Lecture 2.

16 Oct '23

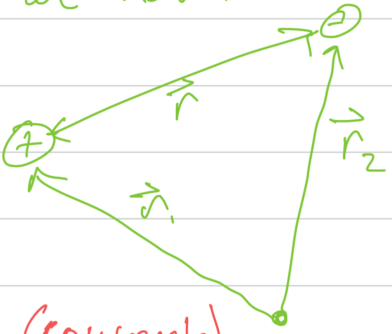
Last time, we 'derived' the Rydberg formula using the Bohr model. This approach manages to get the right energy levels, and had some useful explanatory power (recall the Pickering series), but it can't do much else. In particular, it was clear soon after the Bohr model was developed that it can't handle even the helium atom - but lots of clever generalizations were attempted (see RMP 2000)

So today let's solve it the correct way, using the Schrödinger equation. We'll stick with hydrogen for now, as it is still the only atom with an analytical solution.

But, even though we're just solving (again) the hydrogen atom, we'll do so in a way that nicely illustrates how Rydberg physics is a fun 'playground' in which we can explore the wide world of AMO / Quantum physics.

So, to start with, we have:

$$H = -\frac{\nabla_1^2}{2m_1} - \frac{\nabla_2^2}{2m_2} + V(\vec{r}_2 - \vec{r}_1)$$



* note: atomic units!

where...

- $\nabla^2 = \Delta$ = Laplacian
- $V(\vec{r}_2 - \vec{r}_1) \equiv V(r) = -1/r$, (coulomb)

$$\vec r = \vec r_2 - \vec r_1$$

Evidently, this is a 6D equation, but the interaction only depends on a single distance! This encourages us to try a change of coordinate system...

$$\vec R = \frac{m_1 \vec r_1 + m_2 \vec r_2}{\underbrace{m_1 + m_2}_{M}}, \quad \vec r = \vec r_1 - \vec r_2.$$

So: $\partial_{r_1} = \frac{\partial R}{\partial r_1} \cdot \frac{\partial}{\partial R} + \frac{\partial r}{\partial r_1} \cdot \frac{\partial}{\partial r} = \frac{m_1}{M}\partial_R + \partial_r$

$\partial_{r_2} = \frac{\partial R}{\partial r_2} \cdot \partial_R + \frac{\partial r}{\partial r_2}\partial_r = \frac{m_2}{M}\partial_R - \partial_r$

And then:

$$-\frac{\nabla_1^2}{2m_1} - \frac{\nabla_2^2}{2m_2} = -\left[\frac{1}{2}\frac{m_1}{M^2}\partial_R^2 + \cancel{\frac{\partial_R \partial_r}{M}} + \partial_r^2/m_1 + \frac{1}{2}\frac{m_2}{M^2}\partial_R^2 + -\cancel{\frac{\partial_R \partial_r}{M}} + \partial_r^2/m_2\right]$$

$$= -\frac{1}{2}\frac{m_1 + m_2}{M}\partial_R^2 + -\frac{m_2 \partial_r^2 + m_1 \partial_r^2}{m_1 m_2}$$

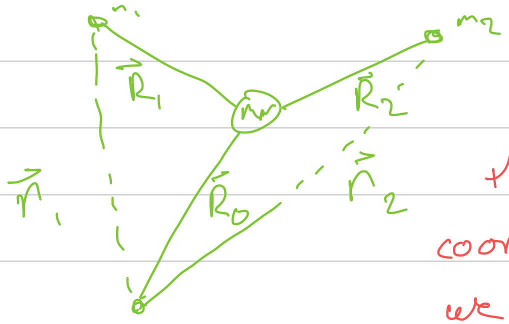
$$= -\frac{1}{2M}\nabla_R^2 - \frac{1}{2\mu}\nabla_r^2$$

So in the end our transformed Ham is:

$$H = \underbrace{-\frac{\nabla_R^2}{2M}}_{\text{CoM}} \underbrace{-\frac{\nabla_r^2}{2\mu} - \frac{1}{r}}_{\text{relative}}.$$

Such transformations should be fairly routine, but it's good to remember how to do it when more complicated situations emerge, for ex:

He: $H = -\frac{\nabla_1^2}{2m_e} - \frac{\nabla_2^2}{2m_e} - \frac{\nabla_0^2}{2M_n} - \frac{1}{|\vec{R}_1 - \vec{R}_0|} - \frac{1}{|\vec{R}_2 - \vec{R}_0|} + \frac{1}{|\vec{R}_1 - \vec{R}_2|}$



After transforming to the obvious relative coordinates (try it @ home), we get...

$$H = -\frac{\nabla_0^2}{2M} - \frac{\nabla_1^2}{2\mu} - \frac{\nabla_2^2}{2\mu} - \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{|\vec{r}_1 - \vec{r}_2|} \underbrace{- \frac{\nabla_1 \cdot \nabla_2}{M_n}}$$

- $M = M_n + 2m_e$
- $\mu = \frac{m_e M_n}{M_n + m_e}$

this 'mass-polarization' term adds a type of kinetic correlation/interaction b/w the electrons

So, generic many-particle systems will have their overall dimensionality reduced when we choose clever coordinates, but this often comes at a cost via these terms. In atoms it is almost always a perturbation, but similar terms in molecules cannot be ignored.

