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PAPER

Casimir stresses in active nematic films

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Abstract

We calculate the Casimir stresses in a thin layer of active fluid with nematic order. By using a stochastic hydrodynamic approach for an active fluid layer of finite thickness L, we generalize the Casimir stress for nematic liquid crystals in thermal equilibrium to active systems. We show that the active Casimir stress differs significantly from its equilibrium counterpart. For contractile activity, the active Casimir stress, although attractive like its equilibrium counterpart, diverges logarithmically as L approaches a threshold of the spontaneous flow instability from below. In contrast, for small extensile activity, it is repulsive, has no divergence at any L and has a scaling with L different from its equilibrium counterpart.

Introduction

It is well-known that although the zero-point energy of the electromagnetic field inside a cavity bounded by conducting walls is formally diverging, its variation upon displacements of the boundaries remains finite. It corresponds to a weak but measurable attractive force, known as the Casimir force [1]. For example, in the case of two parallel conducting plates at a distance L, the attractive Casimir force per unit area, or the Casimir stress is given by $C_F = -\frac{\pi^2}{240}\frac{h2\pi c}{L^4}$ [1]. It is of purely quantum origin.

Subsequently, thermal analogs of the Casimir stress associated to various fluctuating fields at a finite temperature T have been studied. In nematic liquid crystals confined between two parallel plates, the thermal fluctuations of the director field that describes the nematic order, play the role of the electromagnetic fluctuations in the electromagnetic Casimir effect. In all such classical systems, the boundary conditions on the relevant fields (e.g. the director field for nematic liquid crystals) constrain their thermal fluctuations and lead to a thermal analog of the Casimir stress. For instance, for a nematic liquid crystal between parallel confining plates separated by a distance L with the director field rigidly anchored to them, one again obtains an attractive Casimir stress that varies with the thickness L of the liquid crystal film as $1/L^3$ [2].

Studies on non-equilibrium analogs of thermal Casimir stresses are relatively new. In [3], Casimir stresses between two parallel plates due to non-thermal noises are calculated. Further, embedding objects or inclusions in a correlated fluid are shown to generate effective Casimir-like stresses between the inclusions [4]. In a more recent study, [5] numerically examined run-and-tumble active matter particles in Casimir geometries composed of two finite parallel walls and found an attractive Casimir force depending, rather unusually, exponentially on the plate separation. There are direct biologically relevant examples as well: recently [6], studied the Casimir-like forces felt by inclusions in active fluids, in particular their dependence on active noise and hydrodynamic interaction. Subsequently, [7] studied the role of active Casimir effects on the deformation dynamics of the cell nucleus and showed the appearance of a fluctuation maximum at a critical level of activity, a result in agreement with recent experiments [8]. The active fluid models considered by [6, 7] are effectively one-dimensional and hence do not include any soft orientational fluctuations. More recently, [9] studied one-dimensional suspension of active particles, and uncovered a Casimir-like attractive force mediated by the active suspension.

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In this article, we calculate the Casimir stress between two parallel plates confining a layer of an active nematic fluid with a uniform macroscopic orientation [10–14]. The active fluid is driven out of equilibrium by a local constant supply of energy. Our work directly generalizes thermal Casimir stresses in equilibrium nematics [2] into the nonequilibrium domain.

The hydrodynamic active fluid model [10, 11] has been proposed as a generic coarse-grained model for a driven orientable fluid with nematic or polar symmetry. One of the main features of an active fluid is the existence of an active stress, resulting from the constant energy consumption driving the system out of equilibrium. Due to its very general nature, the active fluid model is able to describe a broad range of phenomena, observed in very different physical systems and at very different length scales [10–12]. Notable examples include the dynamics of actin filaments in the cortex of eukaryotic cells or bird flocks and bacterial biofilms. In particular, in the case of actin filament dynamics, the active stress results from the release of free energy due to the chemical conversion of Adenosine-Triphosphate (ATP) to Adenosine-Diphosphate.

In this article, we study Casimir stresses using a stochastically driven coarse-grained hydrodynamic approach for active fluids [10–12] with a nematic order, described by a unit director field p_α, $\alpha = x, y, z$. The film is infinite along the x, y plane, but has a finite thickness L in the z-direction. A typical example of ordered active nematic where our results may apply is the cortical actin layer in a cell where the orientation of the actin filaments can have a component parallel to the cell membrane. It has been recently shown that a liquid contractile active film of thickness L with polarization either parallel or perpendicular to its surface has a spontaneous flow instability, above a critical value of the activity [15, 16]. This is the nonequilibrium analog of the 'Frederiks transition' in equilibrium classical nematic liquid crystals, in which a uniform magnetic field wins over the direction imposed by boundaries, above a critical value depending on sample thickness [17]. It is driven by the coupling between the director orientation and the active stress. We here calculate $C_{\rm act}$, the *active* analog of the thermal equilibrium Casimir stress, that we formally define below.

Active Casimir stress

We consider a thin film of active fluid with a fixed thickness L along the z-direction confined between the planes $z = 0$ and $z = L$. In the passive case, i.e. without any activity, the Casimir stress $C_{\rm eq}$ is defined as [2]

$$C_{\rm eq} = \langle \sigma_{zz}^{\rm eq} \rangle|_{z=L} - \langle \sigma_{zz}^{\rm eq} \rangle|_{z=\infty}. \tag{1}$$

Here, $\sigma_{zz}^{\rm eq}$ is the normal component of the equilibrium stress that diverges for all z (or, all L); $C_{\rm eq}$ however is finite for any non-zero L[2]. Here, $\langle .. \rangle$ implies averages over thermal noise ensembles (see below).

The average total normal stress $\langle \sigma_{zz}^{\rm tot} \rangle$ depends *a priori* on the slab thickness L. It also has a piece independent of L. We define the L dependent part as the 'Casimir stress'. We show below that this allows us to extract an *active Casimir effect* $C_{\rm act}$ by

$$C_{\rm act} = \langle \sigma_{zz}^{\rm tot} \rangle|_{z=L} - \langle \sigma_{zz}^{\rm tot} \rangle|_{\Delta\mu=0, z=L}, \tag{2}$$

where the L-independent parts of the stresses are ignored. Here, $\Delta\mu$ is the chemical potential difference between ATP and its hydrolysis products. It is the thermodynamic force conjugate to the rate of hydrolysis of the number of ATP molecules per unit time and per unit volume. Further, $\sigma_{zz}^{\rm tot}|_{\Delta\mu=0} = \sigma_{zz}^{\rm eq}$, the normal component of the equilibrium stress.

For simplicity, we assume a one Frank constant description with K as the Frank constant (in which the Frank constants for the splay, bend and twist modes are equal). The Frank free energy density in three dimensions is given by $(\partial_\alpha p_\beta)^2 K/2$, where $\alpha, \beta = x, y, z$. By using stochastic hydrodynamic descriptions for orientationally ordered active fluids, we show below that, for a given reference orientation state $p_z = 1$ and up to the quadratic order in orientation fluctuations, $C_{\rm act}$ as given by (2) reduces to

$$C_{\rm act} = -\frac{K}{2}\langle (\partial_z p_i)^2 \rangle|_{z=L} + \frac{K}{2}\langle (\partial_z p_i)^2 \rangle|_{\Delta\mu=0, z=L}, \tag{3}$$

where $i, j = x, y$. The quantity $C_{\rm act}$ is difficult to measure directly. However changes of $C_{\rm act}$ due to changes in L can in principle be measured.

When the thickness L of a contractile active fluid layer approaches the critical thickness L_c for the spontaneous flow instability from below [15], we show that $C_{\rm act}$ remains attractive, though it does not scale like its equilibrium counterpart: there is a log correction to the equilibrium result as L approaches L_c from below. We also calculate $C_{\rm act}$ for extensile activity, and contrast it with $C_{\rm act}$ for the contractile case: in this case, $C_{\rm act}$ is found be repulsive unlike the contractile case, has no divergence at any finite L, and scales with L *differently* from the equilibrium result.

