Geometric Floquet Theory

- recall: $i\partial_t |\psi(t)\rangle = H(t)|\psi(t)\rangle$ Schr. eq. w/ $H(t)=H(t+T)$

Floquet: $|\psi(t)\rangle = P(t)\, e^{-it H_F[0]} |\psi(0)\rangle$

effective object, does not exist w/o drive

micromotion $P(t+T)=P(t)$

time-indep. Floquet Hamiltonian

- physical meaning: there exists a distinct rotating frame, defined by $P(t)$, where dynamics is governed by time-indep.:

$$H_F[0] = P^\dagger(t)\, H(t)\, P(t) - P^\dagger(t)\, \partial_t P(t)$$

note: (1) at all times, stroboscopic & non-strobo!

(2) it's a rot frame, i.e. at $t_\ell = \ell T$ dynamics coincides w/ lab-frame dynamics

(3) $P(t)$ difficult to find!

- Floquet e'value problem:

$$H_F[t_0]|u_F[t_0]\rangle = \varepsilon_F^{(n)} |u_F[t_0]\rangle$$

Floquet states

quasienergies, def. up to $\varepsilon_F \to \varepsilon_F + m\omega$, $m\in\mathbb{Z}$

$\Rightarrow$ H_F not unique!

Floquet "gauge" (physical): dependence on initial time t_0 or phase of drive φ $H[t_0]$

• Floquet states depend on t_0:

$|u_F[t_0]\rangle = P(t_0)|u_F[0]\rangle \Rightarrow H_F[t_0] = P(t_0) H_F[0] P^\dagger(t_0)$

• q'energies indep. of t_0: ε_F

– recap: CD driving: Hamiltonian $H(\lambda)$ & ctrl. parameter $\lambda(t)$; $t \in [0, T]$

$H(\lambda)|n[\lambda]\rangle = E(\lambda)|n[\lambda]\rangle$ inst. e'states

– adiabatic limit: $\dot\lambda \to 0$, $T \to \infty$, $\dot\lambda T \to$ const.

$|n(t)\rangle = \mathcal{T} e^{-i\int_0^t ds\, H(\lambda(s))}|n[\lambda(0)]\rangle \to e^{i\phi_n(t)} e^{i\gamma_n(t)} |n[\lambda(t)]\rangle$

system evolves in inst. e'state, if gapped

dyn. phase: $\phi_n(t) = -\int_0^t ds\, E(\lambda(s))$

geom. phase: $\gamma_n(t) = \int_{\lambda(0)}^{\lambda(t)} d\lambda\, \langle n[\lambda]| i\partial_\lambda |n[\lambda]\rangle$

– CD driving: away from ad. limit: excitations

· inst. diag.: $U_\lambda^\dagger H_\lambda U_\lambda = \mathcal{D}_\lambda$

· co-moving frame: $\tilde H = U_\lambda^\dagger H_\lambda U_\lambda - \dot\lambda U_\lambda^\dagger i\partial_\lambda U_\lambda$

$= \mathcal{D}_\lambda - \dot\lambda \tilde{\mathcal{A}}_\lambda$ gauge pot.

• add term to counteract excitations:

$H_{CD}(\lambda(t)) = H(\lambda(t)) + \dot\lambda \mathcal{A}_\lambda(t)$, $\mathcal{A}_\lambda = (i\partial_\lambda U_\lambda) U_\lambda^\dagger$

$$|n(t)\rangle = \mathcal{T} e^{-i\int_0^t ds\, H_{CD}(\lambda(s))} |n[\lambda(0)]\rangle \overset{(*)}{=} e^{i\gamma_n(t)}\, e^{i\phi_n(t)} |n[\lambda(t)]\rangle$$

recall: $\tilde{H}_{CD} = \mathcal{D}_\lambda$

$\Rightarrow$ in co-moving frame $U_{\text{co-mov}}(t,0) \overset{\substack{\text{for } t\text{-indep } \mathcal{D} \\ \downarrow}}{=} e^{-it\mathcal{D}}$

