27 Nov 123 Lecture 8 We want to unlerstand the energy dependence of our scattering problem veur zero energy in a systematic way. First, we notice that our fig sollus that we wonted to care one not analytic at gnell k. fae ~ k l+1/2 gee ~ k - l - 1/2 This such for the ANALYTIC CONTINUATION that we love to do! But, if we just pick Aus different, but still valed, golation, everything is nice and analytic! $f_{qq}^{0} = (kr_{0})^{-l-tr_{2}} f_{qq}$ $g_{qq}^{0} = (kr_{0})^{2+y_{2}} g_{qq}$ To choracteristic length Scale ~ orberrong-ish.

Not only does this care the ronandy the behavior as hr->0, but one can prove that they are analytic and entire further of energy for all r.

-> (fill, ger) -> energy normalized base pain -> (fra, gre) -> onalytic base poir.

And naturally - Since we defined tandre using (fre, que) - we can define on everyg-analytic phases wift using (fre, gee)

This defines on analytic golution



And clearly for SER = ton SER (note: SER KERTI MERSINDER SER is now also a smooth function of could creage > we can analytically continue! he!

Let's do this! The crucial thing to determine the discrete apectrum is as always, to construct golins where the exponential growthat v > is leiled off. This dependence $\begin{aligned}
\varphi \left\{ or \ E = \frac{u^2}{2} \right\}^{2} &= \left(\frac{2m}{\pi}\right)^{1/2} u^{-l-1} \left(\frac{1}{2u}\right) \left(e^{ikn} + \frac{-ikn}{2}\right) \\
&= \left(\frac{2m}{\pi}\right)^{1/2} u^{-l-1} \left(e^{kn} - (-1)^{k} e^{-kn}\right) \\
&= \frac{1}{2u^2} u^{-l-1} \left(e^{kn} - (-1)^{k} e^{-kn}\right)
\end{aligned}$ We do the same for $g_{Ee}^{o}(r)$ to get: $g_{Ee}^{o}(r) \rightarrow -(\frac{m}{2\pi})^{1/2} \ltimes l((-1)^{l}e^{kr}+e^{-kr})$

So, we have: Sae, gre, and fre all as smooth functions of E. => We have left! $U_{\xi\ell}(r) \longrightarrow \left(\frac{2m}{\pi}\right)^{1/2} \begin{cases} \epsilon & (\cos \delta_{\xi\ell} & -1-r + \sin \delta_{\xi\ell} & (-r)^{\ell} & k^{\ell} \\ + e^{-kr} & (-r-1)^{k} \cos \delta_{\xi\ell} & k^{\ell} & k^{\ell} \\ + e^{-kr} & (-r-1)^{k} \cos \delta_{\xi\ell} & k^{\ell} & k^{\ell} \\ \end{cases}$ $mat \quad vanigh \quad this \quad the m : ! ! !$

50: $fan S_{ko} + (-1)^{l} (Kr_{o})^{-2l-1} = 0$ This gives us the crucial quoitity, the swave scattering length $\begin{cases} Q_{2=0} = \lim_{E \to 0} \left(-v_0 + \cos \delta_{E,2}^0 \right) \\ E \to 0 \end{cases}$ $= \lim_{E \to 0} -tan \delta_{E,R=0}$ And the quartization condition reads a= . (So a>0 naturally!) And this in turn implies that the bound state has every $E_p = -\frac{4^2}{2ma^2}$ Thus: a scottering parameter, a, determines a spectroscopy value, Ep!

