Periodically driven Systems

- driven systems
  $\rightarrow$ time-dep. ext. fields, perturbations, etc.
- model using t-dep. Hamiltonian:

$$H(t) = H_0 + f(t)\, H_1$$

$$H(t) = H_0 + g(t)\, V + g^*(t)\, V^+$$

$$H(t) = H_0 + H_1(t)$$

- time evolution : solve Schrödinger eq.

$$i\partial_t |\psi(t)\rangle = H(t)\,|\psi(t)\rangle \qquad ; \quad [H(t_1), H(t_2)] \neq 0$$

$\hookrightarrow$ solution: time-ordered exponential

$$|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle$$

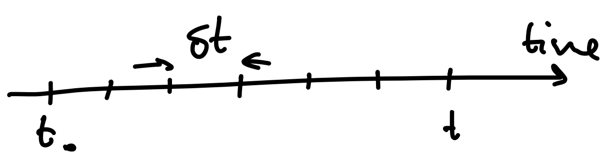
$$U(t, t_0) = \mathcal{T} \exp\left(-i \int_{t_0}^{t} ds\, H(s)\right)$$

computational definition

$$= \lim_{N\to\infty} e^{-i \frac{t-t_0}{N} H\left(N \frac{t-t_0}{N}\right)}\, e^{-i \frac{t-t_0}{N} H\left((N-1)\frac{t-t_0}{N}\right)} \times \dots \times e^{-i \frac{t-t_0}{N} H\left(\frac{t-t_0}{N}\right)}$$

$$= \lim_{N\to\infty} \prod_{n=1}^{N} e^{-i \frac{t-t_0}{N} H\left(n \frac{t-t_0}{N}\right)}$$

$\uparrow$ earlier times come first!



$\delta t \to 0$
$N \to \infty$
$t - t_0 =$ const. fixed

$$\delta t = \frac{t-t_0}{N}$$

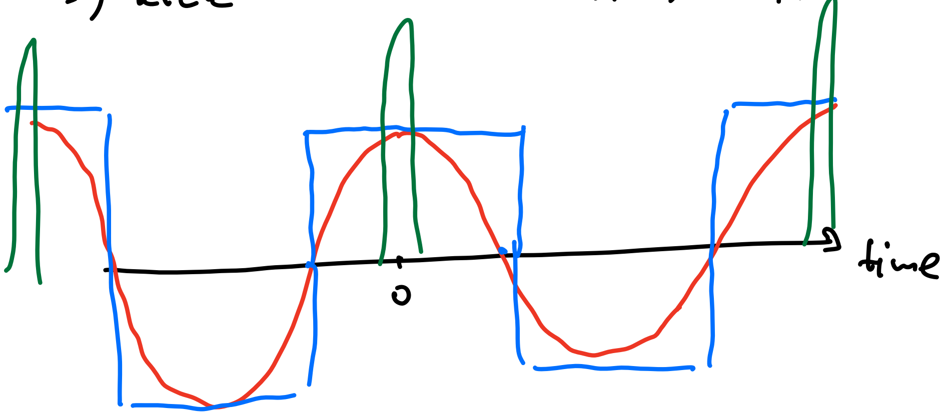
$$U(t, t_0) \overset{\text{analytical def.}}{=} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \dots \int_{t_0}^{t} dt_n\; \mathcal{T}\left(H(t_1)\, H(t_2) \dots H(t_n)\right)$$

$$= \sum_{n=0}^{\infty} (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \dots \int_{t_0}^{t_{n-1}} dt_n \; H(t_1) H(t_2) \cdot \dots H(t_n)$$

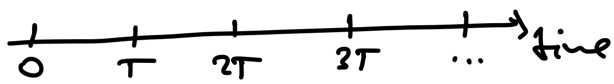
$$= \underline{\mathbb{1}} - i \int_{t_0}^{t} dt_1 \, H(t_1) - \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \; H(t_1) H(t_2)$$

$$+ i \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \; H(t_1) H(t_2) H(t_3) + \dots$$