Our next move is to solve the relative-motion Ham, $H_e = -\frac{\nabla_r^2}{2} - \frac{1}{r}$. Here we took the sensible limit $m_p \gg m_e$ to write $\mu = 1$ a.u. To solve this <u>3D</u> PDE, we exploit the spherical symmetry of $1/r$.

We look up ∇^2 in sph. coords, and find:

$$\underbrace{\nabla^2 = \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{\vec{L}^2}{r^2},}_{\text{where}}$$

$$\vec{L}^2 \underset{\text{Spherical Harmonics}}{Y_{\ell m}}(\theta,\varphi) = \overset{\ell = 0, 1, 2, \ldots}{\ell}(\ell+1)\, Y_{\ell m}(\theta,\varphi) \quad ①$$

Our TISE to solve is therefore:

$$\left(-\frac{1}{2r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\vec{L}^2}{2r^2} - \frac{1}{r} - \varepsilon\right)\psi(\vec{r}) = 0.$$ ②

Let's think about the form $\psi(\vec{r})$ should take.

The first thing to notice is that the eigenstates $Y_{\ell m}$ of $\vec{L}^2$ define a complete and orthonormal basis in $\Omega = \hat{r} = \theta, \varphi$.

perhaps "transform" is better

So, we can reduce the complexity of this 3D problem into infinitely many coupled 1D problems by expanding $|\psi\rangle$ into this basis!

Even though this is a silly example, it illustrates a process that we will repeat many, many times in this course!

So, we write

$$\psi(\vec{r}) = \sum_{\ell m} \frac{u_{E\ell m}(r)}{r} Y_{\ell m}(\hat{r})$$ ③

maybe a helpful analogy: this is just the same argument as Fourier analysis: here we just get coefficients at each r!

Remember: this is exact!

Note that this form of radial function means we only have to deal with 1^{st} derivatives:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{u}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(ru' - u\right) = \frac{1}{r^2}\left(u' + ru'' - u'\right)$$

$$= \frac{1}{r}u''.$$

So everywhere the $1/r$ cancels out and ②, with our wavefunction ③, is ...

$$\left(-\frac{1}{2}\frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2} - \frac{1}{r} - E\right)\sum_{lm} u_{Elm}(r)\,Y_{lm}(\hat{r}) = 0.$$

Now we can utilize $\langle lm|l'm'\rangle = \delta_{ll'}\delta_{mm'}$ by projecting (left-multiplying and integrating) this eq. onto $\langle l'm'|$. This gives an equation for each l, m:

$$\left(-\frac{1}{2}\frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2} - \frac{1}{r} - E\right)u_{El}(r) = 0$$

→ Our problem of solving the S.E. for a Rydberg atom boils down to the solution of a radial S.E. for each l

These equations decoupled because our potentials are spherically symmetric → conservation of angular momentum really saves a lot of time!

This will not generically be the case, as we will see, and many of the problems we encounter will require a lot of clever thinking to avoid solving infinitely-many coupled equations!

Aside: an electric field $\vec{F}$ gives a potential $\vec{r}\cdot\vec{F} = rF\cos\theta$.

→ $\langle l'm'|rF\cos\theta|lm\rangle = \sqrt{\frac{4\pi}{3}}\,rF\,\underbrace{\langle l'm'|Y_{10}|lm\rangle}$

- this term couples equations with l to those with $l'=l\pm1$.

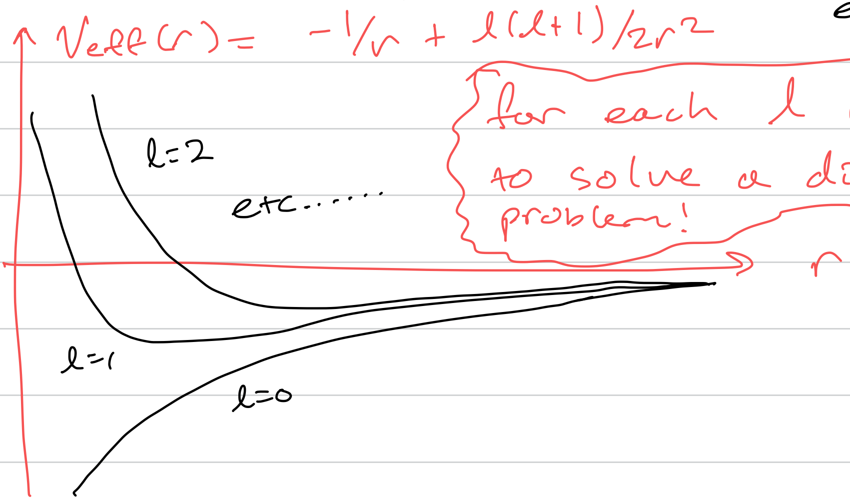
For now, though, we will have to settle for just solving ∞ decoupled 1D equations, $(H_l - E)u_{El}(r) = 0$.

