

Coherent quantum states from classical oscillator amplitudes

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In the first days of quantum mechanics Dirac pointed out an analogy between the time-dependent coefficients of an expansion of the Schrödinger equation and the classical position and momentum variables solving Hamilton's equations. Here it is shown that the analogy can be made an equivalence in that, in principle, systems of classical oscillators can be constructed whose position and momenta variables form time-dependent amplitudes which are *identical* to the complex quantum amplitudes of the coupled wave function of an N -level quantum system with real coupling matrix elements. Hence classical motion can reproduce quantum coherence.

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I. INTRODUCTION

In their first formulation of quantum mechanics, both Schrödinger and Dirac were strongly influenced by connections to the Hamiltonian formulation of classical mechanics. Indeed, in one of the very first applications of Schrödinger's time-dependent equation (TDSE), Dirac [1] indicated a close parallel between the coupled first-order set of equations arising from the TDSE and the coupled first-order Hamilton equations of classical mechanics. Dirac introduced the time-dependent basis set expansion,

$$|\Psi(t)\rangle = \sum_n c_n(t) |\pi_n\rangle, \quad (1)$$

where the c_n are complex coefficients and $|\pi_n\rangle$ denotes an arbitrary basis. In the TDSE (where, for the moment, we put $\hbar = 1$) this expansion leads to the set of first-order coupled equations,

$$i\dot{c}_n = \sum_m H_{nm} c_m, \quad (2)$$

where H_{nm} are the matrix elements of the quantum Hamiltonian. Dirac then remarked that by considering the coefficients $q_n \equiv c_n$ and $p_n \equiv i c_n^*$ to be canonical variables and assuming a "Hamiltonian function"

$$\mathcal{H} = \sum_{nm} c_n^* H_{nm} c_m, \quad (3)$$

the quantum equations are equivalent to the classical Hamilton equations

$$\dot{q}_n = \frac{\partial \mathcal{H}}{\partial p_n}, \quad \dot{p}_n = -\frac{\partial \mathcal{H}}{\partial q_n}. \quad (4)$$

Note, however, that this is still fully quantum mechanical since the matrix elements and the amplitudes appearing in the "classical" Hamiltonian are all complex objects. Hence, there is no obvious classical counterpart. Dirac also made a transformation to real variables by using amplitude and phase of the quantum coefficients; namely,

$$c_n = \sqrt{\rho_n} e^{i\theta_n}, \quad (5)$$

where $\rho_n = c_n^* c_n$. Although with this transformation the variables are real, the "classical" Hamiltonian still contains complex quantities and hence does not obviously correspond to any real physical system.

Forty years later, Strocchi [2] approached the question of quantum-classical equivalence slightly differently. He first formulated classical mechanics in terms of complex variables $z_n = (q_n + ip_n)/\sqrt{2}$, with p_n and q_n real, to give the Hamiltonian equations (4) in the form

$$i\dot{z}_n = \frac{\partial \mathcal{H}}{\partial z_n^*}, \quad (6)$$

and its complex conjugate. Here \mathcal{H} is a real classical Hamiltonian. Then it is remarked that the TDSE coupled equations (2) are equivalent to these classical equations; that is, $c_n = z_n$ if the "classical" Hamiltonian function is taken as the expectation value of the quantum Hamiltonian; that is,

$$\mathcal{H} = \langle \Psi(t) | H | \Psi(t) \rangle = \sum_{nm} c_n^*(t) H_{nm} c_m(t), \quad (7)$$

as in the treatment of Dirac. Although ostensibly time-dependent and complex, it is easy to prove that \mathcal{H} becomes constant and real for Hermitian Hamiltonians. Identifying c_n with z_n and reverting to the real variables (q_n, p_n) , one has, however, still the apparently complex form quoted by Strocchi [2],

$$\mathcal{H} = \frac{1}{2} \sum_{nm} H_{nm} (q_n q_m + p_n p_m - i q_n p_m + i p_n q_m). \quad (8)$$

Of course, for H Hermitian one does achieve a real "classical" Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{nm} [H_{nm} (q_n q_m + p_n p_m) + 2\text{Im}(H_{nm}) q_n p_m], \quad (9)$$

and in the special case that all coupling matrix elements are real the Hamiltonian,

$$\mathcal{H} = \frac{1}{2} \sum_{nm} H_{nm} (q_n q_m + p_n p_m), \quad (10)$$

which is that of coupled real harmonic oscillators. Note that the coupling is of a very special form in which there is both linear position and momentum off-diagonal coupling with exactly the same coupling strengths.

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The mapping of the TDSE onto classical-like equations up until now has been regarded as something of a curiosity, although it has been used as a starting point to treat some problems of molecular electronic dynamics semiclassically [3,4]. Here we show that systems of coupled classical oscillators corresponding to the Hamiltonian Eq. (10) can be realized whose position and momentum variables reproduce *exactly* the time-dependent coefficients of an expansion of the quantum wave function. We call these the *p*-and-*q*-coupled oscillators. Each oscillator plays the role of a state in Hilbert space and the coupling between the oscillators plays the role of quantum coupling matrix elements. Unfortunately, for a general quantum system the corresponding exactly equivalent classical system involves rather complicated coupling schemes between oscillators. However, we show that, in a weak-coupling approximation, simpler sets of coupled oscillators, in which only the *q* coupling in the Hamiltonian Eq. (10) occurs, provide an exceedingly good approximation of the exact result.

