An Information Theoretic Analysis of Sequential Decision-Making

Meik Dörpinghaus*§, Édgar Roldán†§‖, Izaak Neri†‡§, Heinrich Meyr*§¶, and Frank Jülicher†§
*Vodafone Chair Mobile Communications Systems, Technische Universität Dresden,
Dresden, Germany, meik.doerpinghaus@tu-dresden.de
†Max-Planck-Institute for the Physics of Complex Systems, Dresden, Germany, {edgar, izaak, julicher}@pks.mpg.de
‡Max-Planck-Institute of Molecular Cell Biology and Genetics, Dresden, Germany
§Center for Advancing Electronics Dresden cfAED, Technische Universität Dresden, Germany
¶Institute for Integrated Signal Processing Systems, RWTH Aachen University, Germany, meyr@iss.rwth-aachen.de
‖GISC – Grupo Interdisciplinar de Sistemas Complejos, Madrid, Spain

***Abstract*—We provide a novel analysis of Wald's sequential probability ratio test based on information theoretic measures for symmetric thresholds, symmetric noise, and equally likely hypotheses. This test is optimal in the sense that it yields the minimum mean decision time. To analyze the decision-making process we consider information densities, which represent the stochastic information content of the observations yielding a stochastic termination time of the test. Based on this, we show that the conditional probability to decide for hypothesis $\mathcal{H}_1$ (or the counter-hypothesis $\mathcal{H}_0$) given that the test terminates at time instant k is independent of time k. An analogous property has been found for a continuous-time first passage problem with two absorbing boundaries in the contexts of non-equilibrium statistical physics and communication theory. Moreover, we study the evolution of the mutual information between the binary variable to be tested and the output of the Wald test. Notably, we show that the decision time of the Wald test contains no information on which hypothesis is true beyond the decision outcome.**

I. Introduction

In many decision problems it is important to make decisions as fast as possible but with a given reliability. This problem has been first studied by A. Wald who introduced a sequential probability ratio test to enable fast decisions between two possible hypotheses [1]. For independent and identically distributed observations this test yields the minimum mean decision time for a decision with a given probability of error [2]. The test accumulates the likelihood ratio given by the sequence of time discrete observations. The decision for one of the hypotheses is made as soon as the cumulative likelihood ratio reaches a given threshold which depends on the required decision reliability. The Wald test therefore corresponds to a first-passage problem with two absorbing boundaries in discrete time. The key characteristics of the Wald sequential test is that its termination time is a random quantity that depends on the actual realization of the random sequence of observations.

Sequential probability ratio tests can also be studied in the limit where observations occur continuously over time. In this situation, the probability ratio test becomes a continuous-time first passage problem. For continuous processes an important property of this first passage problem is that the threshold is hit exactly at a specific time. For certain systems described by a Langevin equation, it was shown previously that the probability for the process to be terminated at one of the two symmetric boundaries is independent of the time of absorption, see, e.g., [3], [4].

In the present paper, we present a novel analysis of the Wald test based on information theoretic measures for symmetric thresholds, symmetric noise, and equally likely hypotheses. We show that for such a discrete-time sequential probability ratio test the probability for the test to be terminated at one of two boundaries is, as in the time continuous case, independent of the time at which the decision is taken. This result is found even though the cumulative likelihood ratio typically does not hit the threshold exactly but can overshoot. To obtain this result, we describe the behavior of the test by a recursive expression for information densities. This measure describes the statistical dependencies for every individual sample process. Differently, mutual information fails to analyze the realization dependent termination behavior as it takes an average over all possible realization of the observation process. Using this recursive equation we show that a key property of the Wald test is that the conditional probability to decide for hypothesis $\mathcal{H}_1$ (or the counter-hypothesis $\mathcal{H}_0$) given that the test terminates at time instant k is independent of time k. Moreover, we show that the decision time τ of the Wald test contains no information on which hypothesis is true beyond the decision outcome. Finally, we provide an expression characterizing the evolution of the mutual information between the binary variable to be tested and the decision outcome of the Wald test.

II. System Model

We consider the following decision problem on the binary random variable $\mathsf{X} \in \{-1, 1\}$ based on a sequence of noisy observations of X where the kth observation is given by

$$\mathsf{Y}_k = \sqrt{\rho}\mathsf{X} + \mathsf{Z}_k, \qquad k \in \mathbb{N}. \tag{1}$$

Here, Z_k is additive noise with zero mean and unit variance. The individual noise samples are assumed to be independent identically distributed (i.i.d.) with density p_{Z}. In addition, we assume that the noise distribution is symmetric with respect to zero, i.e., $p_{\mathsf{Z}}(-z) = p_{\mathsf{Z}}(z)$ which is for example fulfilled by a zero-mean Gaussian distribution. Moreover, the parameter

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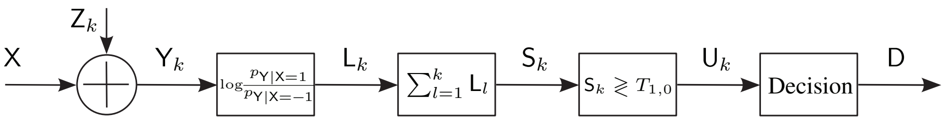

Fig. 1. Extended model of the Wald test

ρ can be interpreted as the signal-to-noise ratio. We consider equally likely hypotheses $P(\mathsf{X}=1)=P(\mathsf{X}=-1)=\frac{1}{2}$.