where $\boxed{H_l \equiv -\frac{1}{2}\frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2} - \frac{1}{r}}$.

Let's solve this equation,
subject to b.c.'s

$$u_{E\ell}(r=0) \sim r^{\ell+1} = 0 \qquad (\nabla^2 \text{ behavior})$$

$$u_{E\ell}(r=\infty) = 0 \qquad (\text{finite probability everywhere})$$



for each ℓ we have to solve a different problem!

But let's not solve them with a garbage method like power series... let's solve them all at once using

Super-Symmetric Quantum Mechanics!

woohoo!

Let us begin with the generic 1D Ham:

$$H = -\frac{1}{2} d^2/dx^2 + V'(x) \qquad ①$$

where $V'(x)$ has been shifted so that the ground state of H sits at zero energy, i.e. $V'(x) = V(x) - \varepsilon_0$.

This is a nice trick, since $\psi_0(x)$, the ground state, is nodeless → so we have no pesky poles in:

$$H\psi_0 = 0$$

$$\rightarrow -\frac{1}{2}\psi_0'' + V'\psi_0 = 0$$

$$\rightarrow \underline{V' = \frac{1}{2}\frac{\psi_0''}{\psi_0}}.$$

That's cool!!

What other weird ways can we write $V'(x)$? How about -

$$V'(x) = W(x)^2 - \frac{1}{\sqrt{2}} W'(x),$$

where $W(x)$ is the SUPER potential!

... But why?

It's simple - we can use $\hat{W}$ to define $\hat{A}$:

$$\hat{A} = \frac{1}{\sqrt{2}}\frac{d}{dx} + W(x)$$

$$\hat{A}^{\dagger} = -\frac{1}{\sqrt{2}}\frac{d}{dx} + W(x).$$

And these have the groovy property that:

$$H = A^{\dagger}A \quad !$$

How? Just check:

$$H\psi = \left(-\frac{1}{\sqrt{2}}d_x + W\right)\left(\frac{1}{\sqrt{2}}d_x + W\right)\psi$$

$$= -\frac{1}{2}\frac{d^2}{dx^2}\psi - \frac{1}{\sqrt{2}}\left(W'\psi + \cancel{W\psi'}\right) + \cancel{\frac{1}{\sqrt{2}}W\psi'} + W^2\psi$$

$$= \left(-\frac{1}{2}d^2/dx^2 + \underbrace{W^2 - \frac{1}{\sqrt{2}}W'}_{\text{this was how we defined } U!}\right)\psi$$

A has allowed us to <u>factor</u> any Ham!

But there's more! check out:

$$H^{(2)} = AA^{\dagger}.$$

$H^{(2)}$ has an eigenspectrum:

$$H^{(2)}\psi_n = \varepsilon_n\psi_n$$

$$\rightarrow AA^{\dagger}\psi_n = \varepsilon_n\psi_n$$

$$\rightarrow A^{\dagger}A(A^{\dagger}\psi_n) = \varepsilon_n(A^{\dagger}\psi_n)$$

$$\rightarrow \underline{A^{\dagger}A\tilde{\psi}_n = \varepsilon_n\tilde{\psi}_n}$$

Amazing!
H, $H^{(2)}$ have
<u>degenerate</u>
<u>spectra!</u>

Why amazing? Well, checkout:

$$AA^{\dagger}\varphi = \left(\frac{1}{\sqrt{2}}\partial_x + W\right)\left(-\frac{1}{\sqrt{2}}d_x + W\right)\varphi$$

$$= -\frac{1}{2}\varphi'' + \frac{1}{\sqrt{2}}\left(W'\varphi + \cancel{W\varphi'}\right) - \cancel{\frac{1}{\sqrt{2}}W\varphi'} + W^2\varphi$$

$$= -\frac{1}{2}\varphi'' + \underbrace{\left(\frac{W'}{\sqrt{2}} + W^2\right)}\varphi$$

And this is <u>not</u> the same as V^1!