In previous papers [5,6], where we studied energy transfer along dipole-dipole interacting molecules, we called this approximation the “realistic-coupling approximation” (RCA) since for this case, either classical or quantum, to be realistic, it is essential that the oscillators and quantum entities largely retain their character when coupled (i.e., the coupling is weak compared to their internal forces). Although in the present work we consider a general Hamiltonian where the diagonal elements in a chosen basis are not necessarily associated with concrete physical entities and thus there is no *a priori* reason for the off-diagonal elements to be small, we will still keep the term RCA for this weak-coupling approximation.

It is interesting that, with reference to specific systems, several authors have already recognised the similarity between *q*-coupled classical oscillators and few-level quantum systems. Indeed, already in one of the the first quantum treatments of resonant electronic energy transfer between identical atoms, Frenkel in 1930 [7] remarked on the essential equivalence of the quantum treatment to the earlier 1925 classical treatment of Holtsmark [8]. In particular, he showed that the normal-mode frequencies of the classical coupled equations reduce to the quantum eigenvalues in the approximation that we call the RCA.

In 1977 McKibben [9] constructed a system of three coupled mechanical pendula and showed that, in a weak-coupling approximation similar to our RCA, their equations of motion become equivalent to the quantum equations describing the operation of a spin filter on three levels of the hydrogen atom. Later Hemmer and Prentiss [10] used the same three-pendulum classical system to interpret the quite different three-level atomic problem of the stimulated resonance Raman effect and pointed out “the strong mathematical similarities between the pendulum amplitude equations and the resonance Raman equations.” More recently, Marx and Glaser [11] have derived an exact correspondence (in the RCA) between the dynamics of three isotropically coupled spins and those of three coupled classical oscillators. Here the Liouville equation for the density matrix and spin correlation functions derived from it were considered.

In a rather different application, Jolk *et al.* [12] and Kovaleva *et al.* [13] have pointed out the similarity of the

noncrossing behavior of the eigenfrequencies of a pair of dissimilar classical oscillators under varying coupling to the noncrossing behavior of the eigenenergies of two quantum levels subject to a varying perturbation, as in the celebrated Landau-Zener problem. Similarly, Spreeuw *et al.* [14] have discussed a classical optical system as analog to a driven two-level atom.

In previous communications [5,6] we have shown that the time-dependent Schrödinger equation for an aggregate of *N* coupled monomers having a single electronic transition and the classical equations for coupled electric dipoles are equivalent in the RCA. There we demonstrated that the quantum coherence in the transfer of electronic excitation along a linear chain of monomers is reproduced by the transfer of electric dipole strength along the equivalent classical array. The equivalence proof was restricted to this special problem of coupled classical electric dipoles and its relevance for energy transfer in photosynthesis.

These many and varied applications point to a general property of equivalence of classical oscillators and quantum systems in finite Hilbert spaces. However, none of the above authors discuss their findings in the context of Dirac and Strocchi’s analysis. Also, they all use classical oscillators that are *q* coupled; that is, they work only in the RCA and do not recognize that it is possible to construct classical oscillators that *exactly* mimic the quantum system. In the present paper we show how to construct such *p*-and-*q*-coupled oscillators and then clarify under what conditions the RCA allows simpler *q*-coupled sets of oscillators to be used as analogs of coupled-state quantum systems.

The structure of the paper is as follows: In Sec. II A we derive the classical *p*-and-*q*-coupled equations whose variables give directly the coefficients of the time-dependent Schrödinger wave function. We also examine the derivation of the RCA and the meaning of this approximation in the context of the Dirac and Strocchi mapping of the Schrödinger equation to Hamilton’s equations. As a specific example, in Sec. III we consider excitation transfer on a quantum aggregate of *N* coupled monomers and its classical equivalent of *N* coupled pendula. Special attention is given to the *N* = 2 dimer case whose wave function is considered a fundamental example of quantum entanglement. In this case the quantum (and therefore also the *p*-and-*q*-coupled classical) solution can be obtained in simple closed form. Then the RCA is tested by comparing the *q*-coupled classical solution with the *p*-and-*q*-coupled solution (i.e., with the *exact* quantum solution). Oscillatory energy transfer on the dimer is shown to correspond classically to the well-known “beating” phenomenon of coupled pendula from whose motion the entangled wave function can be extracted.

The classical analysis is given, for example, for the specific case of coupled undamped vertical pendula. However, it is clear that the analysis also applies to other sets of oscillators, particularly classical electrical *LC* circuits, as shown in Appendix B. It is also demonstrated in Appendix C that the inclusion of a velocity-dependent damping term in the classical equations corresponds exactly to the inclusion of damping via a complex eigenvalue in the quantum equations, as used for example in Ref. [10].