The aim is to decide as fast as possible, i.e., with the lowest possible number of observations Y_k, if $\mathsf{X}=1$ (hypothesis $\mathcal{H}_1$) or if $\mathsf{X}=-1$ (hypothesis $\mathcal{H}_0$) with a given reliability. In his seminal paper [1] Wald solved this problem by providing a sequential probability ratio test (the *Wald* test), which is optimal in the sense that it minimizes the mean decision time for a given reliability [2]. The decision time τ is itself a random variable, which depends on the actual realization of the observation sequence. For this purpose, the Wald test collects observations Y_k until the cumulated log-likelihood ratio

$$\mathsf{S}_k=\sum_{l=1}^{k}\mathsf{L}_k=\sum_{l=1}^{k}\log\left(\frac{p_{\mathsf{Y}|\mathsf{X}}(\mathsf{Y}_l|\mathsf{X}=1)}{p_{\mathsf{Y}|\mathsf{X}}(\mathsf{Y}_l|\mathsf{X}=-1)}\right) \tag{2}$$

exceeds (falls below) a prescribed threshold T_1 (T_0). The decision time τ is the minimum value of k for which $\mathsf{S}_k\notin(T_0,T_1)$. The test decides for $\mathcal{H}_1$ ($\mathcal{H}_0$) when S_k first crosses T_1 (T_0), where $\mathsf{D}=1$ ($\mathsf{D}=-1$) is the decision of the test. In (2), $p_{\mathsf{Y}|\mathsf{X}}$ denotes the probability density function of the observations Y_k conditioned on the event X. The thresholds T_1 and T_0 depend on the maximum allowed probabilities for making a wrong decision $\alpha_1\geq P(\mathsf{D}=1|\mathsf{X}=-1)$ and $\alpha_0\geq P(\mathsf{D}=-1|\mathsf{X}=1)$. Here, $P(\mathsf{D}=1|\mathsf{X}=-1)$ ($P(\mathsf{D}=-1|\mathsf{X}=1)$) denotes the probability that the test decides for hypothesis $\mathcal{H}_1$ ($\mathcal{H}_0$) although $\mathcal{H}_0$ ($\mathcal{H}_1$) is true. The thresholds T_1 and T_0 are functions of the maximum allowed error probabilities α_1 and α_0. As their determination is rather involved, in the following we use the approximations $T_0\simeq\log\frac{\alpha_0}{1-\alpha_1}$ and $T_1\simeq\log\frac{1-\alpha_0}{\alpha_1}$ with $\alpha_0,\alpha_1<0.5$ [1]. This choice still guarantees that the actual error probabilities are not larger than the maximum allowed error probabilities. If the test terminates exactly on one of the thresholds, the actual error probabilities $P(\mathsf{D}=1|\mathsf{X}=-1)$ and $P(\mathsf{D}=-1|\mathsf{X}=1)$ coincide with the maximum allowed error probabilities α_1 and α_0. If the mean and the variance of the increments L_k are small in comparison to the thresholds the test ends close to one of the thresholds [1, pp. 132-133]. In the present paper we focus on the important special case of $\alpha_0=\alpha_1=\alpha$, yielding symmetric thresholds $T_1=-T_0=T$, and symmetric noise.

A. Three state representation of the Wald test

For the analysis of the Wald test we introduce the model in Fig. 1. Here, the ternary variable $\mathsf{U}_k\in\mathcal{U}=\{-1,\epsilon,1\}$ with

$$\mathsf{U}_k=\begin{cases}1 & \text{if } k\geq\tau \text{ and } \mathsf{D}=1\\ -1 & \text{if } k\geq\tau \text{ and } \mathsf{D}=-1\\ \epsilon & \text{if } k<\tau\end{cases} \tag{3}$$

and $\mathsf{U}_0=\epsilon$ describes the initial state of the Wald test. The state ϵ denotes the undecided state of the test. Until U_k reaches ±1 it is not possible to decide with the required reliability. The evolution of the state variable U_k can also be described by the trellis and the state transition diagram in Fig. 2.