This is only approximate since a is the E-O limit of ton SER/4. But if we make a energy-dependent, then the quant 1 zet len condition reads $A(E) - \frac{1}{\kappa} = 0$ $\rightarrow E = -t^2$ $2ma(E)^2$ where $\alpha(\Xi) = -r_0 + cm S_{E,0}^{\circ}$ a has a natural connection to the zero-energy wave burtha. A+ lorger, we had u(r) -> C Sin(kr+S) From Wigner: S->(kro)S->-ka(o) at very small k. $\rightarrow u(r) \rightarrow \frac{C}{k} \sin(kr - ka) \rightarrow C(r - a)$

 v_{r} v_{r} $node D_{r=a}$

Note what this means: a positive scottering length means that the zero-energy scatering wave func has a node at r=ce. For r>a, the wave function is identical to one where we had instead on infinitely hand wall at r=a! 50: When a >0, the interaction between the colliding particles is repulsive! EVEN THOUGH the potential 25 actually attract me .. This concept, doing with the existence of tools (Feshbach resonances) to tune a, is at the heart of ultracold atoms.

We will see how this comes about by deriving the Fermi pseudopotential - which can be applied to problems like the following example:

->] Rb atoms in a BEC approach eachother. In the relative internuclear coordinate, the Born-Oppenheimer potential + scattering wave function near E=0 looks something like: Aulde UScilladians V(r) r = 0the long wave length b/c low T.

For just 2 atoms, we can solve (numerically at least) this problem with the real Vcr) without too much deficulty. But what if we want to add even a 3rd Rb? Or Hor S or 100053? Now dealing with many Vig(1) interactions pets very hardespecially gince all that work goes into what

is essentially just a phase shift for the scuttering golures. Moreover, we can see that some of the key properties of this collision one not really obvious from this picture! We know that the fact that a 30 have meany that our particles electively repel one onother despite the V(r) being attractive. But if we make U(r) just a hit shallower, so that this high-lying bound state (recall @20 =) Ebr - Enaz) is no longer their, then all and the wave function : s suched in - the potential ledy attractive:



Clearly it would be nece to scannowlize the actual interaction V(1) with something easier to compute and more intuitive

which doesn't reed to tall us the complex physics at rero bat just veeds to get the wave function ocutside of this zone right. Such a potential wight look something like: U(r) = ITT a S²(r) + when +, simple, no U(r) details ... and this is exactly the Fermi pseudopot. "Ferni's Favorite Figure ...", Gould + Sharapor EPJH 47,10 (2022). (shows also Farmi's much better sketch of "FigA")

This was the system: A Rydberg atom excited malde of a dense gas ,8 This was actually on experiment done by Amaldi ad Segre - who, along with Fermi, Majorana, und others were nembers of the "Via Panisperna Boys" - in Rome inthe early 30, The bochground deng 2+5 was so large that ~ (000s of atoms were within the Rydberg obit.

Amaldi and Segre antecipated that the no longer be able to observe my could Rydberg spectra, since collections with the backgood alcens must screby destroy the ability to such weakly bound states! ercite

But to their surprise, they did observe Rydberg series-just et slightly different transition Crequeriles...



FIG. 1. Absorption spectra of sodium-hydrogen mixtures at higher pressures (above) and at lower pressures (below) of hydrogen. Note the unshifted mercury line 2537, which lies in the background.

Nature 133, (4((1934)) Depending on the badeground gas this shift in the lines was at times towards higher everyics (blue) lower energies (not) and

(Füchtbauer, Schulz, Brandt Zeitschrift für Physik 20, sometimes none at all! 403 (1934) 10 lue This was also deserved basically contemporaneously leg Fürchtbæhert coworkers; in Rostock! term: quidely realized cm¹⁰ 11 2 3 4 5 6 7 8 9 10 11 2 10 14 15 16 17 18 19 20 21 22 23 24 25 c Fig. 4. Gesamtverlauf der Verschlebung bei 0⁰ und 1 Atm. für die Cäsiumhauptserie. that this was a hint that more was going on in the "Atomi Gonti" than one icipated. Why? His first colculation assumed that the residual ion would polarize and attract the neutral gas atoms through the interaction potentlul $V(R_i) = -\frac{2}{2R_i 4}$ (we will derive pris formale soon). He then calculated the energy shift resulting

from the polarization of the medeum by the ion.