II. QUANTUM N -LEVEL PROBLEM AND COUPLED CLASSICAL OSCILLATORS

A. Exact mapping of quantum to classical motion

First we will show that the first-order Schrödinger equations (2) map simply onto the classical Hamilton equations for a set of coupled classical oscillators. Since the classical problem of coupled oscillators is almost always formulated using the Newton equations rather than the Hamilton equations we also present the corresponding second-order Newton equations. We start with the Strocchi classical Hamiltonian, Eq. (10). The Hamilton equations (4) give

$$\dot{q}_n = \sum_m H_{nm} p_m, \quad \dot{p}_n = -\sum_m H_{nm} q_m. \quad (11)$$

Symbolically writing q and p as vectors and H as a matrix, the above equations are

$$\dot{\mathbf{q}} = \mathbf{H}\mathbf{p}, \quad \dot{\mathbf{p}} = -\mathbf{H}\mathbf{q}, \quad (12)$$

and formally

$$\ddot{\mathbf{q}} = \mathbf{H}\dot{\mathbf{p}} = -\mathbf{H}^2\mathbf{q}, \quad (13)$$

which are a set of coupled oscillator equations and can be solved for $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$. The momenta at time t can then be calculated from

$$\mathbf{p} = \mathbf{H}^{-1}\dot{\mathbf{q}}. \quad (14)$$

The set of complex amplitudes, the vector \mathbf{z} , is constructed as

$$\mathbf{z} = \frac{1}{\sqrt{2}}(\mathbf{q} + i\mathbf{p}). \quad (15)$$

From the Hamilton equations (12) we have

$$\ddot{\mathbf{z}} = -\mathbf{H}^2\mathbf{z}. \quad (16)$$

Similarly, the Schrödinger equation (2) is written as

$$i\dot{\mathbf{c}} = \mathbf{H}\mathbf{c} \quad (17)$$

or

$$\ddot{\mathbf{c}} = -\mathbf{H}^2\mathbf{c}, \quad (18)$$

which is exactly the classical equation (15). Hence the p -and- q -coupled classical equations and the coupled quantum Schrödinger equations are identical and, in particular, they have the same eigenvalues.

To see how to construct a set of real classical oscillators, now we write the formal solution in terms of the individual oscillator amplitudes. First we split off the diagonal term in the Hamiltonian. This quantum energy H_{nn} (divided by \hbar) will be denoted by ω_n . The off-diagonal elements (divided by \hbar) will be denoted by V_{mn} . Then the Hamilton equations give

$$\begin{aligned} \dot{q}_n &= \omega_n p_n + \sum_{m \neq n} V_{nm} p_m, \\ \dot{p}_n &= -\omega_n q_n - \sum_{m \neq n} V_{nm} q_m. \end{aligned} \quad (19)$$

Forming $z_n = \frac{1}{\sqrt{2}}(q_n + ip_n)$ gives

$$i\dot{z}_n = \omega_n z_n + \sum_{m \neq n} V_{nm} z_m, \quad (20)$$

which are identical to the quantum equations (2) by construction. Taking the time derivative of the \dot{q}_n equation and substituting for the \dot{p}_n from the second equation leads to the coupled second-order equations

$$\ddot{q}_n + \omega_n^2 q_n = -\left[\sum_{m \neq n} (\omega_n + \omega_m) V_{nm} - \sum_m W_{nm} \right] q_m, \quad (21)$$

where we defined

$$W_{nm} = \sum_{m' \neq n, m' \neq m} V_{nm'} V_{m'm}. \quad (22)$$

This set of classical equations are the p -and- q -coupled equations. Although the coupling elements appear quite complicated, for a given set of quantum matrix elements V_{nm} they can, in principle, be solved to obtain $q_n(t)$ and $\dot{q}_n(t)$. Then the time-dependent momenta must be calculated by matrix inversion as in Eq. (14) and the complex $z_n(t)$, equal to the quantum coefficients $c_n(t)$, calculated.

1. Hamiltonian in realistic-coupling approximation

To construct a classical analog to a given quantum system the oscillators must be coupled as in Eq (21). However, when the Hamiltonian involves only positional q couplings, the Hamilton equations simplify to

$$\dot{q}_n = \omega_n p_n, \quad \dot{p}_n = -\omega_n p_n - \sum_{m \neq n} H_{nm} q_m, \quad (23)$$

or, equivalently,

$$\ddot{q}_n + \omega_n^2 q_n = -\sum_{m \neq n} \omega_n V_{nm} q_m, \quad (24)$$

which are the q -coupled classical equations. If we define real coupling elements $K_{nm} \equiv -\omega_n V_{nm}$ these are the standard coupled equations; for example, of a set of linearly coupled mechanical or capacitor-coupled electrical oscillators. In Appendix A it is shown that, in the RCA, which corresponds to having $V_{nm} \ll \omega_n, \omega_m \forall n, m$, the solutions of the simple q -coupled equations are a good approximation to the more complicated p -and- q -coupled equations (21).