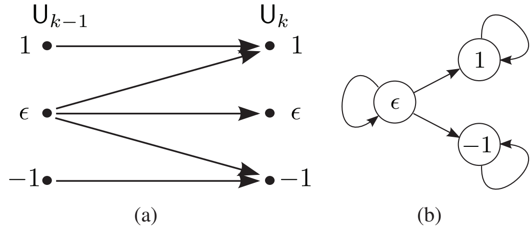

Fig. 2. Trellis (a) and state transition diagram (b) of U_k

To analyze the behavior of the Wald test we derive relations for the probability distribution of the state variables U_k. In the following, we assume without loss of generality that hypothesis $\mathsf{X}=1$ is true. This is possible as we consider symmetric thresholds and a symmetric noise distribution yielding

$$P(\mathsf{U}_k=1|\mathsf{X}=1)=P(\mathsf{U}_k=-1|\mathsf{X}=-1) \tag{4}$$

$$P(\mathsf{U}_k=1|\mathsf{X}=-1)=P(\mathsf{U}_k=-1|\mathsf{X}=1). \tag{5}$$

The probability that the Wald test has already made a decision for the hypothesis $\mathcal{H}_1$ ($\mathcal{H}_0$) corresponding to $\mathsf{X}=1$ ($\mathsf{X}=-1$) at the time instant k or before can be expressed as

$$\begin{aligned}P(\mathsf{U}_k=a|\mathsf{X}=1)&=P(\mathsf{U}_{k-1}=a|\mathsf{X}=1)\\&\quad+P(\mathsf{U}_k=a|\mathsf{U}_{k-1}=\epsilon,\mathsf{X}=1)P(\mathsf{U}_{k-1}=\epsilon)\end{aligned} \tag{6}$$

with $a\in\{1,-1\}$. For (6) we have used that

$$P(\mathsf{U}_{k-1}=\epsilon|\mathsf{X}=1)=P(\mathsf{U}_{k-1}=\epsilon) \tag{7}$$

which follows from

$$\begin{aligned}P(\mathsf{U}_{k-1}=\epsilon)&=\frac{1}{2}\{P(\mathsf{U}_{k-1}=\epsilon|\mathsf{X}=1)+P(\mathsf{U}_{k-1}=\epsilon|\mathsf{X}=-1)\}\\P(\mathsf{U}_{k-1}=\epsilon|\mathsf{X}=1)&=P(\mathsf{U}_{k-1}=\epsilon|\mathsf{X}=-1)\end{aligned} \tag{8}$$

as we assume that both events $\mathsf{X}=1$ and $\mathsf{X}=-1$ are equally likely. Additionally (8) follows from the assumption of a symmetric noise distribution p_{Z} with zero mean. Thus, (6) provides a recursive relation for $P(\mathsf{U}_k=a|\mathsf{X}=1)$. The initial distribution is given by $P(\mathsf{U}_0=\epsilon)=1$ as at time instant $k=0$ no observation has been considered and the Wald test has not yet decided for one of the hypothesis. With (6) we get

$$P(\mathsf{U}_k=a|\mathsf{X}=1)=\sum_{l=1}^{k}P(\mathsf{U}_l=a|\mathsf{U}_{l-1}=\epsilon,\mathsf{X}=1)P(\mathsf{U}_{l-1}=\epsilon)$$

as $P(\mathsf{U}_0=a|\mathsf{X}=1)=0$. Note that $P(\mathsf{U}_k=a|\mathsf{X}=1)$ corresponds to the probability that the Wald test has terminated at the positive (negative) threshold *at any time instant up to the time instant* k. On the other hand, $P(\mathsf{U}_k=a|\mathsf{U}_{k-1}=\epsilon,\mathsf{X}=1)$ is the probability that the Wald test terminates at the positive (negative) threshold *at the time instant* k.

Finally, due to the assumption of symmetric thresholds, symmetric noise, and the assumption that $\mathsf{X} = 1$ and $\mathsf{X} = -1$ are equally likely, it can be shown that for all k

$$P(\mathsf{U}_k = 1) = P(\mathsf{U}_k = -1) \tag{9}$$

$$P(\mathsf{U}_k = 1 | \mathsf{U}_{k-1} = \epsilon) = P(\mathsf{U}_k = -1 | \mathsf{U}_{k-1} = \epsilon). \tag{10}$$

III. Information Theoretic Analysis

We study the mutual information between the binary input X and the sequence of decision variables $\mathsf{U}_0, \mathsf{U}_1, \dots, \mathsf{U}_k$ depending on the number of observations k considered for decision-making so far. In this regard, we also discuss the statistical dependency of the decision time τ and X. In the following, we denote the vector containing the sequence of decision variables up to the time slot k by $\mathbf{U}^k$, i.e., $\mathbf{U}^k = [\mathsf{U}_0, \mathsf{U}_1, \dots, \mathsf{U}_k]$.

Based on the chain rule for mutual information it holds that

$$I(\mathsf{X}; \mathbf{U}^k) = \sum_{l=1}^{k} I(\mathsf{X}; \mathsf{U}_l | \mathbf{U}^{l-1}). \tag{11}$$

The summands on the RHS of (11) can be expressed by

$$I(\mathsf{X}; \mathsf{U}_k | \mathbf{U}^{k-1}) = H(\mathsf{U}_k | \mathbf{U}^{k-1}) - H(\mathsf{U}_k | \mathbf{U}^{k-1}, \mathsf{X}) \tag{12}$$

where $H(\cdot)$ denotes the (Shannon) entropy.