$\Rightarrow$ back to original frame:

$$U_{\text{lab}}(t,0) = \mathcal{T} e^{-i\int_0^t A_\lambda(s)\,ds}\, \overbrace{U_\lambda(0)\, e^{-it\mathcal{D}}\, U_\lambda^\dagger(0)}^{= e^{-itH}} = \overbrace{W(t,0)}^{\text{geom.}}\, \overbrace{e^{-itH}}^{\text{dyn.}}$$

$\rightarrow$ CD driving eliminates necessity for ad. limit

$\rightarrow$ but: still need state to be gapped

- $U(1)$ gauge freedom: rephasing of individual e'states

$$|n[\lambda]\rangle \longrightarrow e^{i\chi_n(\lambda)} |n[\lambda]\rangle$$

Berry connection not gauge invariant:

$$A_n = \langle n[\lambda]| i\partial_\lambda |n[\lambda]\rangle \longrightarrow A_n - \partial_\lambda \chi_n$$

$\Rightarrow$ gauge pot. not unique $\mathcal{A}_\lambda \rightarrow \mathcal{A}_\lambda'$

$\Rightarrow$ H_{CD} not unique

$\rightarrow$ Kato potential is gauge inv.:

$$\mathcal{A}_\lambda^K = \frac{1}{2}\sum_n [\Pi_n(\lambda), i\partial_\lambda \Pi_n(\lambda)] \quad ; \Pi_n(\lambda) = |n[\lambda]\rangle\langle n[\lambda]|$$

$$= \sum_{m,n} (1-\delta_{mn}) \langle n[\lambda]| i\partial_\lambda |m[\lambda]\rangle\, |n[\lambda]\rangle\langle m[\lambda]|$$

unique gauge pot. that satisfies (*)

– phases picked up by inst. e'states:

$$i\partial_t |\psi(t)\rangle = H(t)|\psi(t)\rangle$$

$$|n(t)\rangle = \mathcal{T} e^{-i\int_0^t ds\, H(s)} |n[\lambda(0)]\rangle = e^{i(?)} |n[\lambda(t)]\rangle$$

drive	*Hamiltonian $H(t)$*	*accumulated phase*
adiabatic $T_{\mathrm{ramp}}\to\infty$	H_{ctrl}	$\gamma_n(t)+\phi_n(t)$
Kato counterdiabatic gauge-invariant	$H_{\mathrm{ctrl}}+\mathcal{A}_{\mathrm{K},\lambda}$	$\gamma_n(t)+\phi_n(t)$
generic counterdiabatic χ_n arbitrary	$H_{\mathrm{ctrl}}+\mathcal{A}'_\lambda$	$\chi_n(t)+\gamma_n(t)+\phi_n(t)$
dyn. counterdiabatic $\chi_n(t)=-\gamma_n(t)$	$H_{\mathrm{ctrl}}+\mathcal{A}_{\mathrm{D},\lambda}$ $=(i\partial_\lambda u_\lambda)u_\lambda^\dagger$	$\phi_n(t)$
Kato AGP gauge-invariant	$\mathcal{A}_{\mathrm{K},\lambda}$	$\gamma_n(t)$
periodic AGP $\chi_n(t)=-\gamma_n(t)+2\pi\ell_n t/T$	$\mathcal{A}_{\mathrm{F}}(t)=\mathcal{A}_{\mathrm{F}}(t+T)$	$2\pi\ell_n$, $\ell_n\in\mathbb{Z}$ (at $t=T$)

<u>note</u>: unlike PS 2, here phases are wrt. $\mathcal{A}_\lambda^t$

---

back to Floquet systems:

$$H_F[0] = P^\dagger(t)\,H(t)\,P(t) - \underbrace{P^\dagger(t)\, i\partial_t P(t)}_{} \quad / \; P(\cdot)P^\dagger$$

