_1 \simeq \epsilon/5$ and $\omega_2 \simeq \epsilon^2/50$ here are characteristic for a Hopf bifurcation. Similarly, we find expressions for the correlation function:

$$C(q, \omega = \omega_0^{\mathrm{eff}}) \simeq \frac{1}{q^{z+2-\eta}} \frac{D_{\mathrm{eff}}}{1 + \gamma(q)^2}, \quad (11)$$

and for the frequency dependence of the homogeneous mode $q = 0$ in the regime $(\omega - \omega_0^{\mathrm{eff}})\xi^z \gg 1$ [22].

Because of the formal analogy of the RG fixed point discussed here with the one of an equilibrium field theory, an FD relation appears exactly at the critical point and relates the functions $G_{\alpha\beta}$ and $\gamma_{\alpha\beta}$. Since the physical correlation and response functions $C_{\alpha\beta}$ and $\chi_{\alpha\beta}$ can be determined from $G_{\alpha\beta}$ and $\gamma_{\alpha\beta}$ using the scale-dependent variable transformations of Eq. (8), a relation between correlation and response functions appears:

$$\cos\theta_{\mathrm{eff}}\chi_{11}'' - \sin\theta_{\mathrm{eff}}\chi_{12}'' = \frac{1}{2\Lambda_{\mathrm{eff}} D_{\mathrm{eff}}}(\omega C_{11} + i\omega_0^{\mathrm{eff}} C_{12})$$
$$\cos\theta_{\mathrm{eff}}\chi_{12}' + \sin\theta_{\mathrm{eff}}\chi_{11}' = \frac{1}{2\Lambda_{\mathrm{eff}} D_{\mathrm{eff}}}(\omega_0^{\mathrm{eff}} C_{11} + i\omega C_{12}). \quad (12)$$

Here, $\chi_{\alpha\beta} = \chi_{\alpha\beta}' + i\chi_{\alpha\beta}''$ has been separated in its real and imaginary parts. The relation (12) is asymptotically satisfied in the long-time and wavelength limits at the critical point.

The physical correlation and response functions $C_{\alpha\beta}$ and $\chi_{\alpha\beta}$, however, do not obey the equilibrium FD relation. The degree of this violation can be characterized by a frequency-dependent effective temperature $T_{\mathrm{eff}}$:

$$\frac{T_{\mathrm{eff}}(\omega)}{T} = \frac{\omega}{2k_B T} \frac{C_{11}(\omega, q = 0)}{\chi_{11}''(\omega, q = 0)}. \quad (13)$$

Here, $k_B$ denotes the Boltzmann constant and $T$ is the temperature of the system. We find that $T_{\rm eff}/T \sim (\omega - \omega_0^{\rm eff})^{-\sigma}$ diverges at the critical point with a universal exponent $\sigma$. For the particular case $c_a = 0$ and $u_a = 0$, $\sigma = 1$, while otherwise $\sigma \simeq 1 - \epsilon/5$ to first order in $\epsilon$. The power-law divergence of $T_{\rm eff}$ as a function of frequency reveals a violent breaking of the FD relation. This divergence at the oscillation frequency has been experimentally observed on a single active oscillating system [23].

We have shown that the critical point in $d = 4 - \epsilon$ dimensions of locally coupled oscillators is formally related to the equilibrium phase transition in the $XY$ model. From this analogy it follows that on the oscillating side of the Hopf bifurcation, the system of coupled oscillators exhibits long range phase order and coherent oscillations for $d > 2$. We can speculate how our results are modified in lower dimensions $d$. In analogy with the equilibrium $XY$ model, we expect the phase order of the oscillations to vanish for $d < 2$, and to be quasilong range exactly at the lower critical dimension $d = 2$. In the last case, spectral peaks on the oscillating side of the Hopf bifurcation are expected to exhibit power-law tails with nonuniversal exponents. If the formal analogy with the equilibrium critical point found here in $d = 4 - \epsilon$ persists in $d = 2$, we would expect to see features of the Kosterlitz-Thouless universality class [24] in systems of coupled oscillators in this dimension.

Critical oscillators are ideally suited for nonlinear signal detection and amplification. Indeed, close to the critical frequency, the linear response function exhibits divergent behaviors, indicative of a high sensitivity of the system. It has been suggested that the ear of vertebrates uses critical oscillations of mechanosensitive hair cells for the detection of weak sounds and that the properties of the critical point provide the basis to explain the observed compressive nonlinear response to mechanical stimuli and to frequency selectivity in the ear [9,25]. The correlation and response functions of single mechanosensory hair bundles have been determined experimentally [23]. These single cell experiments detected vibrations at the scale of tens of nanometers. There, the Hopf bifurcation was concealed by finite-size effects but its signature could be observed. In the cochlea of mammals, power-law responses over several orders of magnitude have been seen [26]. This suggests that in such systems, a large number of oscillating degrees of freedom operate collectively and bring the system closer to true criticality.

The critical oscillations discussed here can in principle be realized in artificial systems. Nanotechnology aims to build functional units on the submicrometer scale. Large arrays of nanorotators or oscillators on patterned substrates coupled to their neighbors by elastic or viscous effects would provide a two-dimensional realization of our field theory. This could permit in the future experimental studies of the critical phenomena discussed here.

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