Steady state stresses in a fluctuating active fluid

We consider an incompressible viscous active fluid film with nematic order. Our analysis below closely follows the physical discussion of [18], where the diffusion coefficient of a test particle immersed in an active fluid with nematic order is calculated. The force balance in an incompressible active fluid is given by

$$\partial_\beta(\tilde{\sigma}_{\alpha\beta} + \sigma^a_{\alpha\beta} - P\delta_{\alpha\beta} + \sigma^e_{\alpha\beta}) = 0, \tag{4}$$

where fluid inertia is neglected [10, 19]. Here, $\tilde{\sigma}_{\alpha\beta}$ denotes the traceless part of the symmetric deviatoric stress and the antisymmetric deviatoric stress is given by

$$\sigma^a_{\alpha\beta} = \frac{1}{2}(p_\alpha h_\beta - p_\beta h_\alpha). \tag{5}$$

Here $h_\alpha = -\delta F/\delta p_\alpha$ is the orientational field conjugate to the nematic director p_α, where $F = \int \mathrm{d}^3 r f$ denotes the nematic director free energy with a free energy density f. Furthermore, P denotes the hydrostatic pressure. Note that in a nematic system the equilibrium stress includes the Ericksen stress

$$\sigma^e_{\alpha\beta} = -\frac{\partial f}{\partial(\partial_\beta p_\gamma)}\partial_\alpha p_\gamma = -K(\partial_\alpha p_\gamma)(\partial_\beta p_\gamma). \tag{6}$$

Here, $\alpha, \beta = x, y, z$. The total normal stress is thus given by

$$\sigma^{\mathrm{tot}}_{\alpha\beta} = \tilde{\sigma}_{\alpha\beta} + \sigma^a_{\alpha\beta} - P\delta_{\alpha\beta} + \sigma^e_{\alpha\beta}; \tag{7}$$

see equation (4) above.

In the following, we impose for simplicity a constant amplitude of the nematic director $p_\gamma p_\gamma = 1$. The constitutive equations of a single-component active fluid then read [19]

$$\left\{\tilde{\sigma}_{\alpha\beta} + \zeta\Delta\mu q_{\alpha\beta} + \frac{\nu_1}{2}\left(p_\alpha h_\beta + p_\beta h_\alpha - \frac{2}{3}p_\gamma h_\gamma \delta_{\alpha\beta}\right)\right\} = 2\eta v_{\alpha\beta} + \xi^\sigma_{\alpha\beta}, \tag{8}$$

$$\frac{D}{Dt}p_\alpha = \frac{1}{\gamma_1}h_\alpha - \nu_1 p_\beta \tilde{v}_{\alpha\beta} + \xi_{\perp\alpha}, \tag{9}$$

where $q_{\alpha\beta} = \left(p_\alpha p_\beta - \frac{1}{3}\delta_{\alpha\beta}\right)$ is the nematic tensor. The symmetric velocity gradient tensor is $\tilde{v}_{\alpha\beta} = (\partial_\alpha v_\beta + \partial_\beta v_\alpha)/2$, where v_α is the three-dimensional velocity field of the active fluid ($\alpha = x, y, z$). The term $\zeta\Delta\mu q_{\alpha\beta}$ in (8) is the traceless part of the active stress. The term $\frac{\nu_1}{2}\left(p_\alpha h_\beta + p_\beta h_\alpha - \frac{2}{3}p_\gamma h_\gamma \delta_{\alpha\beta}\right)$ is the traceless symmetric part of a stress contribution that originates in the orientation fluctuations and is required to be present due to the Onsager symmetry relations. The shear viscosity is denoted by η, γ_1 is the rotational viscosity and ν_1 the flow alignment parameter which is a number of order one. Noises $\xi^\sigma_{\alpha\beta}$ and $\xi_{\perp\alpha}$ are assumed to be thermal noises of zero-mean and variances given by

$$\langle\xi^\sigma_{\alpha\beta}(t, \mathbf{x})\xi^\sigma_{\gamma\delta}(t', \mathbf{x}')\rangle = 2k_{\mathrm{B}}T\eta\left[\left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{2}{3}\delta_{\alpha\beta}\delta_{\gamma\delta}\right]\delta(t - t')\delta(\mathbf{x} - \mathbf{x}'), \tag{10}$$

$$\langle\xi_{\perp\alpha}(t, \mathbf{x})\xi_{\perp\beta}(t', x')\rangle = 2\frac{k_{\mathrm{B}}T}{\gamma_1}[\delta_{\alpha\beta} - p_\alpha p_\beta]\delta(t - t')\delta(x - \mathbf{x}'), \tag{11}$$

where k_{B} is Boltzmann constant and T denotes temperature. Notice that the noises $\xi_{\perp\alpha}(t, \mathbf{x})$ are *multiplicative* in nature (see noise variance (11)). However, since we are interested in a linearized description about uniform ordered states (see below), the multiplicative nature of these noises do not affect our calculations. Furthermore, we do not consider any athermal or active noises for simplicity. We consider an incompressible system imposed by the constraint $\partial_\alpha v_\alpha = 0$.

The pressure P plays the role of a Lagrange multiplier used to impose the incompressibility constraint $\partial_\alpha v_\alpha = 0$. The incompressibility leads to the following equation for P:

$$\nabla^2 P = -\frac{\nu_1}{2}\partial_\alpha\partial_\beta\left(p_\alpha h_\beta + p_\beta h_\alpha - \frac{2}{3}\mathbf{p}\cdot\mathbf{h}\delta_{\alpha\beta}\right) - \zeta\Delta\mu\partial_\alpha\partial_\beta(p_\alpha p_\beta) + \partial_\alpha\partial_\beta\sigma^e_{\alpha\beta} + \partial_\alpha\partial_\beta\xi^\sigma_{\alpha\beta}. \tag{12}$$

We consider a film of the active fluid with a fixed thickness L along the z direction, confined between the planes $z = 0$ and $z = L$. We consider a quiescent reference state together with $p_z = 1$, which is a steady state solution of (4) and (9). We study small fluctuations $\delta\mathbf{p} = (p_x, p_y, 0)$ around this state; $\delta p = |\delta\mathbf{p}|$ (fluctuations of $\mathbf{p}$ along the z direction, δp_z is quadratic in δp: $\delta p_z = (p_x^2 + p_y^2)/2$). We impose boundary conditions $(p_x, p_y) = 0$ and vanishing shear stress at $z = 0$ and $z = L$. See figure 1 for a schematic diagram of the system.

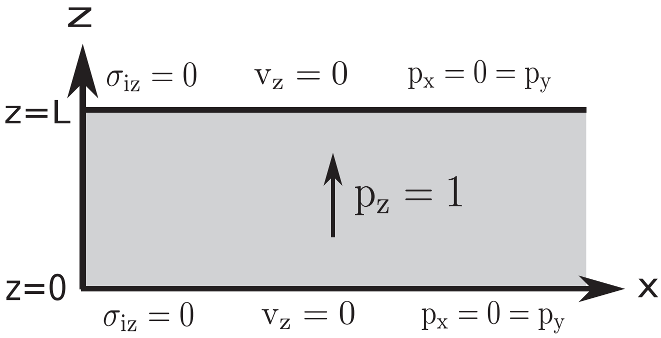

Figure 1. A two-dimensional projection of the model system (in light gray) on the xz-plane. Boundary conditions on the flow field and the orientation fluctuations are indicated. The upward arrow within the system indicates the orientation reference state $p_z = 1$, and $i = x, y$.

The total average normal stress on the surface at $z = L$, $\langle \sigma_{zz}^{\rm tot} \rangle_{z=L}$ reads:

$$\langle \sigma_{zz}^{\rm tot} \rangle_{z=L} = \eta \left\langle \frac{\partial v_z}{\partial z} \right\rangle_{z=L} - \zeta\Delta\mu \langle p_z^2 \rangle |_{z=L} - \frac{\nu_1}{3} \langle p_i h_i \rangle_{z=L} - \frac{\nu_1}{3} \langle p_z h_z \rangle |_{z=L} + \langle \sigma_{zz}^e \rangle_{z=L} - \langle P \rangle_{z=L}, \quad (13)$$

where $\sigma_{zz}^e = -K(\partial_z p_i)^2$. Here, $i, j = x, y$ are the coordinates along the film surface. Using, for simplicity and analytical convenience, a single Frank elastic constant K for the nematic liquid crystals, the Frank free energy density is given by $f = K(\nabla_\alpha p_\beta)^2/2$. Below we evaluate the pressure P which obeys equations (12). The remaining terms in (13) are also to be evaluated using the relevant equations of motion and then averaging over the various noise terms The contributions to the stress that are linear in small fluctuations (p_x, p_y) vanish upon averaging; therefore, a non-vanishing Casimir stress is obtained from contributions to the stress quadratic in $\delta \mathbf{p}$ in (13). It is instructive to analyze the different contributions in (13) to $C_{\rm tot}$ term by term. This will allow us to considerably simplify (13) as we will see below.