=: $\bar{\mathcal{A}}_F(t)$ Floquet gauge pot in rot. frame

<u>recall</u>: $H_F[t] = P(t)\, H_F[0]\, P^\dagger(t)$

$\Rightarrow H_F[t] = H(t) - \mathcal{A}_F(t)$

$\Rightarrow P(t) = \mathcal{T} e^{-\int_0^t ds\, \mathcal{A}_F(s)}$ //

$\Rightarrow \boxed{H(t) = H_F[t] + \mathcal{A}_F(t)} = H_{CD}(t)$

→ interpret time t / initial time as ramp parameter: $\lambda = t$

lab frame Hamiltonian $H(t)$ generates CD driving w.r.t. the Floquet states

$$|u_F(t)\rangle = \mathcal{T} e^{-i\int_0^t ds\, H(s)} |u_F[0]\rangle \overset{\text{Floquet}}{=} P(t)\, e^{-it H_F[0]} |u_F[0]\rangle$$

$$= e^{-it\varepsilon_F^{(n)}} P(t)\, |u_F[0]\rangle = e^{-it\varepsilon_F^{(n)}} |u_F[t]\rangle$$

inst. Floquet e'state at Floquet gauge t, up to phase

- gauge transf. for periodic states:

$$|u_F[t]\rangle \longrightarrow e^{i\chi_n(t)} |u_F[t]\rangle$$

periodicity $\Rightarrow$ $e^{i\chi_n(t+T)} \overset{!}{=} e^{i\chi_n(t)}$

init. cond. $\Rightarrow$ $\chi_n(0) = 0$

$$\Rightarrow \chi_n(t) = m_n\omega t + \sum_{\ell\neq 0} c_\ell e^{i\ell\omega t}, \quad m_n \in \mathbb{Z} \ \forall n$$

- apply to $H(t) = H_F[t] + A_F(t)$

$$\Rightarrow \quad H_F[t] \longrightarrow H_F[t]$$

$$A_F(t) \longrightarrow A_F(t) - \sum_n \partial_t \chi_n(t)\, \Pi_n(t) \quad ; \quad \Pi_n(t) = |u_F[t]\rangle\langle u_F[t]|$$

$$\Rightarrow H(t) \longrightarrow \underbrace{H_F[t] - \sum_n \left(\partial_t \chi_n(t)\right) \Pi_n(t)}_{= \sum_n \left(\varepsilon_F^{(n)} - \partial_t \chi_n(t)\right) \Pi_n(t)} + A_F(t)$$