As the additive noise samples Z_k are i.i.d. it holds that $p_{\mathsf{S}_k | \mathsf{S}_1, \dots, \mathsf{S}_{k-1}, \mathsf{X}} = p_{\mathsf{S}_k | \mathsf{S}_{k-1}, \mathsf{X}}$. This and the fact that U_k does not change anymore once it reaches ± 1 implies that $P(\mathsf{U}_k | \mathsf{U}_1, \dots, \mathsf{U}_{k-1}, \mathsf{X}) = P(\mathsf{U}_k | \mathsf{U}_{k-1}, \mathsf{X})$. Likewise we get $P(\mathsf{U}_k | \mathsf{U}_1, \dots, \mathsf{U}_{k-1}) = P(\mathsf{U}_k | \mathsf{U}_{k-1})$. Hence, it holds that

$$H(\mathsf{U}_k | \mathbf{U}^{k-1}) = H(\mathsf{U}_k | \mathsf{U}_{k-1}) \tag{13}$$

$$H(\mathsf{U}_k | \mathbf{U}^{k-1}, \mathsf{X}) = H(\mathsf{U}_k | \mathsf{U}_{k-1}, \mathsf{X}) \tag{14}$$

yielding

$$I(\mathsf{X}; \mathbf{U}^k) = \sum_{l=1}^{k} I(\mathsf{X}; \mathsf{U}_l | \mathsf{U}_{l-1}). \tag{15}$$

Thus, the increase in mutual information between the sequence $\mathbf{U}^k$ and X from the time $k-1$ to time k is given by

$$\begin{aligned} I(\mathsf{X}; \mathbf{U}^k) - I(\mathsf{X}; \mathbf{U}^{k-1}) &= I(\mathsf{X}; \mathsf{U}_k | \mathsf{U}_{k-1}) \\ &= I(\mathsf{X}; \mathsf{U}_k, \mathsf{U}_{k-1}) - I(\mathsf{X}; \mathsf{U}_{k-1}) \\ &= I(\mathsf{X}; \mathsf{U}_{k-1} | \mathsf{U}_k) + I(\mathsf{X}; \mathsf{U}_k) - I(\mathsf{X}; \mathsf{U}_{k-1}). \end{aligned} \tag{16}$$

where we applied twice the chain rule for mutual information.

Lemma 1. *For the Wald test on the model defined in Section II it holds that*

$$I(\mathsf{X}; \mathsf{U}_{k-1} | \mathsf{U}_k) = 0. \tag{17}$$

Proof. In Appendix A we show that

$$P(\mathsf{X} | \mathbf{U}^k) = P(\mathsf{X} | \mathsf{U}_k). \tag{18}$$

Using Bayes' rule and (18) we obtain

$$P(\mathsf{X}, \mathsf{U}_{k-1} | \mathsf{U}_k) = P(\mathsf{X} | \mathsf{U}_k) P(\mathsf{U}_{k-1} | \mathsf{U}_k). \tag{19}$$

Hence, X and U_{k-1} are conditionally independent and the following Markov chain holds $\mathsf{X} \leftrightarrow \mathsf{U}_k \leftrightarrow \mathsf{U}_{k-1}$. Thus, (17) immediately follows. □

Lemma 1 implies that U_k carries all information on X that is contained in U_{k-1}.

Using Lemma 1 and (16) it follows that

$$I(\mathsf{X}; \mathsf{U}_k) = I(\mathsf{X}; \mathsf{U}_{k-1}) + I(\mathsf{X}; \mathsf{U}_k | \mathsf{U}_{k-1}). \tag{20}$$

Eq. (20) describes the increase of the mutual information between X and the decision variable U_k over one time step. As U_k can be considered as the output of the Wald test at time k, $I(\mathsf{X}; \mathsf{U}_k)$ is the information the Wald test gives on X at time k averaged over all possible realizations of the Wald test.

Theorem 1. *Under the same conditions as in Lemma 1 the Wald test satisfies that*

$$I(\mathsf{X}; \tau | \mathsf{U}_k) = 0. \tag{21}$$

Proof. It holds that $P(\mathsf{X} | \mathbf{U}^k) = P(\mathsf{X} | \mathsf{U}_k, \tau)$ as the last element of $\mathbf{U}^k$ given by U_k and the decision time τ contain the same information as $\mathbf{U}^k$, cf. Fig. 2. Thus, with (18) eq. (21) follows immediately. □

Theorem 1 states that the decision time τ gives no additional information on X beyond U_k, i.e., $I(\mathsf{X}; \mathsf{U}_k, \tau) = I(\mathsf{X}; \mathsf{U}_k)$.

A. Information Densities

As the mutual information by definition is an average over all involved random quantities, (20) does not reflect the fact that the termination time of the Wald test depends on the actual realization of the observation sequence. Differently, (20) describes the behavior of averages over all observation sequences $\mathsf{Y}_1, \dots, \mathsf{Y}_k$. To be able to resolve the actual termination behavior, we need an expression reflecting the realization dependent termination time corresponding to (20) but being more restrictive in the sense that it holds for every individual observation process. For this purpose we state the following.