- difficult to compute for arbitrary t-dep.
  - $\rightarrow$ pert. theory (truncate to first few orders)
    - only valid at short times
- periodic time-dependence: $H(t) = H(t+T)$
  - $\rightarrow$ relevant parameters
    - amplitude A
    - frequency $\omega = 2\pi/T$
    - [phase of drive φ / starting time t_0]
      sin vs cos
- examples of periodic drives:
  1) continuous drives: $H(t) = H_0 + A \cos \omega t \; H_1$
  2) step/square drives: $H(t) = H_0 + A \operatorname{sign}(\cos \omega t) \; H_1$
  3) kick-drives: $H(t) = H_0 + A \sum_{n=-\infty}^{\infty} \delta(t - nT) \; H_1$



- integer multiples of drive period, ℓT, $\ell \in \mathbb{N}$ — <u>stroboscopic times</u>



- some intuitive limits

1) weak-coupling limit : $A \ll \omega_0$ $\neq \omega$
↑ natural energy scale of H_0
→ apply pert. theory
→ captures short time dynamics

2) high-freq. limit : $\omega \gg \omega_0$
→ system "sees" time-averaged Hamiltonian

$$H_{ave} = \frac{1}{T}\int_0^T dt\, H(t)$$

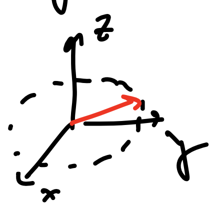
↳ apply inverse-frequency expansions (→ next time)

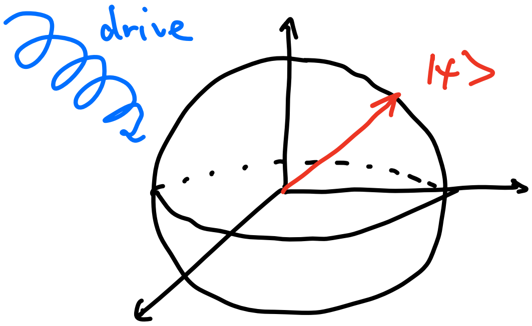
3) slow-frequency / adiabatic limit
→ system follows drive instantaneously (↳ later lecture)

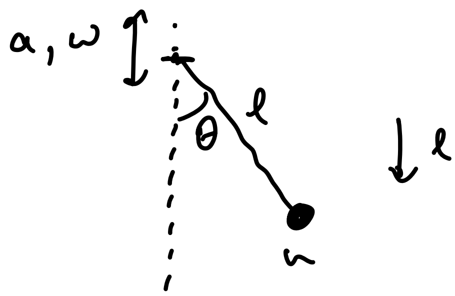
— examples of periodically driven systems

1) two-level system (2LS) in circularly polarized light

$$H(t) = B_z\sigma^z + B_{\parallel}\left(\sigma^x\cos\omega t + \sigma^y\sin\omega t\right)$$







$\omega_0 = \sqrt{g/\ell}$

2) <u>Kapitza pendulum</u> : $x(t) = \ell\sin\theta(t)$
$y(t) = \ell\cos\theta(t) + a\cos\omega t$

Lagrangian: $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + mgy$

$$= \frac{m\ell^2}{2}\left(\dot{\theta}^2 + \frac{2a\omega}{\ell}\underline{\sin\omega t\,\dot{\theta}\cos\theta}\right) + m\ell^2\omega_0^2\cos\theta$$

consider action: $S = \int_0^T dt\, L(\theta, \dot{\theta}, t) \hookrightarrow$ EOM trick: $\int_0^T \sin\omega t\,\dot\theta\,\sin\theta\,dt = \int_{\theta(0)}^{\theta(T)} -\sin\omega t\,\frac{d}{dt}\cos\theta \overset{\text{by parts}}{=} \int \omega\cos\omega t\,\cos\theta\,dt + \overset{0}{\cancel{\text{bdry terms}}}$

$$L' = \frac{m\ell^2}{2}\dot\theta^2 + m\ell^2\left(\omega_0^2 + \frac{a\omega}{\ell}\,\omega\cos\omega t\right)\cos\theta$$