$\to H^{(1)} = -d^2/dx^2 + V^{(1)}$ and $H^{(2)} = -d^2/dx^2 + V^{(2)}$

Little aside:

Since $V^{(1)}(x) = \frac{1}{2}\frac{\varphi_0''}{\varphi_0}$,

and $V^{(1)}(x) = -\frac{W'}{\sqrt{2}} + W^2(x)$,

we have ✱ $\varphi_0''/\varphi_0 = -\sqrt{2}W'(x) + 2W(x)$.

kool trick: $(\varphi_0''/\varphi_0) = \left(\frac{\varphi_0'}{\varphi_0}\right)^2 + \left(\frac{\varphi_0'}{\varphi_0}\right)'$?

$$= \cancel{\varphi_0'^2/\varphi_0^2} + \varphi_0''/\varphi_0 - \cancel{\varphi_0'^2/\varphi_0^2}$$ ✓

So: ✱ is:

$$\underbrace{\left(\frac{\varphi_0'}{\varphi_0}\right)^2} + \underbrace{\left(\frac{\varphi_0'}{\varphi_0}\right)'} = \underbrace{\left(-\sqrt{2}\,W(x)\right)'} + \underbrace{\left(-\sqrt{2}\,W(x)\right)^2}$$

So: $W(x) = -\frac{1}{\sqrt{2}}\varphi_0'/\varphi_0 = -\frac{1}{\sqrt{2}}\frac{d}{dx}\ln\varphi_0(x)$.

We can solve for all E_n, φ_n of V^1, then use φ_0 to get W, then W to get V^2, and its spectrum for free!!!!!

to the degenerate spectrum!
↓
The one exception is the zero-energy state.
Notice that if $A^\dagger A \psi_0 = 0$, $\rightarrow A\psi_0 = 0$.
this is a 1st-order DE!

$$\frac{1}{\sqrt{2}}\psi_0' + W(x)\psi_0 = 0$$

$$\rightarrow \psi_0(x) = \exp\left[-\int_{-\infty}^{x}\sqrt{2}\,W(y)\,dy\right]$$

is a solution! It can be normalized as long as $\int_{-\infty}^{x} W(y)\,dy$ goes to ∞ as $x \to \infty$.
So far, so good – so why can't $AA^\dagger\psi_0 = 0$ too?

This would require $\psi_0(x) = \exp\left[+\int_{-\infty}^{x} W(y)\,dy\right]$

↑ now a +, not – !!

These cannot be simultaneously normalized!

PUNCHLINE:

$H_1 = A^\dagger A$ and $H_2 = AA^\dagger$ have the same spectrum even though they correspond to very different potentials!

And: ψ_0 is always obtained from a simple first order DE!

Let's do some examples. First, take

$$W(x) = a x^3.$$

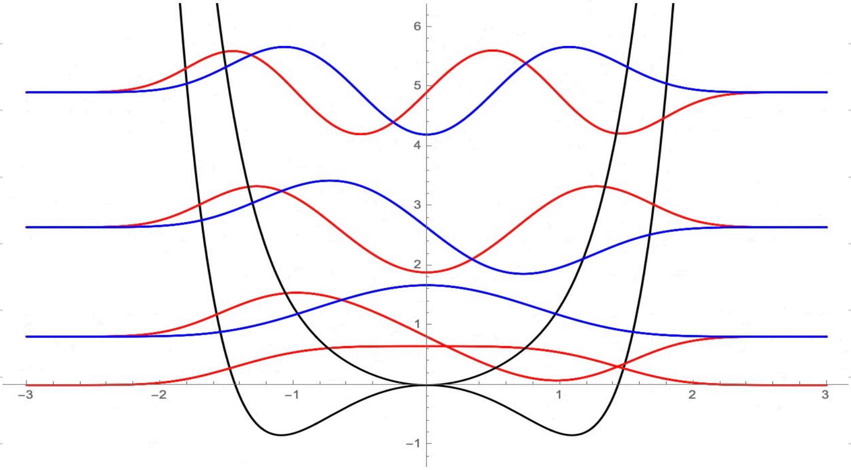
This gives:

- $V^1 = a^2 x^6 + \frac{3}{\sqrt{2}} a x^2$
- $V^2 = a^2 x^6 - 3/\sqrt{2}\, a x^2$
- $\psi_0^{(x)} = N \exp\left[-\frac{\sqrt{2}}{4} a y^4 \Big|_{-\infty}^{x}\right]$

$$= \underbrace{N e^{-\infty}}_{\text{normalization}} \exp\left(-\frac{\sqrt{2}}{4} a x^4\right)$$

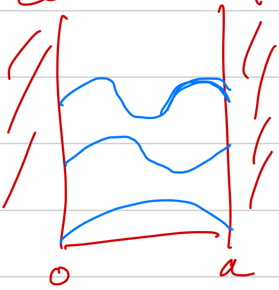
- $W = -\frac{1}{\sqrt{2}} \frac{\psi_0'}{\psi_0} = -\frac{1}{\sqrt{2}}\left(-\frac{4}{4} a x^3 \sqrt{2}\right) e^{-} / e^{-}$

$$= a x^3 \checkmark$$



This shows that two <u>very</u> different pots can have the same spectrum! But a little abstract...