B. Eigenfunctions and eigenmodes

1. Quantum result

In the basis $|\pi_n\rangle$ defined in Eq. (1) the Hamiltonian defined in Eq. (2) is not diagonal in the general case. To find the eigenfunctions and eigenenergies we solve

$$H|\psi_k\rangle = E_k|\psi_k\rangle. \quad (25)$$

The eigenstates $|\psi_k\rangle$ can be expressed in the original basis via

$$|\psi_k\rangle = \sum_{n=1}^N B_{kn} |\pi_n\rangle. \quad (26)$$

Denoting the diagonal elements $H_{nn} = \langle \pi_n | H | \pi_n \rangle \equiv \epsilon_n$ and the off-diagonal elements by $H_{nm} = \langle \pi_n | H | \pi_m \rangle \equiv \mathcal{V}_{nm}$ the coefficients B_{kn} can be obtained from the coupled set of equations

$$(E_k - \epsilon_n) B_{kn} = \sum_m \mathcal{V}_{nm} B_{km}. \quad (27)$$

A general time-dependent wave function (coherent wave packet) can be expanded either in the basis $|\pi_n\rangle$ or in the eigenbasis $|\psi_k\rangle$; namely,

$$|\Psi(t)\rangle = \sum_n c_n(t)|\pi_n\rangle = \sum_k b_k(t)|\psi_k\rangle. \quad (28)$$

Since the eigenbasis diagonalizes the Hamiltonian, the coefficients $b_k(t)$ are given simply by

$$b_k(t) = A_k \exp\left(-\frac{i}{\hbar} E_k t\right), \quad (29)$$

where the time-independent complex coefficients A_k are decided by the initial conditions. Then using Eq. (26) in Eq. (28) one has the amplitude of state n ,

$$c_n(t) = \sum_k A_k B_{kn} \exp\left(-\frac{i}{\hbar} E_k t\right). \quad (30)$$

2. Exact mapping

The Newton equations of motion for the exactly equivalent set of classical oscillators are the coupled equations (21). The eigenfrequencies of these classical equations are identical to the E_k of the diagonalized quantum problem.

Since the eigenvalues and therefore the eigenfunctions of the quantum and classical systems are identical, we see that the classical time dependence can be used to construct the quantum wave function. The normal modes arising from the diagonalization of Eqs. (21) are of the form $q_k = \beta_k \cos(\Omega_k t + \alpha_k)$ where β_k and α_k are real constants. From these normal modes we derive the velocities

$$\dot{q}_k(t) = -\Omega_k \beta_k \sin(\Omega_k t + \alpha_k). \quad (31)$$

Then the complex classical amplitudes are $z_k(t) = \frac{1}{\sqrt{2}}[q_k(t) + ip_k(t)] = \frac{1}{\sqrt{2}}[q_k(t) + (i/\Omega_k)\dot{q}_k(t)]$ to obtain the general solution as a sum of normal modes,

$$z_n(t) = \sum_k B_{kn} A_k \exp[-i(\Omega_k t)], \quad (32)$$

where we have to set the $t = 0$ initial conditions so that $A_k = (\beta_k/\sqrt{2}) \exp(i\alpha_k)$ and put $\Omega_k = E_k/\hbar$. Comparing this result with Eq. (30), we see that the classical amplitudes $z_n(t)$ reproduce the quantum coefficients $c_n(t)$ and the classical motion can be used to reconstruct the quantum entangled wave function of Eq. (28).

3. Coupled equations in realistic-coupling approximation

As we show in Appendix A that, in the RCA, Eqs. (21) reduce to become identical in form to the simpler linearly q -coupled classical Eqs. (24), which we write as

$$\ddot{q}_n + \omega_n^2 x_n = - \sum_{m \neq n} K_{nm} q_m. \quad (33)$$

Now we show that, again in the RCA, these equations have eigenfrequencies which closely approximate those of the quantum problem and therefore also the eigenfrequencies of Eqs. (21). Substitution of the special eigenmode solution

$q_n = C_{kn} \cos(\Omega_k t)$ gives

$$(\Omega_k^2 - \omega_n^2) C_{kn} = \sum_m K_{nm} C_{km}. \quad (34)$$

This equation can be written as

$$(\Omega_k - \omega_n) C_{kn} = (\Omega_k + \omega_n)^{-1} \sum_m K_{nm} C_{km}. \quad (35)$$

Now we make a second realistic coupling approximation (RCA) by considering that the spread of eigenfrequencies (bandwidth of the dispersion relation) resulting from diagonalization of Eq. (34) is small compared to the mean natural frequency $\omega \equiv \bar{\omega}_n$ of the oscillators. Similarly, for nonidentical oscillators, the width of the ω_n distribution must be small compared to ω . Then on the right-hand side of Eq. (34) we can approximate Ω_k and ω_n by ω to give

$$(\Omega_k - \omega_n) C_{kn} = \sum_m \frac{K_{nm}}{2\omega} C_{km}. \quad (36)$$

Furthermore, if we make the identification $E_k \equiv \hbar\Omega_k$, $\epsilon_n \equiv \hbar\omega_n$, and $\mathcal{V}_{nm}/\hbar \equiv K_{nm}/(2\omega)$, Eqs. (27) and (36) are identical and we can put in the RCA

$$C_{kn} = B_{kn} \quad \forall k, n. \quad (37)$$

Note that the RCA is synonymous with the condition that the couplings K_{nm} are small compared to ω and, equivalently in the quantum case, that all \mathcal{V}_{nm} are small compared to the mean energy ϵ of the various eigenvalues ϵ_n of H_0 . Note also that this in no way implies that perturbation theory must be applicable since, as in the examples given below, the analysis applies to the case where all ϵ_n are degenerate, where perturbation theory is invalid.