Erret seen in I This procedure is not rigorous, Ferni, Il Nuovo Einet seen in Ferni, II Nuovo (1930) Units grueedure units unt gives an answer units Units grueedure in untson, pup 8 cc (1936) Ulique Screb $\Delta E = \sum_{r} \frac{\alpha}{2R_{r}4}$ Jugt have toald up all possible con flywatting ER.3! Assume a uniform distribution - every atom occupies a "Wigner Seitz" sphere of volume 1p, where g is the density. Thus, we can go from = to jdR, where the lower lamit is not zero (consing a divergence!) but laterd $\frac{4\pi}{3}R_{ws}^{3} = \frac{1}{\beta} \rightarrow R_{ws}^{2} = \left(\frac{4\pi}{3}\right)^{-1/3}$ $= \sum_{i} \frac{1}{R_{i}^{\mu}} = \int_{R_{ms}}^{\infty} \frac{f}{R^{\mu}} \cdot R^{2} dR \int \Delta \Omega = -4T \int_{R}^{\infty} \int_{R_{ms}}^{\infty} \frac{1}{R_{ms}} = 4T \left(\frac{4T}{3}\right)^{1/3} f^{4/3} = 20 f^{4/3}$ The R⁻⁴ dependence means: we can just som over all space since contributions outside the Ryd orbit are regligible.

In sam, this gives an energy shift of about DZ= -10xg4/3. This is always regative as & is (in the ground Gate) dways positive > this predicts a RED shift. So we obviously read to see what the electron is

really doing , and we have to go beyond the somewhat naive picture of the electron-atom interaction being incapsulated as U~ - 2/17-714. (as usual, more on this point later ...)

To start the derivation, notice that she Rydney dectron is very glow: v - l(n30 :+ has a very long de Groglie wavelength: When it encounters a perturber, it does so at rearly zero every => 5-vove is all that natters!

The electronic TISE is: (-1/r in simplest ros) (-2/r-nel⁴ in simplest (-2/r-nel⁴ in simplest (-2/r-nel⁴ in simplest (-3/r-nel⁴ in simplest (-3/r) Now let's think about what he elec wf. "Should" look (the rea an atom: From this picture, we can see that a good idea might be to consider averaging this WF over a region approx edual to the size of the postrbes. Outglde, such a procedure just

gives the same WF back (maybe a bit "pixellated' since it is so clowly verying over Kus.

That nears: take O and integrate it over a Uvolume dn; n. $i = \frac{1}{2}i$ $i = \frac{1}{2$ small volume dx; wound the postubers: To calculate \$, recall the Car of Y new a pertuber: it is S-wave locally and nearly $u \neq 200 \text{ energy}:$ $u''(v) = \frac{u(r)}{r} \int_{00}^{0} (\theta_{1}\theta)$ $u''(v) = \frac{2m}{t^{2}} \left(V(r)u(r) - Eu(r) \right)$ E=0...From a few payes ago: outside the range vo of U(r), we KNOW U(r)!!

 $u(v) = C(v-a), r \geq v_0,$ Twe don't get know this. Idea: outside U(1), the angueF I is the same as 4. Let's case this to find C.

Start w/ the E=O SE: $\frac{\pi^2}{2n} \frac{\nabla^2 \Psi}{\nabla^2} = V \Psi, \quad \Psi = \frac{M(r)}{r} \frac{V_{(o)}(\vec{r})}{(o_0(\vec{r}))}.$ Futcy rate it over our averaging volume: $\int \nabla^2 y d^3 r = \frac{2^n}{n^2} \int V \psi d^3 r$ $= 4\pi loo \int_{\partial}^{n} u''(r)r dr = u''r + u'$ $= 4\pi V_{00} \left[u'(v) r \right]_{0}^{0} - \int_{0}^{0} u'(v) dr \right]$ $= 4\pi V_{00} \left[nu'(r_0) - 0 - u(r_0) - (-0) \right]$ $= 4\pi V_{00} \left(Cr_0 - Cr_0 + C_0 \right)$