Corollary 1. *Under the same conditions as in Lemma 1 the following recursive expression for information densities holds*

$$i(\mathsf{X}; \mathsf{U}_k) = i(\mathsf{X}; \mathsf{U}_{k-1}) + i(\mathsf{X}; \mathsf{U}_k | \mathsf{U}_{k-1}) \tag{22}$$

where the information densities are defined as $i(\mathsf{X}; \mathsf{U}_k) = \log\left(\frac{P(\mathsf{U}_k|\mathsf{X})}{P(\mathsf{U}_k)}\right)$ *and* $i(\mathsf{X}; \mathsf{U}_k | \mathsf{U}_{k-1}) = \log\left(\frac{P(\mathsf{U}_k|\mathsf{U}_{k-1},\mathsf{X})}{P(\mathsf{U}_k|\mathsf{U}_{k-1})}\right)$ *[5].*

Proof. Eq. (22) for information densities is equivalent to

$$\frac{P(\mathsf{U}_k|\mathsf{X})}{P(\mathsf{U}_k)} = \frac{P(\mathsf{U}_{k-1}|\mathsf{X})}{P(\mathsf{U}_{k-1})} \frac{P(\mathsf{U}_k|\mathsf{U}_{k-1},\mathsf{X})}{P(\mathsf{U}_k|\mathsf{U}_{k-1})} \tag{23}$$

which follows from (19). □

Note that $I(\mathsf{X}; \mathsf{U}_k) = \mathrm{E}_{\mathsf{X},\mathsf{U}_k}[i(\mathsf{X}; \mathsf{U}_k)]$. Hence, we get (20) by taking the expectation of (22) with respect to all random quantities and, thus, (22) implies (20) but not vice versa.

To describe the behavior of the Wald test to terminate when the process of cumulative log-likelihood ratios S_k reaches one of the thresholds at $\pm T$, (22) must hold for all combinations of events of U_k, U_{k-1}, and X.

B. Time-Independence of Decision Probabilities

Corollary 1 allows to prove the following new Theorem.

Theorem 2. *For the system model in Section II the following holds. In case the Wald test terminates at time k the probability* *to decide for hypothesis $\mathcal{H}_1$ is independent of time, i.e.*

$$P(\mathsf{U}_k = 1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1, \mathsf{U}_k \neq \epsilon) = \kappa, \quad \forall k \in \mathbb{N} \quad (24)$$

with the constant $\kappa = P(\mathsf{D} = 1|\mathsf{X} = 1)$. Equivalently,

$$P(\mathsf{U}_k = -1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1, \mathsf{U}_k \neq \epsilon) = 1 - \kappa, \quad \forall k \in \mathbb{N}.$$

Lemma 2. *The statement in Theorem 2 is equivalent to the property that the ratio of the termination probabilities at both boundaries is independent of the time instant k, i.e.,*

$$\frac{P(\mathsf{U}_k = 1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1)}{P(\mathsf{U}_k = -1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1)} = \gamma, \quad \forall k \in \mathbb{N} \quad (25)$$

with γ being a positive constant.

Proof of Lemma 2. The Wald test fulfills (22) and, thus, (23) for all combinations of $\mathsf{X} \in \{-1, 1\}$, and $\mathsf{U}_k, \mathsf{U}_{k-1} \in \mathcal{U}$. Due to the symmetry of the problem we consider only the case $\mathsf{X} = 1$ w.l.o.g.. Evaluating (23) for different values of U_k and U_{k-1} yields

$$\frac{P(\mathsf{U}_k = 1|\mathsf{X} = 1)}{P(\mathsf{U}_k = 1)} = \frac{P(\mathsf{U}_{k-1} = 1|\mathsf{X} = 1)}{P(\mathsf{U}_{k-1} = 1)} \quad (26)$$

$$\frac{P(\mathsf{U}_k = 1|\mathsf{X} = 1)}{P(\mathsf{U}_k = 1)} = \frac{P(\mathsf{U}_k = 1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1)}{P(\mathsf{U}_k = 1|\mathsf{U}_{k-1} = \epsilon)} \quad (27)$$

$$\frac{P(\mathsf{U}_k = -1|\mathsf{X} = 1)}{P(\mathsf{U}_k = -1)} = \frac{P(\mathsf{U}_k = -1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1)}{P(\mathsf{U}_k = -1|\mathsf{U}_{k-1} = \epsilon)} \quad (28)$$

$$\frac{P(\mathsf{U}_k = -1|\mathsf{X} = 1)}{P(\mathsf{U}_k = -1)} = \frac{P(\mathsf{U}_{k-1} = -1|\mathsf{X} = 1)}{P(\mathsf{U}_{k-1} = -1)}. \quad (29)$$

For (26) to (29) we have used that $P(\mathsf{U}_k = a|\mathsf{U}_{k-1} = a, \mathsf{X} = 1) = 1$ and $P(\mathsf{U}_k = a|\mathsf{U}_{k-1} = a) = 1$ with $a \in \{1, -1\}$ and (7).