III. TWO SIMPLE EXAMPLES

To illustrate the construction of classical oscillator systems which can reproduce entangled wave functions, both exactly and approximately in the RCA, we consider the quantum problem of N identical two-level systems. Specifically we consider the dynamics when there is exactly one excitation present and take $|\pi_n\rangle$ as the state in which monomer n is excited and all other monomers are in the ground state. This is the exciton model studied originally by Frenkel. Such model quantum systems are of fundamental importance in several areas; for example, in quantum computing and in the modeling of the photosynthetic unit, in addition to applications mentioned in the introduction. Indeed, the quantum dimer of $N = 2$ coupled two-level systems (two qubits), is viewed as having a wave function which is the simplest example of quantum entanglement. This is the case we discuss first.

A. Quantum dimer problem and two classical coupled oscillators

To make the problem concrete we will think of a dimer composed of two identical atoms or molecules each having only a ground state and one excited state. Note that we restrict ourselves to the subspace where exactly one excitation is present. Classically this corresponds to two coupled oscillators.

1. Quantum result

The dimer has + and – eigenstates of the form

$$|\psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|\pi_1\rangle \pm |\pi_2\rangle), \quad (38)$$

with eigenenergies $\epsilon_{\pm} = \epsilon \pm \mathcal{V}$, where ϵ is the monomer transition energy and $\mathcal{V} \equiv \mathcal{V}_{12} = \mathcal{V}_{21}$ is real. The state $|\pi_1\rangle$ has monomer 1 excited and monomer 2 in the ground state and correspondingly for $|\pi_2\rangle$.

We expand a solution of the TDSE as

$$|\Psi(t)\rangle = a_+(t)|\psi_+\rangle + a_-(t)|\psi_-\rangle \quad (39)$$

The two eigenstates propagate independently in time according to

$$a_{\pm}(t) = A_{\pm} \exp[-(i/\hbar)\epsilon_{\pm}t] = A_{\pm} \exp[-(i/\hbar)(\epsilon \pm \mathcal{V})t]. \quad (40)$$

Excitation transfer is described by the initial condition $|\Psi(0)\rangle = |\pi_1\rangle$, which leads to the time-dependence

$$|\Psi(t)\rangle = c_1(t)|\pi_1\rangle + c_2(t)|\pi_2\rangle \quad (41)$$

with $c_1 = \frac{1}{2}[\exp(-i\epsilon_+t/\hbar) + \exp(-i\epsilon_-t/\hbar)]$ and $c_2 = \frac{1}{2}[\exp(-i\epsilon_+t/\hbar) - \exp(-i\epsilon_-t/\hbar)]$. These expressions can be simplified further to give

$$\begin{aligned} c_1(t) &= \exp[-(i/\hbar)\epsilon t] \cos[\mathcal{V}t/\hbar], \\ c_2(t) &= -i \exp[-(i/\hbar)\epsilon t] \sin[\mathcal{V}t/\hbar], \end{aligned} \quad (42)$$

which are the exact quantum solutions and describe a periodic transfer of excitation between the two monomers. Note that, in forming the density matrix of coefficients $c_i^*c_j$, the pure phase factor $\exp[-(i/\hbar)\epsilon t]$ disappears.

2. Exact mapping

The mapped Hamilton equations, with $\omega = \epsilon/\hbar$ and $V = \mathcal{V}/\hbar$ and for the case $N = 2$, are

$$\begin{aligned} \dot{q}_1 &= \omega p_1 + V p_2, & \dot{p}_1 &= -\omega q_1 - V q_2, \\ \dot{q}_2 &= \omega p_2 + V p_1, & \dot{p}_2 &= -\omega q_2 - V q_1, \end{aligned} \quad (43)$$

to give the coupled oscillator equations

$$\begin{aligned} \ddot{q}_1 + (\omega^2 + V^2)q_1 &= -2\omega V q_2, \\ \ddot{q}_2 + (\omega^2 + V^2)q_2 &= -2\omega V q_1. \end{aligned} \quad (44)$$

In the usual way these symmetric equations can be diagonalized by the transformation $q_{\pm} = q_1 \pm q_2$ to give normal modes

$$\ddot{q}_{\pm} + (\omega \pm V)^2 q_{\pm} = 0, \quad (45)$$

with eigenfrequencies $\Omega_{\pm} = \omega \pm V$, where we take the positive square root and assume $|V| < \omega$. As they should, these reproduce exactly the eigenenergies $\epsilon_{\pm} = \epsilon \pm \mathcal{V}$ of the quantum dimer problem.