For something a bit more down to earth, consider the inf. square well:



this has solutions $\sim \sin(kx)$ where $k = n\pi/a \rightarrow \mathcal{E} = \frac{k^2}{2} = \frac{n^2\pi^2}{2a^2}$.

The SUPERPOT is thus:

$$W(x) = -1/\sqrt{2} \cdot \frac{\pi}{a} \cdot \cot\left(\frac{\pi x}{a}\right).$$

(set $a=\pi$ to make the rest easy..)

And this gives 2 potentials...

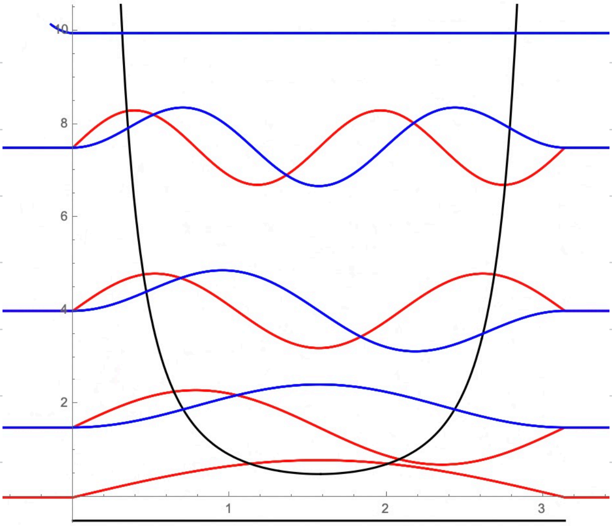
$$U^{(1,2)}(x) = W^2 \pm \tfrac{1}{\sqrt{2}} W'(x)$$

$$= \frac{1}{2}\cot^2(x) \mp \frac{1}{2}\left(-\csc^2(x)\right)$$

$$= \frac{1}{2}\left(\frac{\cos^2(x) \mp 1}{\sin^2(x)}\right)$$

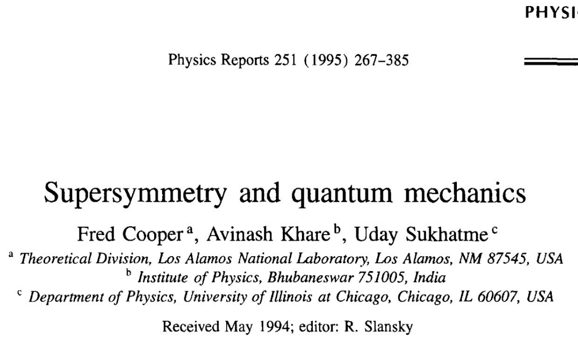
$$= -\frac{1}{2} \text{ or } \frac{1}{2}\left(2\csc^2(x) - 1\right).$$

→ the first term gives the inf. sq. well, shifted by $\mathcal{E}_0$. Cool! And the second is something else - something weird - but something w/ spectrum $\underline{\frac{n^2\pi^2}{2a^2}}$, $n = 2, 3, \ldots$



Here we see the states of both systems...

This is a ⟶ good review article if you want to learn more!



PHYSI

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Supersymmetry and quantum mechanics

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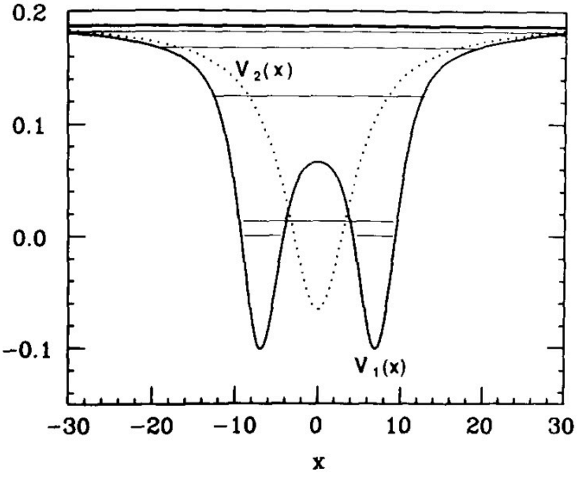


e.g. how to solve wacky susy systems like this!