Hamiltonian: $p_\theta = \partial L'/\partial\dot\theta$, $\quad H = p_\theta\dot\theta - L'$

$$H(t) = \frac{1}{2m\ell^2}p_\theta^2 - m\ell^2\left(\omega_0^2 + \frac{a\omega}{\ell}\,\omega\cos\omega t\right)\cos\theta$$

re-define: $m\ell^2 \to m$

$a\omega/\ell \to A$

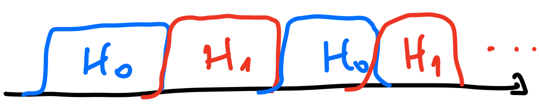
Hamiltonian for Kapitza pendulum:

$$H(t) = \frac{p_\theta^2}{2m} - m\left(\omega_0^2 + A\omega\cos\omega t\right)\cos\theta$$

- recall movie
- stabilization of inverted equilibrium position at high enough freq. ω ; how high?

3) periodically kicked spin chain:

$$H(t) = \begin{cases} H_0 = \sum_j J\,\sigma^z_{j+1}\sigma^z_j + h_z\sigma^z_j, & t\in[0,T/2) \bmod T \\ H_1 = \sum_j h_x\,\sigma^x_j, & t\in[T/2,T) \bmod T \end{cases}$$



$$U(T,0) = \mathcal{T} e^{-i\int_0^T ds\,H(s)} \overset{\text{comp. def!}}{=} e^{-i\frac{T}{2}H_1}\,e^{-i\frac{T}{2}H_0}$$

- if $J = J_{ij}$ random numbers / disordered ($\to$ MBL)

$\to$ Floquet time crystal ($\to$ later) recall:

- static unitary changes the basis (observable expectations remain intact)
- time-dep. unitary changes the reference frame (expect. values of obs. may change: e.g. energy, current, etc.)

$$\begin{tikzcd} \text{lab frame} \arrow[rr, bend right, "V^\dagger(t)"'] & & \text{rot. frame} \arrow[ll, bend right, "V(t)"'] \end{tikzcd}$$

$H_{lab}(t)$ $\qquad\qquad$ $H_{rot}(t)$

$U_{lab}(t_2, t_1)$ $\qquad\qquad$ $U_{rot}(t_2, t_1)$

$|\psi_{lab}(t)\rangle$ $\qquad\qquad$ $|\psi_{rot}(t)\rangle$

$\rightarrow$ relation b/w states: $|\psi_{rot}(t)\rangle = V^\dagger(t)\,|\psi_{lab}(t)\rangle$

$\rightarrow$ — // — evo op's:

$$U_{lab}(t_2, t_1) = V(t_2)\; U_{rot}(t_2, t_1)\, V^\dagger(t_1)$$

$$U_{rot}(t_2, t_1) = \mathcal{T}\exp\left(-i\int_{t_1}^{t_2} ds\; H_{rot}(t)\right)$$

$\rightarrow$ rel'n b/w Hamiltonians:

$$i\partial_t |\psi_{rot}(t)\rangle = H_{rot}(t)|\psi_{rot}(t)\rangle \qquad (*)$$

$$\rightarrow i\partial_t\, |\psi_{lab}(t)\rangle = H_{lab}(t)\,|\psi_{lab}(t)\rangle$$

$$i\partial_t\left(V(t)\,|\psi_{rot}(t)\rangle\right) = H_{lab}(t)\,V(t)\,|\psi_{rot}(t)\rangle$$

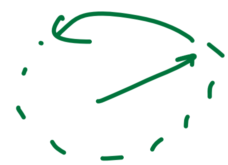
$$V(t)\,i\partial_t|\psi_{rot}(t)\rangle + \left(i\partial_t V\right)|\psi_{rot}(t)\rangle = H_{lab}(t)\,V(t)\,|\psi_{rot}(t)\rangle \quad \Big/\, V^\dagger(t)\times$$

$$i\partial_t|\psi_{rot}(t)\rangle = \underbrace{\Big(V^\dagger(t)\,H_{lab}(t)\,V(t) - \underbrace{V^\dagger(t)\,i\partial_t V(t)}_{\text{Gallilean "force"}}\Big)}_{\overset{(*)}{=}\; H_{rot}(t)}|\psi_{rot}(t)\rangle$$