The momenta are obtained from the Hamilton equations as

$$\begin{aligned} p_1 &= \frac{\omega}{(\omega^2 - V^2)} \left(\dot{q}_1 - \frac{V}{\omega} \dot{q}_2 \right), \\ p_2 &= \frac{\omega}{(\omega^2 - V^2)} \left(\dot{q}_2 - \frac{V}{\omega} \dot{q}_1 \right), \end{aligned} \quad (46)$$

which combine conveniently to give

$$p_{\pm} = p_1 \pm p_2 = \frac{1}{\Omega_{\pm}} \dot{q}_{\pm}. \quad (47)$$

Thus in the eigenmodes we have the simple expression for the quantum amplitudes in terms of the classical amplitudes

$$c_{\pm} = z_{\pm} = q_{\pm} + \frac{i}{\Omega_{\pm}} \dot{q}_{\pm}. \quad (48)$$

Note that all the above results are invariant to a change of sign of V which only serves to flip the Ω_{\pm} eigenvalues.

As an example, in the beating mode the wave function is exactly reproduced by the classical amplitudes. The initial conditions for beating are $q_1(0) = \beta$, $q_2(0) = 0$, and $\dot{q}_1 = \dot{q}_2 = 0$ leading to amplitudes

$$\begin{aligned} q_1(t) &= (\beta/2)[\cos(\Omega_+t) + \cos(\Omega_-t)], \\ q_2(t) &= (\beta/2)[\cos(\Omega_+t) - \cos(\Omega_-t)]. \end{aligned} \quad (49)$$

Forming the velocities and from them the momenta, after some algebra one finds $p_1(t) = -(\beta/2)[\sin(\Omega_+t) + \sin(\Omega_-t)]$ and $p_2(t) = -(\beta/2)[\sin(\Omega_+t) - \sin(\Omega_-t)]$ so that

$$\begin{aligned} z_1 &= (\beta/2\sqrt{2})[e^{-i\Omega_+t} + e^{-i\Omega_-t}], \\ z_2 &= (\beta/2\sqrt{2})[e^{-i\Omega_+t} - e^{-i\Omega_-t}]. \end{aligned} \quad (50)$$

Choosing $\beta = -\sqrt{2}$ and noting that $\epsilon_{\pm} = \hbar\Omega_{\pm}$, these are exactly the quantum amplitudes c_1 and c_2 of Eq. (41) leading to the explicit beating forms of Eqs. (42).

We now show how to simulate the quantum equation of motion using two coupled classical pendula. The classical equations of motion for two (mathematical, i.e., linearized) pendula with oscillation angle ϕ , natural frequency ω , and coupled by a spring with coupling strength K are

$$\begin{aligned} \ddot{\phi}_1 + \omega^2 \phi_1 + K \phi_1 &= K \phi_2, \\ \ddot{\phi}_2 + \omega^2 \phi_2 + K \phi_2 &= K \phi_1. \end{aligned} \quad (51)$$

Making again the identification $V = K/(2\omega)$, equations (44), which map exactly the Schrödinger equation, can be written in the same form

$$\begin{aligned} \ddot{q}_1 + \omega_s^2 q_1 + K q_1 &= K q_2, \\ \ddot{q}_2 + \omega_s^2 q_2 + K q_2 &= K q_1, \end{aligned} \quad (52)$$

where

$$\omega_s^2 \equiv \omega^2 - K + K^2/(4\omega^2), \quad (53)$$

so that, in this case, $\omega_s = \omega - (K/2\omega) = \Omega_-$. Thus, one takes two pendula of natural frequency ω . One then couples them with strength K and simultaneously adjusts the lengths to give a new natural frequency ω_s . Then the oscillation amplitudes and velocities of this classical system reproduce *exactly* the complex time-dependent amplitudes of the quantum dimer wave function, with transition energy $\epsilon = \hbar\omega$ and coupling matrix element $V = K/(2\omega)$. Clearly, as an alternative one can leave the natural frequency unchanged as ω_s and then, for a given K , infer the transition energy ω of the equivalent quantum system from ω_s . Hence we have shown how to construct a pair of classical oscillators whose motions reproduce the entangled two-qubit quantum wave function time dependence.

3. Dimer in realistic-coupling approximation

Previous works pointing out the equivalence of classical oscillator motion and quantum time development have not used the exact mapping but rather the simpler standard equations (51) or their equivalent and then invoked the RCA. Next we investigate the accuracy of this approach in the simple dimer case.

The eigenfrequencies of the exact mapping [Eqs. (52)] are readily calculated to be $\Omega_+ = \omega_S$ and $\Omega_- = \sqrt{\omega_S^2 + 2K}$. With $K = 2\omega V$ this translates to the eigenfrequencies $\Omega_{\pm} = \omega \pm V$ of Eqs. (44) and of course of the quantum problem. The eigenfrequencies of the standard equations (51) are $\Omega_+ = \omega$ and $\Omega_- = \sqrt{\omega^2 + 2K}$ involving the natural oscillator frequency. However, in the realistic-coupling approximation (RCA) we expand, as in Sec. II B,

$$\Omega_- = \sqrt{\omega^2 + 2K} \approx \omega + (K/\omega) = \omega + 2V, \quad (54)$$

to give eigenfrequency difference $\Omega_- - \Omega_+ = 2V$, as in the exact mapping. This implies that, when the RCA is valid, the solutions to the standard equations (51) will be a good approximation to the exact solution, up to an overall phase caused by the shift in absolute value of the eigenfrequencies. This explains the previously observed close agreement of quantum solutions and standard (q -coupled) oscillator amplitudes.