 $= 4 \pi t \sqrt{60} Ca = 2m \int V 4 d^{3}r$

So: we have VY. Now we need I: $\overline{\Psi} = \int d^3r \, \frac{1}{60} \, u(r)/n - \frac{4\pi 1}{60} \int_0^{r_0} \frac{c(r-a)}{r} r^2 dr$ = 3100 (Cr2 - Cardn $= \frac{3400}{r_{0}^{3}} \left(\frac{2r_{0}^{3}}{3} - \frac{2}{2} \right)$ here we use the fact that the areas ing ~ CY00. volume is large compared to as -> pascel ______ (strongly interact inj note: this gives us portides wild to de dupious!) \mathcal{L} 50:

 $0 = \nabla^2 \Psi + \frac{2m}{m} \left(\xi - U \right) \Psi - \frac{2m}{m} V \Psi$ $= \nabla^{2} \overline{\Psi} + \frac{2m}{\pi^{2}} (E - U) \overline{\Psi} - 4\pi 60 Ca$ And Rinably: => $-\frac{\pi^{2}}{2m} \nabla^{2} \overline{\Psi} + U \overline{\Psi} + \frac{4\pi a \pi^{2}}{2m} \overline{\Psi} = E\overline{\Psi}$ This: s just the SE for I w/ on additional energy

Purchline: the effect of all these pertubers is to shift the electrons energy by $\Delta E = 2\pi h^2 a g$ m This term can be pos a neg, following the sign of a, and sypically dominates the other due to the g^{1/3} factor! More globally useful purchlue: we showed that: 2n (1423r = 4tta4 - This equation would hold for any 4 it the real potential V was replaced by V_{Feemi}(r,R) = 2tt th² a S(r-R). this is the FAMOUS Contact (Fami potantie!

Key physics insight from Formi: Squeeze everything you don't know into an increasingly smaller volume and characterize you like of knowledge by a single number a! Some other checks/points: 1) The Fermi pseudoporential also guarantees that we get the right scattering length in the Born approx: $d = - + \alpha S_{e} = + \pi \pi \int_{ulei}^{2} \frac{f_{Ee}(kr)}{r^{2}} \left[\frac{V(r)}{V(r)} \frac{V(r)}{r^{2}} \right]_{ulei}^{2} dr dede$ $= \frac{1}{100} \frac{2 \left(V_{em}^{(2)} \right)^{2} (hr)^{2l+2}}{100} V(\vec{r}) (\vec{r})^{2} \sin \theta dr d\theta d\theta$ N(2)= S(r) 8/0 /8/10) - 2719

⇒ l=0: $\begin{array}{c} \alpha = 2\pi \pi \pi 2 \left[\frac{2}{\sqrt{2}} \left[\frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{$ 2) The pseudopatential is otherwise too gingular to use in more exact calculations, for example when we trz to use the actual with (with phase shift) verther then fac(r). That means we have (for E->0 and l=0): 4(v) > C - - 0 V rather then $\psi(t) = \frac{f_{se}(r)}{r} \rightarrow \frac{r}{r} \rightarrow 1.$ (2) For D, we mught colculate matrix elements (i.e.: $\int V(r) P(r) dr = \frac{r-q}{r} \Big|_{r=0} = \infty$)

is coned by a more vigorous de l'united yields the regularization operator: This which $V_{\text{parmin}} = \frac{2\pi \alpha}{m} S'(\hat{r}) \frac{\partial}{\partial r}$ Notrce that this gives: $\int u(r) \psi lr = \int g(r) \frac{\partial}{\partial r} (r-a) dr$ $= \int \delta(r) [i] dr = 1.$

This behaves as necessary. However, it's not recessory to include this very. op. when we one doing mean-field of post- hat the cales, for example when the Fermi PIP :5 used to Decive: 1) The Gross-Pitaeusler: Eq. for BECS (see other course material) 2) Rylkerg Molecules