Due to the symmetry of the test and the considered scenario reflected by (9) based on (26) and (29) we get

$$\frac{P(\mathsf{U}_k = 1|\mathsf{X} = 1)}{P(\mathsf{U}_k = -1|\mathsf{X} = 1)} = \frac{P(\mathsf{U}_{k-1} = 1|\mathsf{X} = 1)}{P(\mathsf{U}_{k-1} = -1|\mathsf{X} = 1)} = \gamma. \quad (30)$$

I.e., the ratio between the probability that the Wald test terminates at the positive boundary and the probability that it terminates at the negative boundary *at any time instant up to the time instant k* is constant over k. We denote this constant as γ. Using (27), (28), (9), (10), and (30) we get

$$\frac{P(\mathsf{U}_k = 1|\mathsf{X} = 1)}{P(\mathsf{U}_k = -1|\mathsf{X} = 1)} = \frac{P(\mathsf{U}_k = 1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1)}{P(\mathsf{U}_k = -1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1)} = \gamma \quad (31)$$

stating that the ratio of the termination probabilities on the positive and on the negative threshold *at the time instant k* is also a constant independent of the time instant k. □

Proof of Theorem 2. As

$$\begin{aligned} &P(\mathsf{U}_k = 1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1, \mathsf{U}_k \neq \epsilon) \\ &= \frac{P(\mathsf{U}_k = 1, \mathsf{U}_k \neq \epsilon|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1)}{P(\mathsf{U}_k \neq \epsilon|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1)} \\ &= \frac{P(\mathsf{U}_k = 1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1)}{P(\mathsf{U}_k = 1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1) + P(\mathsf{U}_k = -1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1)} \\ &= \frac{1}{1 + \frac{P(\mathsf{U}_k = -1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1)}{P(\mathsf{U}_k = 1|\mathsf{U}_{k-1} = \epsilon, \mathsf{X} = 1)}} = \frac{1}{1 + \frac{1}{\gamma}} \quad (32) \\ &= P(\mathsf{D} = 1|\mathsf{X} = 1) = \kappa \quad (33) \end{aligned}$$

where for (32) we have used Lemma 2. Finally, (33) holds as

$$\begin{aligned} P(\mathsf{D} = 1|\mathsf{X} = 1) &= \sum_{k=1}^{\infty} P(\mathsf{U}_k = 1|\mathsf{U}_{k-1} = \epsilon, \mathsf{U}_k \neq \epsilon, \mathsf{X} = 1) \\ &\quad \times P(\mathsf{U}_{k-1} = \epsilon, \mathsf{U}_k \neq \epsilon|\mathsf{X} = 1) \\ &= \frac{1}{1 + \frac{1}{\gamma}} \sum_{k=1}^{\infty} P(\mathsf{U}_{k-1} = \epsilon, \mathsf{U}_k \neq \epsilon|\mathsf{X} = 1) = \frac{1}{1 + \frac{1}{\gamma}} \quad (34) \end{aligned}$$

where for (34) we have used (32) and the fact that the Wald test terminates almost surely [6, Th. 6.2-1]. □

Equation (25) for a discrete-time problem is similar to a continuous-time result on the first passage problem with two absorbing boundaries: For stopping time distributions of stochastic entropy production an analogous expression to (25) has been found for nonequilibrium steady states [4, Eq. (11) and Append. S2 in its Suppl. Material], [7]. Moreover, in communication theory such a symmetry has been found to show that the probability of cycle slips to the positive/negative boundary in phase-locked loops used for synchronization is independent of time [3, Eq. (74)]. While in the continuous-time first passage problem with two absorbing boundaries the random trajectory always terminates exactly on one of the thresholds, in the discrete-time setup considered in the present paper this does not hold. Nevertheless, even in this case the ratio of the termination probabilities at both boundaries in (25) is time independent.

C. Evolution of Mutual Information

At time k the mutual information between X and the decision variable U_k of the Wald test is given by

$$\begin{aligned} I(\mathsf{X}; \mathsf{U}_k) &= \{P(\mathsf{U}_k = 1|\mathsf{X} = 1) + P(\mathsf{U}_k = -1|\mathsf{X} = 1)\} \\ &\quad \times \log\left(2/(P(\mathsf{U}_k = 1|\mathsf{X} = 1) + P(\mathsf{U}_k = -1|\mathsf{X} = 1))\right) \\ &\quad + P(\mathsf{U}_k = 1|\mathsf{X} = 1) \log\left(P(\mathsf{U}_k = 1|\mathsf{X} = 1)\right) \\ &\quad + P(\mathsf{U}_k = -1|\mathsf{X} = 1) \log\left(P(\mathsf{U}_k = -1|\mathsf{X} = 1)\right). \quad (35) \end{aligned}$$