To assess the accuracy of the RCA we have calculated exact quantum [exactly equivalent to the solution of the p -and- q -coupled classical equations (44)] and compared to solutions of the standard q -coupled classical equations (51) for the $N = 2$ dimer case with “beating” initial conditions. The results are shown in Figs. 1 and 2. In Fig. 1 the case $\omega = 1$, $K = 0.01$ is shown and one sees excellent agreement between exact, p -and- q -coupled (blue, solid) and standard q -coupled (red, dashed) results over several periods of the transfer time [the time is given in units of $K/(2\omega)$ or equivalently V/\hbar]. Only a small shift in relative phase is perceptible. In the middle column, the q -coupled results are multiplied by an overall phase factor $\exp[i(\Omega_- - \omega)t]$ to compensate for the overall energy shift between classical and quantum eigenvalues. The upper figures show the absolute values squared of the classical q -coupled and quantum coefficients where this phase difference disappears. Similarly, the lower panel shows the imaginary part of the off-diagonal density matrix elements $c_1^*c_2$ and $z_1^*z_2$ where the overall phase factor cancels also (the real part is zero). By contrast to the good agreement for $K = 0.01$, the case $K = 0.1$ is shown in Fig. 2, where one sees the beginning of the breakdown of the RCA in that, although still of the same shape, the phase difference between quantum and classical q -coupled curves is becoming more pronounced.

B. Circular array of N interacting monomers

For $N = 2$ we have shown that, by adjusting the natural oscillator frequency to be ω_S , one can readily construct a classical system to mimic exactly the quantum wave function. To illustrate the complexity of the equivalent classical system that begins to arise for quantum systems of larger Hilbert space dimension, we consider next the extension to N interacting quantum monomers. This is a model for Frenkel exciton energy transfer in molecular crystals [15], on dye aggregates [16,17],

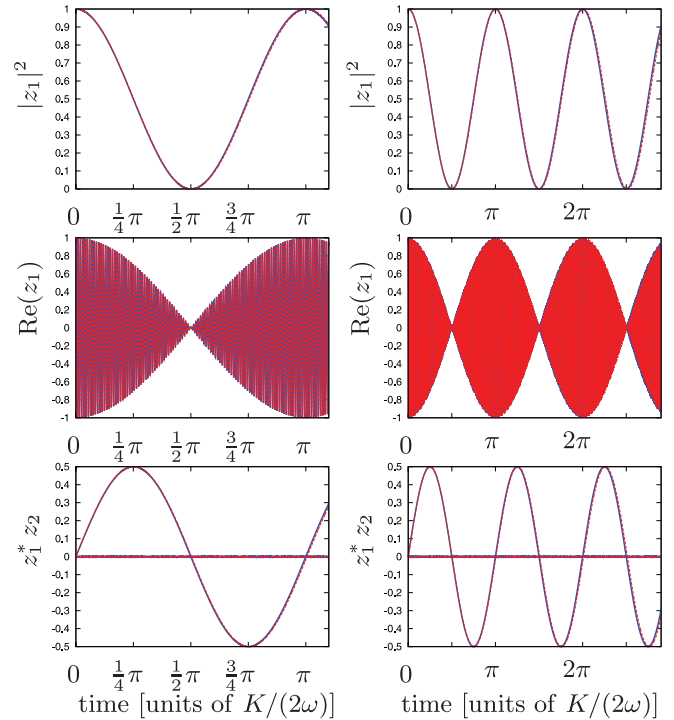


FIG. 1. (Color online) Time dependence of quantum or p -and- q -coupled classical (blue, solid) and q -coupled classical (red, dashed) motion. The left column shows details for short times and the right column is for longer times. The upper row shows the absolute value squared and the middle row shows the real part of the amplitudes. The bottom panel shows $c_1^*c_2$ and $z_1^*z_2$. The time is given in units of $K/(2\omega)$ or, equivalently, V/\hbar .

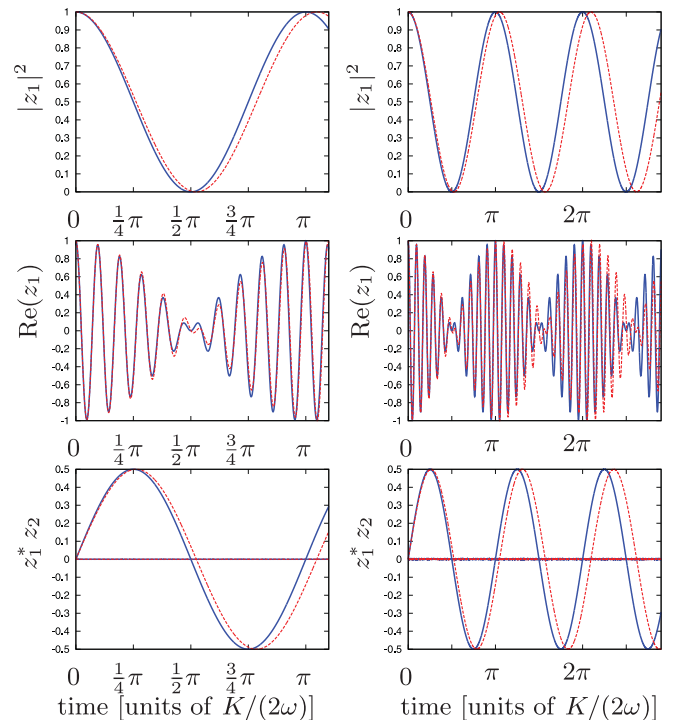


FIG. 2. (Color online) Same as Fig. 1 except that the coupling strength is $K = 0.1$.