We first consider the contribution $\eta \left\langle \frac{\partial v_z}{\partial z} \right\rangle_{z=L}$ in (13). Using the condition of incompressibility $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, this may be written as

$$\eta \left\langle \frac{\partial v_z}{\partial z} \right\rangle_{z=L} = -\eta \langle \boldsymbol{\nabla}_\perp \cdot \mathbf{v}_\perp \rangle_{z=L} = -\eta \boldsymbol{\nabla}_\perp \cdot \langle \mathbf{v}_\perp \rangle_{z=L} = 0, \quad (14)$$

since there is no flow on average. Here, $\boldsymbol{\nabla}_\perp = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ is the two-dimensional gradient operator and $\mathbf{v}_\perp = (v_x, v_y)$ is the in-plane component of the three-dimensional velocity $\mathbf{v}$.

Secondly, $\langle p_i h_i \rangle_{z=L} = 0$ since $p_i = 0$ at $z = L$. Further, $\langle p_z h_z \rangle_{z=L} = \langle h_\parallel \rangle$, since $p_z = 1$ at $z = L$. Here, $h_\parallel$ is a Lagrange multiplier, which must be introduced to impose $p^2 = 1$, or to the leading order $p_z = 1$ in the geometry that we consider. Using $p_z = 1$ in equation (9) and linearizing around $p_z = 1$, we obtain $h_\parallel \sim \frac{\partial v_z}{\partial z}$ at all z [19]. Using the incompressibility condition, $\partial v_z / \partial z = -\boldsymbol{\nabla}_\perp \cdot \mathbf{v}_\perp$. This then gives $\langle h_z \rangle = 0$ to the leading order in fluctuations.

To evaluate the pressure P, we consider the equation for the velocity field v_α that obeys the generalized Stokes equation

$$\eta \nabla^2 v_\alpha = \partial_\alpha P + \zeta\Delta\mu \partial_\beta(p_\alpha p_\beta) - \frac{\nu_1}{2} \partial_\beta(p_\beta h_\alpha + p_\alpha h_\beta) - \frac{1}{2} \partial_\beta(p_\alpha h_\beta - p_\beta h_\alpha) - \partial_\beta \sigma_{\alpha\beta}^e - \partial_\beta \xi_{\alpha\beta}^\sigma. \quad (15)$$

We focus on the in-plane velocity v_i, $\alpha = i = x, y$ in (15) above. Now consider the different terms in (15) with $\alpha = i$ at $z = L$ and note that (i) $p_i = 0$ at $z = L$, (ii) in the absence of any mean flow and consistent with the in-plane rotational invariance, velocity fluctuations $+v_i$ and $-v_i$ should be equally likely in the statistical steady state, i.e. the steady state average of any function odd in v_i should be zero. This implies that $\langle v_i \rangle = 0 = \langle \partial_z^2 v_i \rangle$ in steady states. Similarly, in an oriented state having nematic order with $p_z = 1$, fluctuations $+p_i$ and $-p_i$ should be equally likely in the steady state, i.e. any function odd in p_i must have a vanishing average in the steady states. Furthermore, since h_i is odd in p_i, we must have $\langle h_i \rangle = 0$ at steady state. This gives

$$\langle \partial_\beta(p_\beta h_i) \rangle = \langle \partial_z(p_z h_i) \rangle = 0,$$
$$\langle \partial_\beta(p_i h_\beta) \rangle = \langle \partial_z(p_i h_z) \rangle = 0 .$$

Similarly

$$\zeta\Delta\mu\langle\partial_\beta(p_ip_\beta)\rangle|_{z=L} = \zeta\Delta\mu\langle\partial_j(p_ip_j)\rangle|_{z=L} + \zeta\Delta\mu\langle\partial_z(p_ip_z)\rangle|_{z=L} = \zeta\Delta\mu\langle\partial_z(p_ip_z)\rangle|_{z=L} \tag{16}$$

vanishes in the steady state due to the inversion symmetry of p_i. Lastly, we note that

$$\partial_\beta\sigma^e_{i\beta}|_{z=L} = -K[\partial_j(\partial_ip_\gamma\partial_jp_\gamma)|_{z=L} + \partial_z(\partial_ip_\gamma\partial_zp_\gamma)|_{z=L}] = -\frac{1}{2}\partial_i(\partial_zp_j)^2|_{z=L}, \tag{17}$$

where we have used $[(\partial_ip_\gamma)(\partial_z^2p_\gamma)]_{z=L} = [(\partial_ip_j)(\partial_z^2p_j)]_{z=L} + [(\partial_ip_z)(\partial_z^2p_z)]_{z=L} = 0$, since $p_j = 0$ and $p_z = 1$ exactly at $z = L$. Putting together everything and averaging in the steady states, we then obtain at $z = L$

$$\partial_iP = \partial_\beta\sigma^e_{i\beta} = -\frac{K}{2}\partial_i(\partial_zp_j)^2, \tag{18}$$

giving

$$P = -\frac{K}{2}(\partial_zp_j)^2 + a_0 \tag{19}$$

at $z = L$, where a_0 is a constant of integration. Then substituting P in (13)

$$\langle\sigma^{\rm tot}_{zz}\rangle_{z=L} = -\frac{K}{2}\langle(\partial_zp_i)^2\rangle|_{z=L} - \zeta\Delta\mu + a_0 = -\frac{K}{2}\langle(\partial_zp_i)^2\rangle|_{z=L} + \tilde{a}_0, \tag{20}$$

where $\tilde{a}_0$ is another constant that in general can depend upon $\Delta\mu$. Similarly in the passive case [2]

$$\langle\sigma^{\rm tot}_{zz}\rangle|_{\Delta\mu=0,z=L} = \langle\sigma^{\rm eq}_{zz}\rangle_{z=L} = -\frac{K}{2}\langle(\partial_zp_i)^2\rangle|_{z=L,\Delta\mu=0} + a_0^{\rm eq}, \tag{21}$$

where $a_0^{\rm eq}$ is a constant. The constants $\tilde{a}_0$ and $a_0^{\rm eq}$ are prescribed by boundary conditions and do not *a priori* depend upon L.

We find that the total Casimir stress $C_{\rm tot}$ at $z = L$ has two distinct contributions:

$$C_{\rm tot} = C_{\rm eq} + C_{\rm act}. \tag{22}$$

The first term, $C_{\rm eq}$ is the Casimir stress contribution of an equilibrium nematic, i.e. with $\Delta\mu = 0$. It reads

$$C_{\rm eq} = -\frac{1}{8\pi}\frac{k_{\rm B}T}{L^3}\zeta_R(3), \tag{23}$$

where $\zeta_R(3)$ is the Riemann-Zeta function [2].

The additional term is a new contribution to the Casimir effect and is of non equilibrium origin. We find

$$C_{\rm act} = \langle\sigma^{\rm tot}_{zz}\rangle|_{z=L} - \langle\sigma^{\rm tot}_{zz}\rangle|_{\Delta\mu=0,z=L} = -\frac{K}{2}\langle(\partial_zp_i)^2\rangle|_{z=L} + \frac{K}{2}\langle(\partial_zp_i)^2\rangle|_{\Delta\mu=0,z=L}. \tag{24}$$

We show below that $C_{\rm act}$ in an ordered active nematic layer is fundamentally different from its equilibrium counterpart, primarily because the dynamics of orientation fluctuations here is very different from its equilibrium counterpart.

