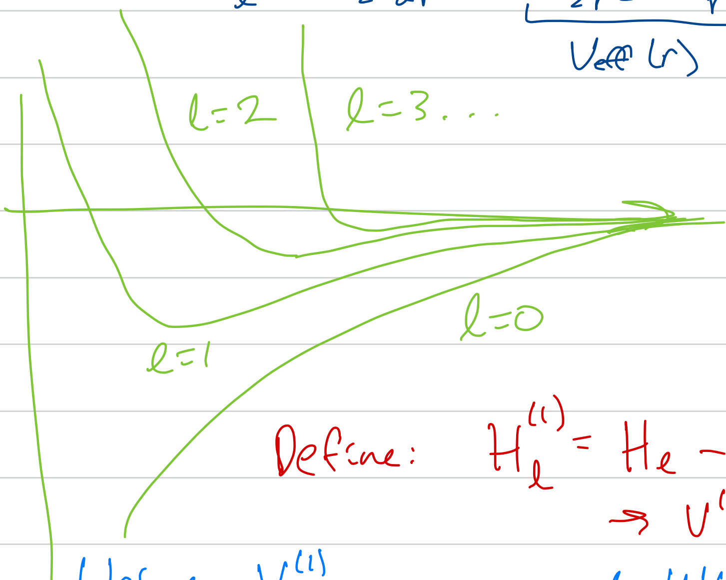
LECTURE 2b/3a

23 Oct '23

Quick review of the SUSYSO (supersymmetric solution) to the hydrogen atom.

- Goal: obtain all $E<0$ eigenstates of the radial schrö eq in each l channel.

$$H_l = -\frac{1}{2}\frac{d^2}{dr^2} + \underbrace{\frac{l(l+1)}{2r^2} - \frac{1}{r}}_{V_{eff}(r)}$$



Define: $H_l^{(1)} = H_l - \varepsilon_0$

$$\rightarrow V^{(1)} = \frac{l(l+1)}{2r^2} - \frac{1}{r}.$$

Using $V^{(1)}$, we guessed $W(r)$ and got:

$$W(r) = \frac{1}{\sqrt{2}}\left[\frac{1}{l+1} - \frac{l+1}{r}\right].$$

This guess also generated for us $\boxed{\varepsilon_{0l} = -\frac{1}{2(l+1)^2}}$.

With W, we defined:

$$\hat{A}_\ell = \frac{1}{\sqrt{2}}\left(d/dr + \frac{1}{\ell+1} - \frac{\ell+1}{r}\right)$$

$$\hat{A}_\ell^\dagger = \frac{1}{\sqrt{2}}\left(-d/dr + \frac{1}{\ell+1} - \frac{\ell+1}{r}\right)$$

These define

ℓ-dependence is normal

$$\bullet\quad H_\ell^{(1)} = A_\ell^\dagger A_\ell = \frac{1}{2}\Big(\underbrace{-d^2/dr^2 + \boxed{\frac{(\ell+1)\ell}{r^2}} - \frac{2}{r}}_{\text{this is the !}} + \frac{1}{(\ell+1)^2}\Big)$$

$$= H_\ell^{coulomb} - \underbrace{\left(-\frac{1}{2(\ell+1)^2}\right)}_{E_{0\ell}} \qquad 0$$

ℓ-dep. is of $\ell+1$

$$\bullet\quad H_\ell^{(2)} = A_\ell A_\ell^\dagger = \frac{1}{2}\left(-d^2/dr^2 + \boxed{\frac{(\ell+1)(\ell+2)}{r^2}} - \frac{2}{r} + \frac{1}{(\ell+1)^2}\right)$$

$$= H_{\ell+1}^{coulomb} + \frac{1}{2(\ell+1)^2}. \qquad 1$$

From $\circ$:

$$\rightarrow H_{\ell+1}^{(1)} = H_{\ell+1}^{coulomb} - \left(-\frac{1}{2(\ell+2)^2}\right)$$

$$\rightarrow H_{\ell+1}^{coul} = H_{\ell+1}^{(1)} - \frac{1}{2(\ell+2)^2} \qquad 2$$

1 + 2 = 3 :

$$\rightarrow H_\ell^{(2)} = H_{\ell+1}^{(1)} + \frac{1}{2(\ell+1)^2} - \frac{1}{2(\ell+2)^2} \qquad 3$$

Now we want to use these to generate excited states.

- Let ③ act on the g.s. of $H^{(1)}_{\ell+1}$:

$$H^{(2)}_{\ell}\phi^{(1)}_{0\ell+1} = \Big(\underbrace{H^{(1)}_{\ell+1} + \frac{1}{2(\ell+1)^2} - \frac{1}{2(\ell+2)^2}\Big)\phi^{(1)}_{0\ell+1}}_{=0}$$

$$\rightarrow H^{(2)}_{\ell}\phi^{(1)}_{0\ell+1} = \Big(0 + \underbrace{\frac{1}{2(\ell+1)^2} - \frac{1}{2(\ell+2)^2}}_{\star}\Big)\phi^{(1)}_{0,\ell+1}.$$

So: $\star$ is an eval of $H^{(2)}_{\ell}$.