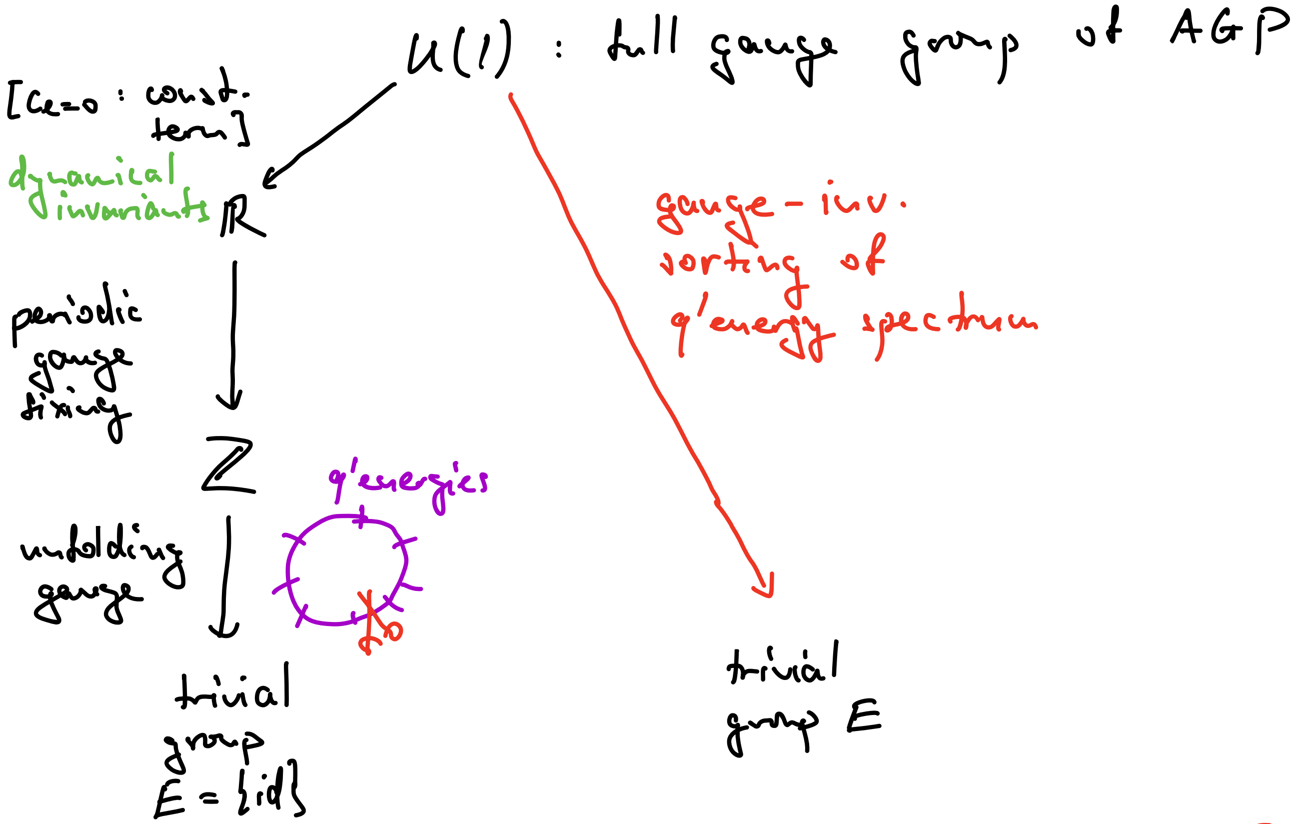
t-indep. $\Rightarrow$ need to also be t-indep.

$$\partial_t \chi_n = m_n\omega + \sum_{\ell\neq 0} i\ell\omega\, c_\ell e^{i\ell\omega t} \Rightarrow c_\ell = 0 \quad \forall \ell\neq 0$$

$$\Rightarrow \chi_n(t) = m_n\omega t$$

$U(1)$ gauge group is broken to $\mathbb{Z}$ $(m_n\in\mathbb{Z})$:

$\rightarrow$ residual $\mathbb{Z}$ gauge freedom is precisely the quasienergy folding ambiguity: $\varepsilon_F^{(n)} \longrightarrow \varepsilon_F^{(n)} + m_n\omega$



Q: what if we use gauge-inv. Kato AGP?

- alternative, Kato decomposition:

$$H(t) = H_F[t] + A_F(t) \overset{!}{=} H_F[t] + \mathcal{D}(t) + \overbrace{A_F(t) - \underbrace{\mathcal{D}(t)}_{= \sum_n A_n^F(t)\,\Pi_n(t)}}^{\overset{!}{=} A_K(t)}$$

← Berry connection

$$= H_K(t) + A_K(t)$$

← gauge-inv.

$$A_K(t) = \frac{1}{2} \sum_n \left[\Pi_n(t),\, i\partial_t \Pi_n(t)\right]$$

& $H_K(t) := H(t) - A_K(t)$ Kato Hamiltonian

- properties of $H_K(t)$:
  - $H_K(t+T) = H_K(t)$, $T = 2\pi/\omega$
  - $H_K(t) = \sum_n \varepsilon_n^K(t)\,\Pi_n(t)$

→ e'states of H_K are the Floquet states $\Pi_n(t)$

→ e'energies are time-dep. & periodic!