We calculate $C_{\rm act}$ for small fluctuations around the chosen reference state by using the dynamical equations (4) and (9). Since $\langle\sigma^{\rm tot}_{zz}\rangle \sim \delta p_\alpha^2$, it suffices to study the dynamics after linearizing about the reference state. Considering a contractile active fluid, i.e. $\Delta\mu < 0$, we find that as thickness L approaches L_c from below, where L_c is the critical thickness for the spontaneous flow instability (see [15]; see also below), akin to the Frederiks transition in equilibrium nematics [17], the Casimir stress C diverges logarithmically. We find that as L approaches L_c from below

$$C_{\rm act} = k_{\rm B}T\frac{-\pi^2}{2L^3}\frac{\Gamma\gamma_1}{8\eta + \gamma_1(\nu_1-1)^2}\ln\left|\frac{[2/\gamma_1 + (\nu_1-1)^2/4\eta]\gamma_1}{2\Gamma}\frac{L_c}{L_c - L}\right|. \tag{25}$$

Here, $\Gamma = 2\eta/\gamma_1 + (\nu_1-1)^2/4$ is a positive dimensionless number. The critical thickness L_c is determined by the relation [19]

$$\frac{K}{\gamma_1}\frac{\pi^2}{L_c^2} + \frac{(\nu_1-1)^2}{4\eta}K\frac{\pi^2}{L_c^2} = -\zeta\Delta\mu\frac{\nu_1-1}{2\eta}. \tag{26}$$

Clearly, L_c diverges as $\Delta\mu \to 0$, consistent with the fact that there are no instabilities at any thickness in equilibrium. We further show below that in this limit $C \to C_{\rm eq}$. Compare $C_{\rm act}$ with the corresponding equilibrium result $C_{\rm eq}$ as given in (23). Clearly, $C_{\rm eq}$ has no divergence at any finite L, in contrast to $C_{\rm act}$ in (25). It follows from (25) to (23) that both $C_{\rm act}$ and $C_{\rm eq}$ are negative. This implies that the surfaces at $z = 0$ and $z = L$ are attracted towards each other. This feature is similar to the equilibrium problem [2]. Although both contributions have a $1/L^3$-dependence, the active contribution has a log correction and hence clearly dominates the corresponding equilibrium contribution for a sufficiently small $|L - L_c|$. In contrast, for an extensile active system, C scales as $\zeta\Delta\gamma_1\mu/(\eta L)$ for small activity, and is repulsive in nature.

In order to better understand the result given by equations (25) and (23), we first present arguments at the scaling level using a simplified analysis of the problem that highlights the general features of the active contributions in (25). This is similar to the scaling analysis of [18]. We provide the results of the full fluctuating hydrodynamic equations in appendix that confirm the scaling analysis and yield (25).

We consider a small perturbation to the non-flowing steady state with $\mathbf{p} = \hat{e}_z$ along the z-axis. To calculate the Casimir stress C_{tot}, we need to solve for $\delta\mathbf{p}$ from the hydrodynamic equations (8) and (9) together with the noise variances (10) and (11), subject to the specified boundary conditions. This requires to linearize (8) and (9) around the quiescent reference state with $p_z = 1$ and then to solve for p_x, p_y up the linear order in fluctuations. Although this is in principle straightforward, the algebra involved is cumbersome and not very illuminating. In order to illustrate the main results and the underlying physics easily, we consider here a simplified picture. In this simplified example, we describe the tilt of the polarity with respect to the z-axis normal to the film surface by a single small angle θ. The rate of variation of the angle θ is driven by the elastic nematic torque with a Frank elastic constant K and couples to the the strain rate u according to equation (9)

$$\frac{\partial\theta}{\partial t} = \frac{K}{\gamma_1}\nabla^2\theta - \nu_1 u + \tilde{\xi}_\perp(\mathbf{x}, t). \tag{27}$$

We have added in this equation the thermal noise of the orientation fluctuations $\tilde{\xi}_\perp(\mathbf{r}, t)$ introduced above. Noise $\tilde{\xi}_\perp$ is a simplified form of $\xi_{\perp\alpha}(t, \mathbf{x})$ in equation (9). It is Gaussian-distributed with zero mean and variance given by

$$\langle\tilde{\xi}_\perp(\mathbf{x}, t)\tilde{\xi}_\perp(0, 0)\rangle = 2\frac{K_B T}{\gamma_1}\delta(\mathbf{x})\delta(t), \tag{28}$$

in analogy with (11). We ignore here for simplicity the tensorial character of the strain rate and represent it by a scalar u which represents one of its typical components.

If the polarization angle θ does not vanish, the active stress is finite and it is compensated by the viscous stress in the film

$$\eta u \simeq \zeta\Delta\mu\theta, \tag{29}$$

where we have for simplicity ignored the noise in the stress. Including this noise does not qualitatively change the final result. The two equations (27) and (29) can be solved by Fourier expansion both in space and time, writing the polarization angle as

$$\theta(\mathbf{x}, t) = \sum_n \sin(n\pi z/L)\int \mathrm{d}\omega\int\frac{\mathrm{d}\mathbf{q}}{(2\pi)^2}\exp\mathrm{i}(\mathbf{q}\cdot\mathbf{r} - \omega t)\tilde{\theta}(n, \omega, \mathbf{q}). \tag{30}$$

Here, the position vector is $\mathbf{x} = (\mathbf{r}, z)$ where $\mathbf{r}$ denotes the position in the plane parallel to the film, and the wave vector is $\mathbf{k} = (\mathbf{q}, n\pi/L)$ where $\mathbf{q}$ denotes the wave vector parallel to the plane, while n describes the discrete Fourier mode along the z direction. The Fourier transform of the orientation angle satisfies the equation

$$-\mathrm{i}\omega\tilde{\theta}(n, \omega, \mathbf{q}) = \frac{\nu_1}{\eta}\left(\zeta\Delta\mu - \zeta\Delta\mu_c(n)) - \frac{\eta K q^2}{\nu_1\gamma_1}\right)\tilde{\theta} + \tilde{\xi}_\perp(n, \omega, \mathbf{q}). \tag{31}$$

Here, $\zeta\Delta\mu < 0$ for a contractile active fluid, where as $\zeta\Delta\mu > 0$ for an extensile active fluid. Equation (31) defines the relaxation time $\tau_n(q)$ of $\tilde{\theta}$:

$$\tau_n(q)^{-1} = -\frac{\nu_1}{\eta}\left[\zeta\Delta\mu - \zeta\Delta\mu_c(n)) - \frac{\eta K q^2}{\nu_1\gamma_1}\right]. \tag{32}$$

Clearly, the system gets unstable for $|\zeta\Delta\mu| > \zeta\Delta\mu_c$. We have defined here $\zeta\Delta\mu_c(n) = \eta K n^2/(\nu_1\gamma_1\pi^2 L^2)$. We further note that in the equilibrium limit, $u = 0$ in our simplified description and hence the equilibrium relaxation time $\tau_{qe,n}$ is given by

$$\tau_{qe,n}^{-1} = \frac{K}{\gamma_1}\left(q^2 + \frac{n^2\pi^2}{L^2}\right). \tag{33}$$

The orientation angle correlation function can be directly calculated form equation (31) leading to

$$\langle\tilde{\theta}_n(\mathbf{q}, \omega)\tilde{\theta}_n(\mathbf{q}', \omega')\rangle = \frac{k_B T\gamma_1}{\omega^2 + \tau_q^{-2}}(2\pi)^3\delta(\mathbf{q} + \mathbf{q}')\delta(\omega + \omega'). \tag{34}$$

Using equation (24), this then yields

$$C = -\frac{\pi^3 k_B T}{L^3}\sum_n\int \mathrm{d}^2 q\left[\frac{1}{\tau_n(q)^{-1}} - \frac{1}{\tau_{qe,n}^{-1}}\right]. \tag{35}$$

Equation (35) applies to both contractile and extensile active fluids. We now consider separately two distinct cases below: (i) contractile active fluids for which the active component of the stress is positive along the nematic director (and $\zeta\Delta\mu < 0$), (ii) extensile active fluids for which the active component of the stress is negative along the nematic director ($\zeta\Delta\mu > 0$). [20] provides a microscopic model for contractile and extensile stress generation. A notable example of contractile active fluid is a collection of *Chlamydomonas Reinhardtii*, a 'puller swimmer' [21], where as a collection of *Bacillus subtilis*, a 'pusher swimmer' [22], forms a good example of an extensile active fluid.