(e.g. centrifugal force)

Ex 1 : solution to 2LS : $H(t) = B_z\sigma^z + B_x(\cos\omega t\,\sigma^x + \sin\omega t\,\sigma^y)$

want to compute dynamics under $H(t)$, i.e.



$$|\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle$$

trick : let's transform to co-rotating frame

use : $V(t) = e^{-i\left(\omega t \frac{\sigma^z}{2}\right)}$

$\Rightarrow H_{rot} = B_z\sigma^z + B_{\parallel}\,\sigma^x - \frac{\omega}{2}\sigma^z$ time-indep!

$\Rightarrow U_{rot}(t_2,t_1) = \exp\left(-i(t_2-t_1)H_{rot}\right)$ easy in rot. frame!

– rotating frame difficult / impossible to identify in general

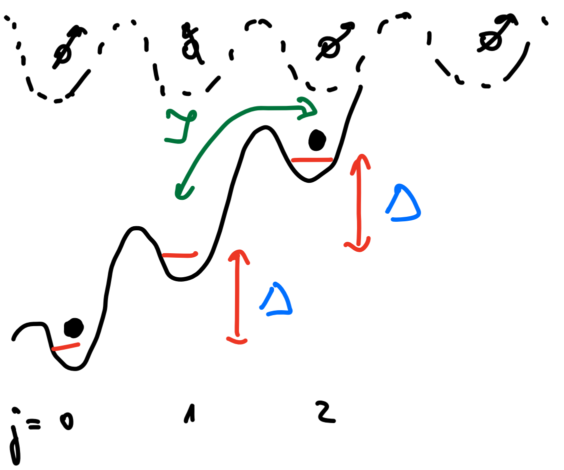
Ex 2 : static systems viewed as periodically driven in rot frame

realization: electron in external electric field

Wannier-Stark ladder

$$H = \sum_j -J\left(a_{j+1}^\dagger a_j + h.c.\right) + j\Delta n_j$$

$$n_j = a_j^\dagger a_j$$



strong $\Delta \gg J$ regime

: particles cannot hop b/c energy released Δ cannot be transformed into kinetic energy J

$\Delta \gg J$

– alternative way to see this:

rot frame: $V(t) = \exp\left(-i\,t\,\Delta\sum_j j\,n_j\right)$

$$H_{rot}(t) = \sum_j -J\left(e^{it\Delta}a_{j+1}^\dagger a_j + h.c.\right) \xrightarrow[\Delta \gg J]{\int_0^T e^{it\Delta} = 0} 0$$

time-periodic $H_{rot}(t)$ w/ freq. Δ

high-freq. limit

→ Can we exploit the time-periodic structure of $H(t)$?

Thm. (Gaston Floquet, 1883) (theory of ODE's)

let $H: \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{C}^{n\times n} \\ t \longmapsto H(t) \end{array}$ be continuous, matrix-valued fn with period T: $H(t+T) = H(t)$

(Hamiltonian)

let $U(t)$ be the fundamental matrix (time-evo operator) to the first-order linear ODE:

$$i\partial_t \psi(t) = H(t)\,\psi(t) \quad ; \quad i\partial_t U(t) = H(t)\,U(t)$$
$$U(0) = \mathbb{1}$$

then:

1) $U(t+T)$ is also a fundamental matrix

2) there exists a non-singular, continuously diff'ble matrix-valued fn:

$P: \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{C}^{n\times n} \\ t \longmapsto P(t) \end{array}$ with period T: $P(t+T) = P(t)$

and a time-independent matrix $H_F \in \mathbb{C}^{n\times n}$, s.t.

$$U(t) = P(t)\exp(-i\,t\,H_F)$$

Corollary: stroboscopically, i.e. at $t = \ell T$

$$U(\ell T) = e^{-i\ell T H_F}$$