But by SUSY, non-zero evals of $H^{(2)}_{\ell}$ are shared by $H^{(1)}_{\ell}$!

$$\rightarrow A_\ell^{+} H^{(2)}_{\ell}\phi^{(1)}_{0\ell+1} = \star A_\ell^{+}\phi^{(1)}_{0\ell+1}$$

$$\rightarrow \underbrace{A_\ell^{+}A_\ell}_{H^{(1)}_{\ell}}\underbrace{(A_\ell^{+}\phi^{(1)}_{0\ell+1})}_{\phi^{(1)}_{1\ell}} = \underbrace{\star}_{\star}\underbrace{(A_\ell^{+}\phi^{(1)}_{0\ell+1})}_{\phi^{(1)}_{1\ell}}$$

So $\star = \dfrac{1}{2(\ell+1)^2} - \dfrac{1}{2(\ell+2)^2}$ is the eval of state $\phi^{(1)}_{1\ell}$!

$$\rightarrow \text{from } ⓪: \quad \mathcal{E}^{\text{coulomb}}_{1\ell} = -\frac{1}{2(\ell+2)^2}.$$

Now we can go again around the loop:

$$H_l^{(2)} \phi^{(1)}_{1\,l+1} = \left(H^{(1)}_{l+1} + \frac{1}{2(l+1)^2} - \frac{1}{2(l+2)^2} \right) \phi^{(1)}_{1\,l+1}$$

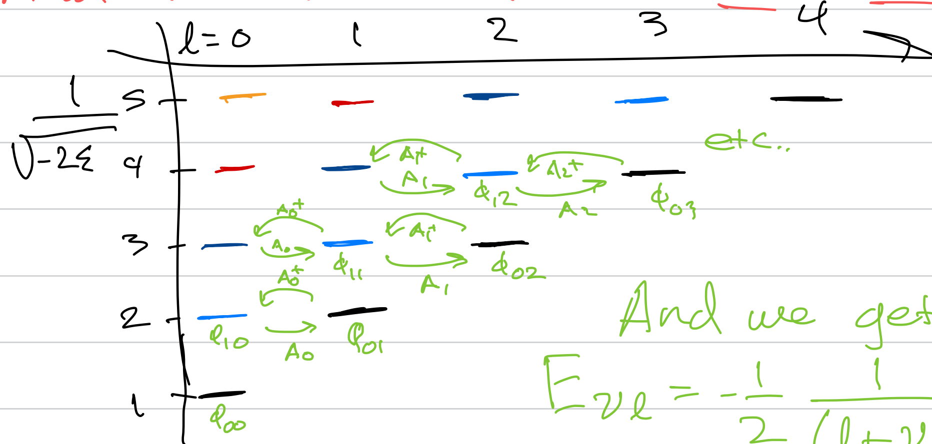
$$= \underbrace{\left(\cancel{\frac{1}{2(l+2)^2}} - \frac{1}{2(l+3)^2} + \frac{1}{2(l+1)^2} - \cancel{\frac{1}{2(l+2)^2}} \right)}_{\star} \phi^{(1)}_{1\,l+1}$$

$$\to \quad H_l^{(1)} \left(A^+ \phi^{(1)}_{1\,l+1} \right) = \star A^+ \phi^{(1)}_{1\,l+1}$$

$$\to \quad H_l^{(1)} \phi^{(1)}_{2\,l} = \star \phi^{(1)}_{2\,l}$$

<u>And</u> then $H_l \phi^{(1)}_{2l} = \underline{-\frac{1}{2(l+3)^2}} \phi^{(1)}_{2l}$.

And finally we get the <u>full Coulomb spectrum</u>



And we get:

$$E_{\nu l} = -\frac{1}{2} \frac{1}{(l+\nu+1)^2},$$

where $\nu = 0, 1, \ldots$

$l = 0, 1, \ldots$

or, letting $n = l + \nu + 1$,

$$\boxed{\varepsilon_{nl} = -\frac{1}{2n^2}}$$

This procedure we just did is part of a much larger context of 'shape invariant potentials', a category of 1D potentials defined by the relationship

indep. of x

$$V^{(2)}(x, a_1) = V^{(1)}(x, a_2) + R(a_1)$$

$a_2 = f(a_1)$.

This lets us define a family of Hams

$$H_s = -\frac{1}{2}\frac{d^2}{dx^2} + V_1(x, a_s) + \sum_{k=1}^{s-1} R(a_k)$$

$$a_s = \underbrace{f(f(f \dots (a_1))) \dots)}_{s-1}$$

Who cares? Well,

$$H_{s+1} = -\frac{1}{2}\frac{d^2}{dx^2} + \underbrace{V_1(x, a_{s+1})}_{= V_2(x, a_s) - R(a_s)} + \sum_{k=1}^{s} R(a_k)$$

$$\rightarrow \quad = -\frac{1}{2}\frac{d^2}{dx^2} + V_2(x, a_s) + \sum_{k=1}^{s-1} R(a_k)$$

H_s and H_{s+1} are SUSY HAMS $\rightarrow$ so their spectra are identical except for H_s's GS, w/ $\mathcal{E}_0^s = \sum_{k=1}^{s-1} R(a_k)$.

Doing this for all s values gives the spectrum

$$\mathcal{E}_n = \sum_{k=1}^{n} R(a_k), \quad \mathcal{E}_0 = 0.$$

Check all this w/ Mr. Coulomb!

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In conclusion, we used SUSY HAMS to solve the hydrogen problem. This introduced a wider concept of factorizable Hamiltonians and SIPs, which are another way of categorizing all analytically solvable systems!

Historically, Mr. Schrö himself was the first to factorize the Coulomb problem!

For more reading:

Cooper, Khare, & Sukhatme Phy. Rep. __251__, 267 (1995)