For a contractile active fluid with $\zeta\Delta\mu < 0$; clearly the system can get unstable for sufficiently large $\zeta\Delta\mu$ for a given L, or equivalently, for sufficiently large L for a fixed $\zeta\Delta\mu$. The nature of $C_{\rm act}$ depends sensitively on whether $|\zeta\Delta\mu| \to \zeta\Delta\mu_c$ from below (near the the threshold for spontaneous flow instability), or $|\zeta\Delta\mu| \ll \zeta\Delta\mu_c$ (far away from the instability threshold). Concentrating first on the near threshold behavior of $C_{\rm act}$, we focus only on the $n = 1$ mode that is dominant near the instability threshold, which gets unstable first as L approaches L_c from below. For ease of notations, we denote $\Delta\mu_c(n = 1) = \Delta\mu_c$, and $\tau_{n=1}(q)^{-1} = \tau(q)^{-1}$, $\tau_{qe,\, n=1} = \tau_{qe}$ with

$$\tau(q)^{-1} = -\frac{\nu_1}{\eta}\left[\zeta\Delta\mu - \zeta\Delta\mu_c) - \frac{\eta K q^2}{\nu_1\gamma_1}\right], \tag{36}$$

$$\tau_{qe}^{-1} = \frac{K}{\gamma_1}\left(q^2 + \frac{\pi^2}{L^2}\right). \tag{37}$$

If the active stress $|\zeta\Delta\mu|$ is larger than this threshold, the non-flowing steady state is unstable and the film spontaneously flows. Retaining only the $n = 1$ mode, $C_{\rm act}$ for an orientationally ordered contractile active fluid is given by

$$C_{\rm act} = -\frac{\pi^3}{L^3}K_{\rm B}T\int {\rm d}^2q\left(\frac{1}{q^2 + q_c^2} - \frac{1}{q^2 + \frac{\pi^2}{L^2}}\right), \tag{38}$$

valid for all $|\zeta\Delta\mu| < \zeta\Delta\mu_c$. Here, we have defined the wave vector q_c such that $q_c^2 = (\gamma_1\nu_1/\eta)(\zeta\Delta\mu_c - |\zeta\Delta\mu|)/K$ and a is a small length-scale cut off. Now, in the vicinity of the spontaneous flow instability, $|\zeta\Delta\mu| \to \zeta\Delta\mu_c$ from below. Then

$$C_{\rm act} = -\frac{\pi^2 k_{\rm B}T}{L^3}\ln\left|\frac{1 + a^2q_c^2}{a^2q_c^2}\right| \sim -\frac{k_{\rm B}T}{L^3}\ln|aq_c|, \tag{39}$$

retaining only the divergent contribution to $C_{\rm act}$ as $q_c^2 \to 0$, or equivalently, $\zeta\Delta\mu \to \zeta\Delta\mu_c$ from below or $L \to L_c = \eta\kappa/(\nu_1\gamma_1\zeta\Delta\mu_c)$ from below. We find that the active Casimir stress (39) diverges logarithmically as $q_c \to 0$ near the instability threshold, i.e. $\zeta\Delta\mu \to \zeta\Delta\mu_c$. The active Casimir stress (39) is clearly attractive. Comparing this with (25) above we note that our simplified analysis does capture the correct sign and the logarithmic divergence near the instability threshold. Compare this with the corresponding equilibrium result given in (23); clearly has the a $1/L^3$-dependence as $C_{\rm act}$, but has no divergence at any finite L.

We now consider the scaling of $C_{\rm act}$ far away from threshold ($|\zeta\Delta\mu| \ll \zeta\Delta\mu_c$) as well. Assuming small $\zeta\Delta\mu$ (i.e. small q_c), we expand the denominator of (35) up to the linear order in $|\zeta\Delta\mu|$. We obtain for

$$C_{\rm act} \sim -k_{\rm B}T\frac{\zeta\Delta\mu\gamma_1}{\eta K L} \tag{40}$$

to leading order in $\zeta\Delta\mu$, valid for $L \ll L_c$. Thus far away from the threshold, the leading contribution to $C_{\rm act}$ scales as $1/L$ with L that is different from both its form near the instability threshold as well as the equilibrium contribution to $C_{\rm tot}$. It remains attractive, however. We thus conclude that $C_{\rm act}$ remains attractive for all $L < L_c$ for a nematically ordered active fluid.

We now discuss the extensile case, i.e. $\zeta\Delta\mu > 0$ for which there are no instabilties at any L. The active Casimir stress $C_{\rm act}$ is still formally defined by equation (24), which yields (35) with the sign of $\zeta\Delta\mu$ reversed. The time-scale τ_q is now given by

$$\tau_n(q)^{-1} = \frac{\nu_1}{\eta}\sum_n\left[(\zeta\Delta\mu + \zeta\Delta\mu_c(n) + \frac{\eta K q^2}{\nu_1\gamma_1}\right] \tag{41}$$

which is positive definite implying stability. The active Casimir stress in this case now reads

$$C_{\rm act} = -\frac{\pi^3 k_{\rm B}T}{2L^3}\frac{\eta}{\gamma_1}\sum_n\int {\rm d}^2q\left[\frac{1}{\zeta\Delta\mu + \zeta\Delta\mu_c(n) + \eta K q^2/(\nu_1\gamma_1)} - \frac{1}{\tau_{qe,n}^{-1}}\right]. \tag{42}$$

We expand in $\Delta\mu$, assuming small activity, and extract the leading order active contribution to $C_{\rm act}$ as

$$C_{\rm act} \sim \frac{k_{\rm B}T}{L}\frac{\gamma_1}{\eta K}\zeta\Delta\mu, \tag{43}$$

that vanishes with $\Delta\mu$, scales with L as $1/L$ and is *positive* in sign. This implies that $C_{\rm act}$ for an extensile active fluid with nematic order is repulsive to the leading order in $\zeta\Delta\mu$, in contrast to $C_{\rm act}$ for a contractile active fluid, or the corresponding equilibrium contribution $C_{\rm eq}$. Furthermore, it does not diverge for any finite L, unlike $C_{\rm act}$ for the contractile case. Note that for an extensile active nematic, the equilibrium contribution always wins over $C_{\rm act}$ for sufficiently small L.

We now heuristically argue why the active Casimir stress $C_{\rm act}$ is attractive for a contractile active nematic, whereas it is repulsive for an extensile active nematic. For both contractile and extensile active nematics, it is given by (3), as the difference between the values of the correlator $K\langle(\partial_z p_i)^2\rangle/2$ for an equilibrium ($\Delta\mu = 0$) nematics and its active counterpart. Equation (35) expresses the Casimir stress as a function of the difference of the relaxation rates of the fluctuations in the active and passive cases. Equation (32) shows that the decay time of typical orientation fluctuations is *smaller* in extensile ($\zeta > 0$) active nematics than its equilibrium value, where as in contractile ($\Delta\mu < 0$) active nematics it is *larger*. As a result, this fixes the sign of the active Casimir stress, $C_{\rm act} < 0$ for contractile active nematics and $C_{\rm act} > 0$ for extensile active nematics.

So far, we have considered a macroscopically oriented state where the reference orientation is assumed to be perpendicular to the film. An alternative choice of boundary condition would be a polarization oriented parallel to the surface of the film: $p_x = 1$ as the reference state for orientation, and $p_z = 0 = p_y$ at $z = 0, L$. Similar arguments show that at the scaling level the active Casimir stress $C_{\rm act}$ in these conditions is still given by equation (39). A third choice for boundary conditions is $p_z = 0$ and p_y to be free at $z = 0, L$ with $p_x = 1$ as the ordered reference state. This is qualitatively different from what we have considered above, owing to the fact that p_y is a soft mode. Further, as discussed in [16], with this choice of the reference state there are no instabilities at any given thickness of the system. Thus, the Casimir stress will be significantly different from (39). We do not discuss this case here.

Summary and conclusions

In this article, we have studied Casimir stresses $C_{\rm tot}$ in a thin layer of ordered nematic liquid crystals. We have shown that it can be written as a sum of the active contribution $C_{\rm act}$ and a contribution $C_{\rm eq}$ equal to that of a passive system. We used the stochastic hydrodynamic theory of active nematics to determine $C_{\rm act}$ for contractile and extensile activities. We find that for contractile active nematics, $C_{\rm act}$ is attractive, just like its equilibrium counterpart. However, $C_{\rm act}$ is fundamentally different from its equilibrium counterpart, because it diverges logarithmically as the threshold thickness for the spontaneous flow instability is approached from below. Thus both the contributions conspire in attraction although with different scalings. For extensile nematics, $C_{\rm act}$ is repulsive and has a scaling with the thickness L that is different from its equilibrium counterpart, and has no divergence at any L. In particular for extensile systems $C_{\rm tot}$ changes sign at a critical thickness which corresponds to an unstable situation. The signs of the active Casimir stress $C_{\rm act}$, which is attractive for contractile activity and repulsive for extensile activity are controlled by the decay times of the orientation fluctuations. Lastly, for small $\Delta\mu$ one might reach large thicknesses L where the physics discussed in [23] could play a role making the passive Casimir stress $C_{\rm eq}$ repulsive; this and its potential effects on active Casimir stress are, however, outside the scope of the present work.