The Wald test terminates almost surely [6, Th. 6.2-1] which means that $\lim_{k\to\infty} P(\mathsf{U}_k = \epsilon) = 0$. Thus, it holds that

$$\lim_{k\to\infty} P(\mathsf{U}_k = -1|\mathsf{X} = 1) = P(\mathsf{D} = -1|\mathsf{X} = 1) = 1 - \kappa \quad (36)$$

$$\lim_{k\to\infty} P(\mathsf{U}_k = 1|\mathsf{X} = 1) = P(\mathsf{D} = 1|\mathsf{X} = 1) = \kappa. \quad (37)$$

Using (31), and $\gamma = \frac{\kappa}{1-\kappa}$, cf. (33), eq. (35) becomes

$$I(\mathsf{X}; \mathsf{U}_k) = \frac{P(\mathsf{U}_k = 1|\mathsf{X} = 1)}{\kappa} I(\mathsf{X}; \mathsf{U}_\infty). \quad (38)$$

with, cf. (36) and (37)

$$I(\mathsf{X}; \mathsf{U}_\infty) = 1 + \kappa \log(\kappa) + (1 - \kappa) \log(1 - \kappa). \quad (39)$$

I.e., $I(\mathsf{X}; \mathsf{U}_k)$ linearly increases with $P(\mathsf{U}_k = 1|\mathsf{X} = 1)$ until it achieves the final value $I(\mathsf{X}; \mathsf{U}_\infty)$. If the test terminates exactly on one of the thresholds it holds that $\kappa = 1-\alpha$, cf. Sect. II, and $I(\mathsf{X}; \mathsf{U}_\infty)$ is the mutual information to be achieved to allow a decision with the predefined error probability.[1]

[1]Note that (38) does not state that $I(\mathsf{X}; \mathsf{U}_\infty)$ is only reached for $k \to \infty$.

IV. Summary

The analysis of the Wald test for symmetric noise, equal error probabilities corresponding to symmetric thresholds, and equally likely hypotheses provides an understanding on the implications of the information processing in optimal sequential decision-making. Mainly, we have shown that for these conditions (i) the decision time contains no information on which hypothesis is true beyond the decision outcome (Theorem 1) and that (ii) the probability to decide for hypothesis $\mathcal{H}_1$ (or $\mathcal{H}_0$) is independent of time (Theorem 2). How far the presented results can be generalized to non-symmetric conditions will be studied in a forthcoming paper.

Appendix A – Proof of (18)

Without loss of generality let us assume that n is the time instant where the state variable U_n changes from ϵ to ± 1. If $n > k$ equation (18) is straightforward, see Fig. 2. For $n \leq k$ and this specific realization of $\mathbf{U}^k$ we can rewrite the LHS of (18) as

$$\begin{aligned} P(\mathsf{X}|\mathbf{U}^{n-1} &= \epsilon \mathbf{1}_{n-1}, \mathbf{U}_n^k = \pm \mathbf{1}_{k-n+1}) \\ &= P(\mathsf{X}|\mathsf{U}_{n-1} = \epsilon, \mathsf{U}_n = \pm 1) \end{aligned} \tag{40}$$

with $\mathbf{1}_n$ being the all one row vector of length n and $\mathbf{U}_n^k = [\mathsf{U}_n, \ldots, \mathsf{U}_k]$. Here, (40) follows from the fact that in case U_{n-1} is in state ϵ it must also have been in state ϵ in all prior time instants and once U_n changes to ± 1 it stays in this state, cf. Fig. 2. Hence, to prove (18) it is sufficient to show that

$$P(\mathsf{X}=1|\mathsf{U}_{l-1}=\epsilon, \mathsf{U}_l=1) = P(\mathsf{X}=1|\mathsf{U}_l=\epsilon, \mathsf{U}_{l+1}=1) \tag{41}$$

holds for an arbitrary l. Here, without loss of generality we assume a transition of the state variable U_l to 1. Expressing (41) in terms of the cumulated log-likelihood ratio in (2) yields

$$P(\mathsf{X}=1|\mathsf{S}_{l-1}<T, \mathsf{S}_l \geq T) = P(\mathsf{X}=1|\mathsf{S}_l<T, \mathsf{S}_{l+1} \geq T). \tag{42}$$

The LHS of (42) can be rewritten as follows

$$\begin{aligned} P(\mathsf{X}=1|\mathsf{S}_{l-1}<T, \mathsf{S}_l \geq T) &= \frac{P(\mathsf{X}=1, \mathsf{S}_{l-1}<T, \mathsf{S}_l \geq T)}{P(\mathsf{S}_{l-1}<T, \mathsf{S}_l \geq T)} \\ &= \frac{1}{1+\frac{P(\mathsf{S}_{l-1}<T, \mathsf{S}_l \geq T|\mathsf{X}=-1)}{P(\mathsf{S}_{l-1}<T, \mathsf{S}_l \geq T|\mathsf{X}=1)}} \qquad (43) \\ &= \frac{1}{1+\frac{P(\mathsf{S}_{l-1}>-T, \mathsf{S}_l \leq -T|\mathsf{X}=1)}{P(\mathsf{S}_{l-1}<T, \mathsf{S}_l \geq T|\mathsf{X}=1)}} \qquad (44) \end{aligned}$$

where for (43) we have used that $P(\mathsf{X}=1) = P(\mathsf{X}=-1)$. Finally, (44) follows from the symmetry of the noise, i.e., $p_{\mathsf{Z}}(-z) = p_{\mathsf{Z}}(z)$, also cf. (5). Thus, to prove (42) we have to show that $\frac{P(\mathsf{S}_{l-1}>-T, \mathsf{S}_l \leq -T|\mathsf{X}=1)}{P(\mathsf{S}_{l-1}<T, \mathsf{S}_l \geq T|\mathsf{X}=1)}$ is independent of l.