But now it's time to get back to our goal of solving the hydrogen atom:

$$H_\ell \psi_\ell = \mathcal{E}_\ell \psi_\ell,$$

where $H_\ell = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{2r^2} - \frac{1}{r}$.

First off, let's figure out our SUPPOT!

From $V^{(1)} = W^2 - \frac{1}{\sqrt{2}} W' = \frac{\ell(\ell+1)}{2r^2} - \frac{1}{r} - \mathcal{E}_0$

we can guess an Ansatz:

$$W(r) = C - D/r$$

$$\rightarrow \boxed{\frac{\ell(\ell+1)}{2r^2}} - \frac{\boxed{1}}{r} - \underline{\mathcal{E}_0} = \underline{C^2} - \frac{\underline{2CD}}{r} + \boxed{\frac{D^2}{r^2} - \frac{1}{\sqrt{2}}\frac{D}{r^2}}$$

$\mathcal{E}_0$: $\underbrace{\mathcal{E}_0 = -C^2}_{\underline{A}}$, $\underbrace{-2CD = -1}_{\underline{B}}$, $\underbrace{D^2 - D/\sqrt{2} = \frac{\ell(\ell+1)}{2}}_{\underline{C}}$

From $\underline{C}$: $D = \frac{\ell+1}{\sqrt{2}}$

→ From $\underline{B}$: $C = \frac{1}{\sqrt{2}(\ell+1)}$

→ From $\underline{A}$: $\mathcal{E}_0 = -\frac{1}{2(\ell+1)^2}$

Wow! We already found the ground state energy for each ℓ!!

We're set now to define

$$\hat{A}_\ell = \frac{1}{\sqrt{2}}\, d/dr + \frac{1}{\sqrt{2}(\ell+1)} - \frac{(\ell+1)}{\sqrt{2}\, r}$$

$$\hat{A}_\ell^{+} = -\frac{1}{\sqrt{2}} \frac{d}{dr} + \frac{1}{\sqrt{2}(\ell+1)} - \frac{\ell+1}{\sqrt{2} r}$$

And their tasty Hams...

$$H_\ell^{(1)}\varphi = A_\ell^{+} A_\ell \varphi = \frac{1}{2}\left[-d_r + \frac{1}{\ell+1} - \frac{\ell+1}{r}\right]\left[\varphi' + \frac{\varphi}{\ell+1} - \frac{(\ell+1)}{r}\varphi\right]$$

$$= \frac{1}{2}\Big(-\varphi'' - \cancel{\frac{\varphi'}{\ell+1}} + \cancel{\frac{\varphi'(\ell+1)}{r}} - \frac{(\ell+1)\varphi}{r^2} + \cancel{\frac{\varphi'}{\ell+1}} + \frac{\varphi}{(\ell+1)^2} - \underline{\frac{\varphi}{r}} - \cancel{\frac{(\ell+1)\varphi'}{r}}$$

$$- \underline{\frac{\varphi}{r}} + \frac{(\ell+1)^2\varphi}{r^2}\Big)$$

$$= \frac{1}{2}\left(-\frac{d^2}{dr^2} + \frac{(\ell+1)(\ell+1-1)}{r^2} - \frac{2}{r} + \frac{1}{(\ell+1)^2}\right]\varphi$$

$$= \underline{H_\ell^{\text{Coulomb}}} + \frac{1}{2(\ell+1)^2} \; !$$

$$H_\ell^{(2)} = A_\ell A_\ell^{+}\varphi = \frac{1}{2}\left[d_r + \frac{1}{\ell+1} - \frac{\ell+1}{r}\right]\left[\varphi' + \frac{\varphi}{\ell+1} - \frac{(\ell+1)\varphi}{r}\right]$$

$$= \frac{1}{2}\Big[-\varphi'' + \cancel{\frac{\varphi'}{\ell+1}} - \cancel{\frac{(\ell+1)\varphi'}{r}} + \frac{(\ell+1)\varphi}{r^2} + \frac{\varphi}{(\ell+1)^2} - \underline{\frac{\varphi}{r}} - \cancel{\frac{\varphi'}{\ell+1}}$$

$$+ \cancel{\frac{(\ell+1)\varphi'}{r}} - \underline{\frac{\varphi}{r}} + \frac{(\ell+1)^2\varphi}{r^2}\Big]$$

$$= \frac{1}{2}\left[-\frac{d^2}{dr^2} + \frac{(\ell+1)(\ell+1+1)}{r^2} - \frac{2}{r} + \frac{1}{(\ell+1)^2}\right]\varphi$$

$$= \underline{H_{\ell+1}^{\text{Coulomb}}} + \frac{1}{2(\ell+1)^2}$$

And so: $$H_\ell^{(2)} = H_{\ell+1}^{(1)} + \frac{1}{2(\ell+1)^2} - \frac{1}{2(\ell+2)^2}. \qquad (17)$$

Kool! We have our factorized Hams!