A potential biological system where the active Casimir stresses could be relevant is the thin cell cortex or the cell lamellipodium. Due to the active Casimir forces acting in the direction of the thickness of the actin layer, because of the overall incompressibility, the active layer tends to stretch along the in-plane directions. This causes the cell membrane to stretch and contributes to the *active tension* of the cell cortex. If the thickness of the system is close to the critical threshold of instability, the Casimir force contribution could become important.

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Appendix

Here, we discuss the full calculation of polarization fluctuations in a stochastically driven active fluid layer. The scheme of the calculations here is very similar to the detailed calculation for the diffusion coefficient of a test particle immersed in an active fluid layer, as given in [18] with full details. Nonetheless, we reproduce the basic outline here for the sake of completeness. We start from the relations (5)–(9) and determine h_α, the conjugate field to the polarity vector from a Frank free energy which describes the energies of splay, bend and twist deformations by parameters K_1, K_2 and K_3. For simplicity we consider here the limit $K_1 \to \infty$ (i.e. the splay modes are suppressed, $\nabla \cdot \mathbf{p} = 0$). We furthermore introduce the constraints $\mathbf{p}^2 = 1$ and $\nabla \cdot \mathbf{v} = 0$, i.e. we ignore fluctuations of the magnitude of $\mathbf{p}$ and we treat the fluid as incompressible. The two constraints $\nabla \cdot \mathbf{p} = 0$ and $p^2 = 1$ are imposed by two Lagrange multipliers $h_\|$ and ϕ in the free energy functional

$$F = \frac{1}{2}\int \mathrm{d}^3x[K_2(\nabla \times \mathbf{p})^2 + K_3(\partial_z\mathbf{p})^2 - h_\| p^2 + 2\phi\nabla \cdot \mathbf{p}], \tag{44}$$

where we have assumed that $\mathbf{p}$ exhibits small fluctuations around a reference state $\mathbf{p}_0 = \hat{\mathbf{e}}_z$, the unit vector along the z-axis. The incompressibility constraint is imposed via the pressure P as Lagrange multiplier. The active fluid is confined between two surfaces at $z = 0$ and $z = L$. We impose the following boundary conditions: no flow across the boundary surfaces $v_z(z = 0) = 0$ and $v_z(z = L) = 0$ and vanishing surface shear stress at the boundaries: $\partial v_\alpha/\partial z = 0$, at $z = 0$ and $z = L$ for $\alpha = x, y$. In addition we impose $\mathbf{p}(z = 0) = \hat{\mathbf{e}}_z$ and $\mathbf{p}(z = L) = \hat{\mathbf{e}}_z$. These boundary conditions are satisfied by the Fourier mode expansions

$$v_\alpha(\mathbf{x}, t) = \int \frac{\mathrm{d}^2q}{(2\pi)^2}\frac{\mathrm{d}\omega}{2\pi}\sum_n \tilde{v}_\alpha^n(\mathbf{q}, \omega)\exp[-\mathrm{i}\omega t + \mathrm{i}\mathbf{r}\cdot\mathbf{q}]\cos\left(\frac{n\pi z}{L}\right), \tag{45}$$

$$v_z(\mathbf{x}, t) = \int \frac{\mathrm{d}^2q}{(2\pi)^2}\frac{\mathrm{d}\omega}{2\pi}\sum_n \tilde{v}_z^n(\mathbf{q}, \omega)\exp[-\mathrm{i}\omega t + \mathrm{i}\mathbf{r}\cdot\mathbf{q}]\sin\left(\frac{n\pi z}{L}\right), \tag{46}$$

$$p_\alpha(\mathbf{x}, t) = \int \frac{\mathrm{d}^2q}{(2\pi)^2}\frac{\mathrm{d}\omega}{2\pi}\sum_n \tilde{p}_\alpha^n(\mathbf{q}, \omega)\exp[-\mathrm{i}\omega t + \mathrm{i}\mathbf{r}\cdot\mathbf{q}]\sin\left(\frac{n\pi z}{L}\right), \tag{47}$$

where $\alpha = x, y$. Here, $\mathbf{r}$ is a vector in the $x - y$ plane and the corresponding wavevector is denoted by $\mathbf{q}$. We linearize the state of the system around a reference state with $v_\alpha = 0$, $v_z = 0$ and $\mathbf{p} = \hat{e}_z$. The force balance equation together with the incompressibility condition and the constitutive equation (8) yield equations for the flow field

$$\begin{aligned}
-\eta\left(q^2 + \frac{n^2\pi^2}{L^2}\right)\tilde{v}_z^n(\mathbf{q}, t) &= \zeta\Delta\mu P_{z\beta}\frac{n\pi}{L}\tilde{p}_\beta^n + \mathrm{i}\zeta\Delta\mu P_{zz}q_\beta\tilde{p}_\beta^n - \frac{\nu_1}{2}P_{zz}\left(\mathrm{i}q_\beta\tilde{h}_\beta - \frac{n\pi}{L}\tilde{h}_z^n\right)\\
&\quad - \frac{\nu_1}{2}\frac{n\pi}{L}(P_{z\beta}\tilde{h}_\beta^n - P_{zz}\tilde{h}_z^n) - \frac{1}{2}\left(\mathrm{i}q_\beta\tilde{h}_\beta^n - \frac{n\pi}{L}\tilde{h}_z^n\right) + P_{z\beta}\tilde{\xi}_\beta^{\sigma,n} + P_{zz}\tilde{\xi}_z^{\sigma,n},\\
-\eta\left(q^2 + \frac{n^2\pi^2}{L^2}\right)\tilde{v}_\alpha^n(\mathbf{q}, t) &= \zeta\Delta\mu P_{\alpha\beta}\frac{n\pi}{L}\tilde{p}_\beta^n + \mathrm{i}\zeta\Delta\mu P_{\alpha z}q_\beta\tilde{p}_\beta^n - \frac{\nu_1}{2}\left(P_{\alpha\beta}\frac{n\pi}{L}\tilde{h}_\beta^n - P_{\alpha z}\frac{n\pi}{L}\tilde{h}_z^n\right)\\
&\quad - \frac{\nu_1}{2}P_{\alpha z}\left(\mathrm{i}q_\beta\tilde{h}_\beta^n - \frac{n\pi}{L}h_z^n\right) + \frac{1}{2}\frac{n\pi}{L}\tilde{h}_\alpha^n + P_{\alpha\beta}\tilde{\xi}_\beta^{\sigma,n} + P_{\alpha z}\tilde{\xi}_z^{\sigma,n}
\end{aligned} \tag{48}$$

where $\alpha, \beta = x$ or y. Here, we have introduced the transverse projection operators $P_{zz} = q^2/(q^2 + n^2\pi^2/L^2)$, $P_{\alpha\beta} = \delta_{\alpha\beta} - q_\alpha q_\beta/(q^2 + n^2\pi^2/L^2) = P_{\beta\alpha}$, and $P_{\alpha z} = -\mathrm{i}q_\alpha(n\pi/L)/(q^2 + n^2\pi^2/L^2) = P_{z\alpha}$ and the pressure P has already been eliminated. The noise terms $\tilde{\xi}_\alpha^{\sigma,n}$ have zero-mean with variance

$$\langle\tilde{\xi}_\alpha^{\sigma,n}(\mathbf{q}, \omega)\tilde{\xi}_\beta^{\sigma,m}(\mathbf{q}', \omega')\rangle = 2\eta k_\mathrm{B}T\left(q^2 + \frac{n^2\pi^2}{L^2}\right)(2\pi)^3\delta(\mathbf{q} + q')\delta(\omega + \omega')\delta_{\alpha\beta}\delta_{nm}, \tag{49}$$

where α and $\beta = x, y, z$.