- $\varepsilon_n^K(t)$ can cross in time, , but no gaps can open up since "t" is the phase of the drive
- $\varepsilon_n^K(t) := \langle u_F[t]|\, H_K(t)\, |u_F[t]\rangle = \langle u_F[t]|\, H(t) - \overbrace{A_K(t)}^{\text{no diag. el. in Floquet e'basis}}\, |u_F[t]\rangle$

$= \langle u_F[t] | H(t) | u_F[t] \rangle$

- H_K describes the correct ad. evo. of Floquet states (in the sense (*)), while $H_F[t]$ does not (extra phases!)

– geometric Floquet theory:

$$U(t,0) = \mathcal{T} e^{-i\int_0^t ds\, H(s)} \underset{\text{Floquet}}{=} \underbrace{\mathcal{T} e^{-i\int_0^t ds\, \mathcal{A}_F(s)}}_{= P(t)} \times e^{-it H_F[0]}$$

$$= \underbrace{\mathcal{T} e^{-i\int_0^t ds\, \mathcal{A}_K(s)}}_{=: W(t,0) = e^{-i\Gamma(t,0)}} e^{\underbrace{-it \mathcal{Æ}(t,0)}_{\text{"dynamical phases"}}}$$

Wilson line operator
contains geom. phases

& Average Energy operator:

e'states are the Floquet states

$$\mathcal{Æ}(t,0) := \sum_n æ_n(t) |u_F[0]\rangle\langle u_F[0]| \text{, where}$$

$$æ_n(t) := \frac{1}{t}\int_0^t ds\, \varepsilon_n^K(s) = \frac{1}{t}\int_0^t ds\, \langle u_F[s] | H(s) | u_F[s]\rangle$$

note: $\mathcal{Æ}(t) \neq \frac{1}{t}\int_0^t ds\, H_K(s)$

– stroboscopic evo:

$$U(T,0) = W(T)\, e^{-iT\mathcal{Æ}(T,0)}$$

where $\mathcal{Æ}(T,0) = \sum_n \bar{æ}_n |u_F[0]\rangle\langle u_F[0]|$

$$\bar{æ}_n = \frac{1}{T}\int_0^T dt\, \langle u_F[t] | H(t) | u_F[t]\rangle$$

$\bar{æ}$ properly sorted (i.e. there exist unique & natural maximum & minimum) & indep. of the phase of the drive (Floquet gauge)

→ can use $\bar{æ}$ to define unique Floquet ground state!

$(\bar{æ}_n,\ |u_F[t]\rangle)$

- $W(T) = W(T,0)$ indep. of $t_0 = 0$
contains gauge-inv. Berry phases of Floquet states over one drive cycle (closed loop)

$$W(T) = \sum_n e^{i\gamma_n(T)} |u_F[0]\rangle\langle u_F[0]|$$ Wilson loop operator

- $[W(T), \mathcal{A}(T,0)] = 0$ commute (only at $t = \ell T$)

– quasienergy decomposition into geom. & dyn. parts:

$$t\,\varepsilon_F^{(n)} = \underbrace{\gamma_n(t)}_{\text{geom. phase}} + \underbrace{t\,\varkappa_n(t)}_{\text{dyn. phase}}$$

– geometric decomposition of $U(t,0)$ is <u>not</u> stroboscopic:

$$U(\ell T, 0) \neq e^{-i\ell T \mathcal{A}(T,0)}$$

every cycle we pick up Berry phases

– <u>remarks</u>:

(1) Floquet's theorem follows from adiabatic thm for periodic drives (+ CD driving)

(2) adiabaticity does not require a Hamiltonian (H_F not unique, proper object is U_F), nor energy: it's about the geometry of the state manifold

(3) inherently noneq. phenomena (e.g. anomalous Floquet topo. insulators or time crystals are geometric!)

→ check out arXiv: 2410:07029

– <u>applications</u>:

1) var'l. principle for H_F? $H_F[t] = H(t) - A_F(t)$

→ locality of H_F is the same as locality of A_F (for $H(t)$ local)

2) better characterization of heating & thermalization in Floquet systems

3) CD driving for Floquet systems: vary A, ω, φ