As a double sanity check, notice that

$$\psi_{0\ell}(r) = e^{-\int_0^r \sqrt{2}\, W(y)\, dy}$$

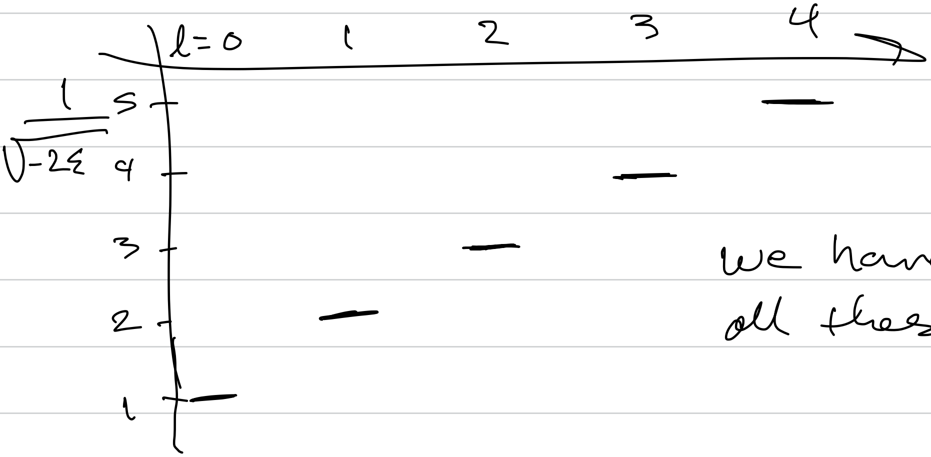
$$= \exp\left[-\int_0^r \frac{1}{\ell+1} - \frac{(\ell+1)}{r}\, dr\right]$$

$$= \exp\left[-r/\ell+1 + (\ell+1)\ln r\right]$$

$$= r^{\ell+1} e^{-r/(\ell+1)},$$ up to normalizatn

This is normalizable and, wow!, agrees w/ what we learned in kindergarten QM!

Since this is the zero-energy GS of $H_\ell^{(1)}$, we confirm that the actual GS of H_ℓ^{coulomb} sits at $\varepsilon_{0\ell} = -\frac{1}{2(\ell+1)^2}$.



We have found all these already!

So far we haven't actually used $H_\ell^{(2)}$ yet!
Let's fix that. From (17):

$$\underset{\text{Ham}}{\underset{\uparrow}{H_\ell^{(2)}}}\;\underset{\text{estate}}{\underset{\uparrow}{\phi_{0,\ell+1}}} = \Big[\overbrace{H_{\ell+1}^{(1)} + \frac{1}{2(\ell+1)^2} - \frac{1}{2(\ell+2)^2}\Big]\phi_{0,\ell+1}}^{\text{gives } 0!}$$

$$= \underset{\text{eval}}{\underset{\uparrow}{\left(\frac{1}{2(\ell+1)^2} - \frac{1}{2(\ell+2)^2}\right)}}\;\underset{\text{estate}}{\underset{\uparrow}{\phi_{0,\ell+1}}}$$

But $H_\ell^{(2)}, H_\ell^{(1)}$ have the same spectra!

$\rightarrow$ $H_\ell^{(1)}$ also has an eval $\frac{1}{2}\left(\frac{1}{(\ell+1)^2} - \frac{1}{(\ell+2)^2}\right)$.

AND $H_\ell^{coul} = H_\ell^{(1)} - \frac{1}{2(\ell+1)^2}$

$\rightarrow$ H_ℓ^{coul} has an eigenenergy

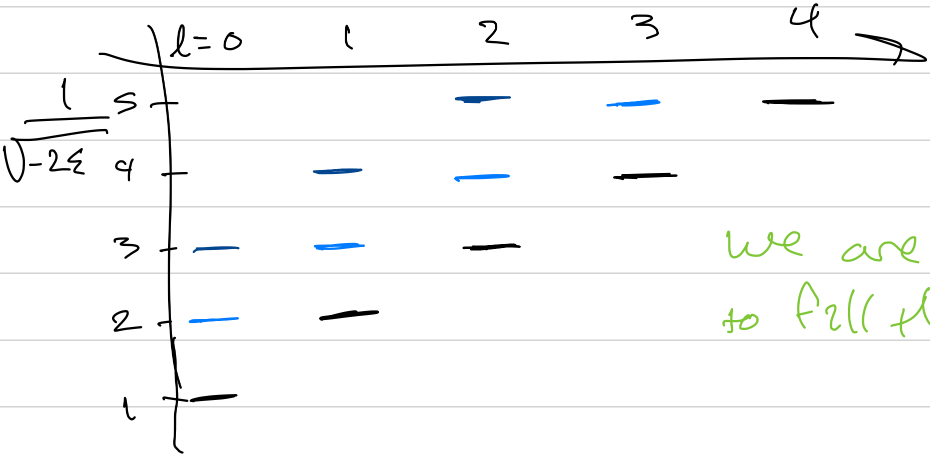
$$\underline{\underline{-\frac{1}{2(\ell+2)^2}}}$$