The dynamic equation for the polarization field reads

$$-\mathrm{i}\omega\tilde{p}_\alpha^n = -\tilde{\omega}_{\alpha z}^n - \frac{K}{\gamma_1}\left(q^2 + \frac{n^2\pi^2}{L^2}\right)\tilde{p}_\alpha^n - \frac{1}{\gamma_1}(h_\|\tilde{p}_\alpha^n - \mathrm{i}q_\alpha\tilde{\phi}^n) - \nu_1\tilde{u}_{\alpha z}^n + \tilde{\xi}_{\perp,\alpha}^n, \tag{50}$$

with $\tilde{u}_{\alpha z}^n = \left(-\frac{n\pi}{L}\tilde{v}_\alpha^n + \mathrm{i}q_\alpha\tilde{v}_z^n\right)/2$, $\tilde{\omega}_{\alpha z}^n = -\left(\frac{n\pi}{L}\tilde{v}_\alpha^n + \mathrm{i}q_\alpha\tilde{v}_z^n\right)$ and noise correlations

$$\langle\tilde{\xi}_{\perp,\alpha}^n(\mathbf{q}, \omega)\tilde{\xi}_{\perp,\beta}^m(\mathbf{q}', \omega')\rangle = \frac{2K_\mathrm{B}T}{\gamma_1}(2\pi)^3\delta(\mathbf{q} + q')\delta(\omega + \omega')\delta_{\alpha\beta}\delta_{nm}. \tag{51}$$

Further, with $K_2 = K_3 = K$ we have $h_\alpha = -\frac{\delta F}{\delta p_\alpha} = K\nabla^2 p_\alpha + h_\| p_\alpha + \nabla_\alpha\phi$ in the real space. Elimination of the Lagrange multipliers $h_\|$ and ϕ finally leads to [18]

$$-\eta\left(q^2 + \frac{n^2\pi^2}{L^2}\right)\tilde{v}_z^n = P_{zz}\tilde{\xi}_z^{\sigma,n} + P_{z\beta}\tilde{f}_\beta^{\sigma,n}, \tag{52}$$

$$-\eta\left(q^2 + \frac{n^2\pi^2}{L^2}\right)\tilde{v}_\alpha^n = \zeta\Delta\mu\frac{n\pi}{L}\tilde{p}_\alpha^n + \frac{\nu_1 - 1}{2}\frac{n\pi}{L}K\left(q^2 + \frac{n^2\pi^2}{L^2}\right)\tilde{p}_\alpha^n + P_{\alpha\beta}\tilde{\xi}_\beta^{\sigma,n} + P_{\alpha z}\tilde{\xi}_z^{\sigma,n}, \tag{53}$$

$$\frac{\partial \tilde{p}_\alpha^n}{\partial t} = -\frac{K}{\gamma_1}\left(q^2 + \frac{n^2\pi^2}{L^2}\right)\tilde{p}_\alpha^n + \frac{\nu_1 - 1}{2}\frac{n\pi}{L}\tilde{v}_\alpha^n + P_{\alpha\beta}\tilde{\xi}_{\perp,\beta}^n + P_{\alpha z}\tilde{\xi}_{\perp,z}^n. \tag{54}$$

Note that $\tilde{v}_z^n$ decouples from $\tilde{p}_\alpha^n$. Equations (53), (54) may be used to obtain expressions for the fluctuations of $\tilde{p}_\alpha^n$:

$$\left(\frac{\partial}{\partial t} + \frac{1}{\tilde{\tau}_q}\right)\tilde{p}_\alpha^n = -\frac{n\pi}{L}\frac{\nu_1 - 1}{2}\frac{P_{\alpha\beta}\tilde{\xi}_m^{\sigma,n} + P_{\alpha z}\tilde{\xi}_z^{\sigma,n}}{\eta\left(q^2 + \frac{n^2\pi^2}{L^2}\right)} + P_{\alpha\beta}\tilde{\xi}_{\perp\beta}^n + P_{\alpha z}\tilde{\xi}_{\perp,z}^n. \tag{55}$$

where we have identified an effective relaxation time $\tilde{\tau}_q$ of the polarization fluctuations $\tilde{p}_\alpha^n$:

$$\tilde{\tau}_q = \left[\frac{K}{\gamma_1}\left(q^2 + \frac{n^2\pi^2}{L^2}\right) + \frac{\nu_1 - 1}{2}\left(\zeta\Delta\mu + \frac{\nu_1 - 1}{2}K\left(q^2 + \frac{n^2\pi^2}{L^2}\right)\right)\frac{n^2\pi^2}{L^2}\frac{1}{\eta\left(q^2 + \frac{n^2\pi^2}{L^2}\right)}\right]^{-1}. \tag{56}$$

For the stability of the assumed oriented state of polarization one must have $\tilde{\tau}_q > 0$. Time-scale $\tilde{\tau}_q$ is the analog of the time-scale $t_p(q)$ that we extract from equation (31). This allows us to calculate the correlation function of p_α^n $(\alpha = x, y)$: we find

$$\langle\sigma_{zz}^e\rangle_{z=L} = -\langle(\partial_z p_i)^2\rangle_{z=L} = -\int\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{\pi}{L}\sum_n\frac{n^2\pi^2}{L^2}\frac{2k_BT}{\Delta_n}\left[\frac{1}{\gamma_1} + \frac{(\nu_1 - 1)^2}{4\eta\left(q^2 + \frac{n^2\pi^2}{L^2}\right)}\frac{n^2\pi^2}{L^2}\right], \tag{57}$$

where

$$\Delta_n = K\left(q^2 + \frac{n^2\pi^2}{L^2}\right)\left[\frac{1}{\gamma_1} + \frac{(\nu_1 - 1)^2}{4\eta\left(q^2 + \frac{n^2\pi^2}{L^2}\right)}\frac{n^2\pi^2}{L^2} + \frac{\zeta\Delta\mu(\nu_1 - 1)}{2\eta\kappa\left(q^2 + \frac{n^2\pi^2}{L^2}\right)^2}\frac{n^2\pi^2}{L^2}\right]. \tag{58}$$

Thus we obtain for the active Casimir stress in an orientationally ordered active fluid: using (24)

$$C_{\mathrm{act}} = -\frac{K}{2}\langle(\partial_z p_i)^2\rangle_{z=L} + \frac{K}{2}\langle(\partial_z p_i)^2\rangle_{z=L,\Delta\mu=0} = \frac{K}{2}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{\pi}{L}\sum_n\frac{n^2\pi^2}{L^2}\frac{2k_BT\zeta\Delta\mu(\nu_1 - 1)}{2\eta K\left(q^2 + \frac{n^2\pi^2}{L^2}\right)\Delta_n}. \tag{59}$$

This holds for both contractile and extensile active fluids and vanishes as $\Delta\mu$ is set to zero.

For a contractile active fluid with nematic order, C_{act} diverges when $\Delta_n = 0$, which can happen with a finite $\Delta\mu < 0$. The minimum thickness for which this can happen is given by the condition

$$\frac{K}{\gamma_1}\frac{\pi^2}{L_c^2} + \frac{(\nu_1 - 1)^2}{4\eta}K\frac{\pi^2}{L_c^2} = -\zeta\Delta\mu\frac{\nu_1 - 1}{2\eta}. \tag{60}$$

We evaluate the active contribution in (59) near the instability threshold (for a finite $\zeta\Delta\mu < 0$), i.e. as $L \to L_c$ from below. In this limit, only the $n = 1$ contribution diverges; the contributions with $n > 1$ are all finite. Therefore, we retain only the $n = 1$ contribution and evaluate it; we discard all higher-n contributions. Define $L = L_c(1 - \delta)$, $\delta > 0$ is a small dimensionless number. Keeping only the divergent term contribution as $\delta \to 0$, we obtain for the active contribution to the Casimir stress C_{tot} as L approaches L_c from below

$$C_{\mathrm{act}} = k_BT\frac{1}{2L_c}\frac{\zeta\Delta\mu(\nu_1 - 1)}{8\eta + \gamma_1(\nu_1 - 1)^2}\ln\left|\frac{[2/\gamma_1 + (\nu_1 - 1)^2/4\eta]\gamma_1}{2\delta\Gamma(\nu_1 - 1)}\right|. \tag{61}$$