The Wald test corresponds to a first passage problem of the discrete-time stochastic process $\{\mathsf{S}_1, \mathsf{S}_2, \ldots\}$ with two absorbing boundaries at $\pm T$. The further argumentation is based on the existence of a corresponding continuous-time process $\{\tilde{\mathsf{S}}(t)\}$ such that $\{\mathsf{S}_k\}$ is a sampled version of $\{\tilde{\mathsf{S}}(t)\}$. Let

$$\tilde{\mathsf{Y}}(t) = \sqrt{\rho}\mathsf{X} + \sum_{l=1}^{\infty} \mathsf{Z}_l \, \frac{\sin(\pi(t-l))}{\pi(t-l)}, \quad t \in \mathbb{R}_+ \tag{45}$$

such that the Y_k in (1) are given by $\tilde{\mathsf{Y}}(k)$. To describe the statistics of $\{\tilde{\mathsf{Y}}(t)\}$ we define the probability spaces $(\Omega, \mathcal{F}, \mathbb{P}_{\mathsf{X}=a})$ for $a \in \{-1, 1\}$, where Ω is the set of all trajectories $\{\tilde{Y}(t)\}$. The set $\mathcal{F}$ of all measurable sets Φ of Ω is a σ-algebra and $\mathbb{P}_{\mathsf{X}=a}$ are measures such that $\mathbb{P}_{\mathsf{X}=a}(\Phi) = \Pr(\{\tilde{Y}(s)\}|_{\mathsf{X}=a} \in \Phi)$. Using a Radon-Nikodým derivative the cumulative log-likelihood ratio $\tilde{\mathsf{S}}(t)$ can be expressed as

$$\tilde{\mathsf{S}}(t) = \log\left(\frac{\mathrm{d}\mathbb{P}_{\mathsf{X}=1}|_{\mathcal{F}(t)}}{\mathrm{d}\mathbb{P}_{\mathsf{X}=-1}|_{\mathcal{F}(t)}}\right) \tag{46}$$

where $\mathbb{P}_{\mathsf{X}=a}|_{\mathcal{F}(t)}$ denotes the restriction of the measure $\mathbb{P}_{\mathsf{X}=a}$ over those events in the sub-σ-algebra $\mathcal{F}(t) \subset \mathcal{F}$ that is generated by trajectories $\{\tilde{Y}(s)\}_0^t$ in the time interval $[0, t]$. It can be shown that the S_k are given by the samples $\tilde{\mathsf{S}}(k)$ of the continuous-time process in (46). In this regard, note that from the definition of $\tilde{\mathsf{Y}}(t)$ in (45) it follows that

$$\frac{\mathrm{d}\mathbb{P}_{\mathsf{X}=1}|_{\mathcal{F}(k)}}{\mathrm{d}\mathbb{P}_{\mathsf{X}=-1}|_{\mathcal{F}(k)}}(\{\tilde{\mathsf{Y}}(s)\}_{s\in[0,k]}) = \prod_{l=1}^{k} \frac{p_{\mathsf{Y}|\mathsf{X}}(\mathsf{Y}_l|\mathsf{X}=1)}{p_{\mathsf{Y}|\mathsf{X}}(\mathsf{Y}_l|\mathsf{X}=-1)}. \tag{47}$$

For the case that the Z_l in (45) are Gaussian distributed $\tilde{\mathsf{Y}}(t)$ is a Brownian bridge [8, pp. 101–104].

For the steady-state log-likelihood ratio process $\{\tilde{\mathsf{S}}(t)\}$ we have shown the following [4], [7]. Let $P(\tau; T)d\tau$ denote the probability that $\{\tilde{\mathsf{S}}(t)\}$ reaches the threshold T for the first time in the time interval $[\tau, \tau + d\tau]$ given that it has not reached $-T$ before. Then it holds that [4, Eq. (11) and Append. S2 in its Suppl. Mat.], [7, Eq. (E16) in Append. E]

$$\frac{P(\tau; T)}{P(\tau; -T)} = \exp(T) \tag{48}$$

i.e., the ratio between the probability that $\{\tilde{\mathsf{S}}(t)\}$ terminates in a given time interval on threshold T and the probability that it terminates in the same time interval on the opposite threshold $-T$ is independent of the time τ. For the discrete-time setup this implies that (44) is independent of l which proves (18).

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