But then: $H_\ell^{(2)}\phi_{1,\ell+1} = \left[H_{\ell+1}^{(1)} + \frac{1}{2(\ell+1)^2} - \frac{1}{2(\ell+2)^2}\right]\phi_{1,\ell+1}$

$$= \frac{1}{2}\left(\cancel{\frac{1}{(\ell+2)^2}} - \frac{1}{(\ell+3)^2} + \frac{1}{(\ell+1)^2} - \cancel{\frac{1}{(\ell+2)^2}}\right)$$

And then! $H_\ell^{(1)}\phi_{2,\ell+1} = \frac{1}{2}\left(\frac{1}{(\ell+1)^2} - \frac{1}{(\ell+3)^2}\right)\phi_{2,\ell+1}$

And then!!! $H_\ell^{coul}\phi_{2,\ell+1} = -\frac{1}{2(\ell+3)^2}\phi_{2,\ell+1}$



we are starting to fill this in!

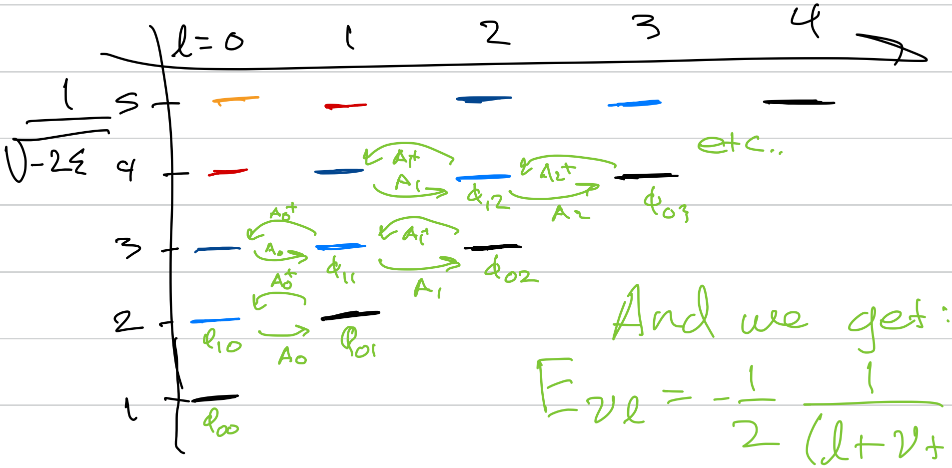
And what we have really done is to solve:

first $0 = A_\ell \phi_{0\ell}$

then $\phi_{1\ell} = A_\ell^+ \phi_{0\ell+1}$

then $\phi_{2\ell} = A_\ell^+ A_{\ell+1}^+ \phi_{0\ell+2}$

then ....



And we get:

$$E_{\nu\ell} = -\frac{1}{2} \frac{1}{(\ell+\nu+1)^2},$$

where $\nu = 0, 1, \ldots$

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or, letting $n = l+\nu+1$,

$$\boxed{E_{nl} = -\frac{1}{2n^2}}$$

This procedure we just did is part of a much larger context of 'shape invariant potentials', a category of 1D potentials defined by the relationship

$$V^{(2)}(x,a_1) = V^{(1)}(x,\underset{\underset{a_2 = f(a_1).}{\uparrow}}{a_2}) + \overset{\overset{\text{indep. of } x}{\downarrow}}{R(a_1)}$$

This lets us define a family of Hams

$$H_s = -\frac{1}{2}\frac{d^2}{dx^2} + V_1(x,\underset{\underset{a_s = \underbrace{f(f(f\dots(a_1)))\dots)}_{\dots s-1}}{\uparrow}}{a_s}) + \sum_{k=1}^{s-1} R(a_k)$$

Who cares? Well,

$$H_{s+1} = -\frac{1}{2}\frac{d^2}{dx^2} + \underbrace{V_1(x,a_{s+1})}_{= V_2(x,a_s) - R(a_s)} + \sum_{k=1}^{s} R(a_k)$$

$$\rightarrow \quad = -\frac{1}{2}\frac{d^2}{dx^2} + V_2(x,a_s) + \sum_{k=1}^{s-1} R(a_k)$$