Substituting for $\zeta\Delta\mu$ from (60), we find

$$C_{\mathrm{act}} = k_BT\frac{-\pi^2}{2L_c^3}\frac{\Gamma\gamma_1}{8\eta + \gamma_1(\nu_1 - 1)^2}\ln\left|\frac{[2/\gamma_1 + (\nu_1 - 1)^2/4\eta]\gamma_1}{2\delta\Gamma}\right|, \tag{62}$$

same as (25) as above. Thus, C_{act} approaches $-\infty$ as $\delta \to 0$. Thus, it is *attractive*, similar to the equilibrium contribution [2]. The equilibrium contribution may be evaluated in straightforward ways by following [2]: One finds, at $L \to L_c$

$$C_{\mathrm{eq}} = -\frac{1}{8\pi}\frac{k_BT}{L_c^3}\zeta_R(3). \tag{63}$$

Thus, following the logic outlined in the main text, the total Casimir stress C_{tot} for an active fluid layer of thickness $L \to L_c$ from below is given by

$$C = C_{\rm act} + C_{\rm eq} = k_{\rm B}T\frac{-\pi^2}{2L_c^3}\frac{\Gamma\gamma_1}{8\eta + \gamma_1(\nu_1 - 1)^2}\ln\left|\frac{[2/\gamma_1 + (\nu_1 - 1)^2/4\eta]\gamma_1}{2\delta\Gamma}\right| - \frac{1}{8\pi}\frac{k_{\rm B}T}{L_c^3}\zeta_R(3), \tag{64}$$

which is, of course, overall attractive.

The scaling of $C_{\rm act}$ with L changes drastically for $L \ll L_c$. We use (59) and focus on the second term on the right-hand side of it which is the active contribution. We extract the $\mathcal{O}(\zeta\Delta\mu)$ contribution for small $\zeta\Delta\mu$ that yields the leading order active contribution to $C_{\rm tot}$ for small $\zeta\Delta\mu$. We find

$$C_{\rm act} = \frac{K}{2}\int\frac{d^2q}{(2\pi)^2}\frac{\pi}{L}\sum_n\frac{n^2\pi^2}{L^2}\frac{2k_{\rm B}T\zeta\Delta\mu(\nu_1 - 1)}{2\eta K^2\left(q^2 + \frac{n^2\pi^2}{L^2}\right)^2\left[\frac{1}{\gamma_1} + \frac{(\nu_1 - 1)^2}{4\eta\left(q^2 + \frac{n^2\pi^2}{L^2}\right)}\frac{n^2\pi^2}{L^2}\right]}. \tag{65}$$

This active contribution, being negative ($\zeta\Delta\mu < 0$), remains attractive and clearly scales as $1/L$, different from both the equilibrium contribution (that scales as $1/L^3$) and the contribution for $L \to L_c$ from below that shows a logarithmic divergence. This is consistent with the predictions from our simplified analysis above, and reminiscent of the $1/L$-dependence of a similar Casimir-like force in a one-dimensional confined active particle system studied in [9].

So far, we have considered only thermal noises above while averaging over the noise ensembles, keeping the active effects only in the deterministic parts of the dynamical model. In general, however, there are active noises present over and above the thermal noises. For simplicity, we supplement the thermal noise in (55) by an active noise that is assumed to be δ-correlated in space and time, with a variance that should scale with $\Delta\mu$. The precise amplitude of the variance should depend on the detailed nature of the stochasticity of the motor movements. We now refer to equation (57): then to the leading order in $\Delta\mu$, the active noises should generate an additional active contribution δC_A to $C_{\rm act}$ in (59) above near $L = L_c$. This is of the form

$$\delta C_A \sim -\frac{D_0\Delta\mu}{L_c^3}\zeta_R(3), \tag{66}$$

where D_0 is a dimensional constant. Thus, this additional contribution is attractive, has the same scaling with L as the equilibrium contribution $C_{\rm eq}$ and has no divergence as $L \to L_c$ from below. We did not consider any active, multiplicative noises that may be important in cell biology contexts as illustrated in [6].

Our analyses above may be extended to obtain $C_{\rm tot}$ just above the the threshold of the spontaneous flow instability for the contractile case [15]. Above the threshold, the steady reference state is given by $v_x = A\cos(z\pi/L)$, $p_z = 1$, $p_{x0} = \epsilon\sin(z\pi/L)$, $v_z = 0 = v_y$, $p_y = 0$, with $A = 4L\zeta\Delta\mu\epsilon/[\pi(4\eta + \gamma_1(\nu_1 + 1)^2)]$ and $\epsilon = \sqrt{1 - L_c/L}$, $L > L_c$ [15]. We discuss the case with $\epsilon \to 0$. We impose the same boundary conditions as above. The viscous contribution to C continues to be zero by the same argument as above, since the spontaneous flow velocity v_x has no in-plane coordinate dependences. Defining δp_x as the fluctuation of p_x around p_{x0}, the new reference state, we note that the boundary condition on δp_x is same as that on p_x before, i.e. for no spontaneous flows; boundary conditions on p_y, having a zero value in the reference state, naturally remains unchanged from the previous case. We, thus, conclude that δp_x and p_y follow the same (linearized) equations (55) for p_x and p_y as in the previous case. Hence, the solutions for δp_x and p_y are identical to those of p_x and p_y in the previous case. It is now straightforward to see that the expression for the Casimir stress $C_{\rm tot}$ as given in (64) now has an additional contribution

$$\delta C = -\frac{K}{2}\langle\partial_z p_{x0}\partial_z p_{x0}\rangle|_{z=L} = -\frac{K}{2}\epsilon^2\frac{\pi^2}{L^2}\cos^2(\pi z/L)|_{z=L} = -\frac{K}{2}\frac{L - L_c}{L}\frac{\pi^2}{L^2}. \tag{67}$$

We note that the additional contribution δC depends on the Frank elastic constant K and has a negative sign, displaying its attractive nature. Further and not surprisingly, it vanishes as $(L - L_c)$ as $L \to L_c$, and hence is small just above the threshold. Thus, even above the threshold of the spontaneous flow instability, the dominant contribution to $C_{\rm tot}$ still comes from (64), its value just below the threshold. Lastly, if we continue to use the above reference states for $L_c \lesssim L$ even for $L \gg L_c$, then δC scales as $1/L^2$ for $L \gg L_c$ and forms the dominant contribution in $C_{\rm tot}$.

In the above we have considered a contractile active fluid. For an extensile system with $\xi\Delta\mu > 0$, there are no divergences in (57) or (59) for any L. Expanding (59) in $\zeta\Delta\mu$, we extract an active contribution linear in $\zeta\Delta\mu$ that scales with L as $1/L$, different from the scaling of $C_{\rm act}$ in the contractile case, or from the equilibrium contribution $C_{\rm eq}$. We find for the leading order active contribution to the Casimir stress

$$C_{\rm act} = \frac{K}{2}\int \frac{d^2q}{(2\pi)^2}\frac{\pi}{L}\sum_n \frac{n^2\pi^2}{L^2}\frac{2k_BT\zeta\Delta\mu(\nu_1-1)}{2\eta K^2\left(q^2+\frac{n^2\pi^2}{L^2}\right)^2\left[\frac{1}{\gamma_1}+\frac{(\nu_1-1)^2}{4\eta\left(q^2+\frac{n^2\pi^2}{L^2}\right)}\frac{n^2\pi^2}{L^2}\right]} \sim \frac{k_BT\zeta\Delta\mu\gamma_1}{\eta KL}, \quad (68)$$

that scales with L as $1/L$; here only. Thus, the active contribution comes with a positive sign ($\zeta\Delta\mu > 0$), i.e. repulsive Casimir stress, a feature obtained in our simplified analysis above. Furthermore given that $C_{\rm eq} < 0$, it is possible that $C_{\rm tot} = C + C_{\rm eq}$ changes sign as the thickness L or the activity parameter $\zeta\Delta\mu$ is varied correspond to an unstable situation, potentially creating an intriguing crossover between a repulsive and an attractive Casimir stress. Lastly, the differences in the active Casimir stress $C_{\rm act}$ for the contractile and extensile cases potentially open up experimental routes to distinguish contractile activity from extensile activity by measuring $C_{\rm tot}$.

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