or in the photosynthetic unit [18]. In a previous paper [5] we have shown that, in the RCA, a set of q -coupled classical oscillators gives energy propagation characteristics indistinguishable from the exact quantum result.

1. Quantum result

For simplicity we take the monomers to be identical with transition energy ϵ . Furthermore we take only nearest-neighbor interaction into account and set $\mathcal{V}_{n,n\pm 1} \equiv \mathcal{V}$. For circular boundary conditions the result of diagonalization is standard [16] and gives

$$E_k = \epsilon + 2\mathcal{V} \cos k, \quad (55)$$

with $k = (2\pi/N)j$, $j = 0, \dots, N-1$. The transformation matrix elements are

$$B_{kn} = \frac{1}{\sqrt{N}} \exp(ikn). \quad (56)$$

If we consider the time development beginning with only monomer 0 excited [i.e., $c_n(0) = \delta_{n0}$], then from the orthogonality of normal modes one has

$$c_n(t) = \frac{1}{N} \exp\left(-\frac{i}{\hbar}\epsilon t\right) \sum_k \exp(ikn) \exp\left(-\frac{i}{\hbar}(2\mathcal{V} \cos k)t\right). \quad (57)$$

2. Exact mapping

The exact mapping of this quantum system to the Hamilton equations gives the p -and- q -coupled equations (21) in the form

$$\ddot{q}_n + (\omega^2 + 2V^2)q_n = -2V\omega(q_{n+1} + q_{n-1}) - V^2(q_{n+2} + q_{n-2}). \quad (58)$$

Note that, although the first-order Schrödinger equations involve nearest-neighbor couplings, the equivalent second-order Newton equations explicitly contain next-nearest-neighbor couplings. Substituting the trial solution $q_{nk} = A_{kn} \exp(i\Omega t + ikn)$ gives the eigenfrequency equation

$$\Omega^2 - \omega^2 - 2V^2 = 4V\omega \cos k + 2V^2 \cos 2k, \quad (59)$$

with solutions

$$\Omega_k = \omega + 2V \cos k, \quad (60)$$

in agreement with the quantum result [Eq. (55)]. Hence for this N -monomer case we see again how to construct a system of classical oscillators to reproduce *exactly* the quantum results. However, the couplings between oscillators is becoming more complicated.

3. N -mer in realistic-coupling approximation

The simpler q -coupled classical equations are those with nearest-neighbor coupling only, and were solved numerically in Ref. [5] to compare with the quantum energy transfer result. Here we show analytically that, in the RCA, one obtains the same eigenfrequencies and time dependence as in the quantum or p -and- q -coupled classical case. We consider a ring of

identical coupled pendula. The coupling is between adjacent pendula only. The standard equations of motion are

$$\ddot{\phi}_n + \omega^2 \phi_n + 2K \phi_n = K(\phi_{n+1} + \phi_{n-1}), \quad (61)$$

leading to the eigenmode equation

$$[\Omega_k^2 - (\omega^2 + 2K)]B_{kn} = -K(B_{k,n+1} + B_{k,n-1}). \quad (62)$$

The eigenvalues are

$$\Omega_k^2 - (\omega^2 + 2K) = -2K \cos k, \quad (63)$$

which, in the RCA, becomes

$$\Omega_k \approx (\omega + K/\omega) - (K/\omega) \cos k. \quad (64)$$

Now we make the identification $E_k \equiv \hbar\Omega_k$, $\epsilon \equiv \hbar\omega$, and $\mathcal{V}/\hbar \equiv -K/(2\omega)$ to give

$$E_k = (\epsilon - 2\mathcal{V}) + 2\mathcal{V} \cos k. \quad (65)$$

Comparison to the quantum dispersion relation [Eq. (55)] shows an overall shift of eigenenergies by $2\mathcal{V}$. This is immaterial because it leads only to an overall phase factor. Hence, as was shown numerically in Ref. [5] for a particular set of initial conditions, the q -coupled classical oscillators describe to very good approximation the exciton dynamics on an N -monomer chain.

IV. CONCLUSIONS

We have shown how the eigenenergies and eigenfunctions of an N -level quantum system with real coupling matrix elements can be reproduced by a suitable array of coupled classical oscillators. Hence, by observing the classical motion, one can reconstruct the quantum mechanical time-dependent wave function or density matrix. The difficulty of realization of a classical system mimicking the quantum entanglement is mainly in the condition that a realistic system must reproduce *all* elements of the quantum couplings faithfully. This will be a task of increasing difficulty as the complexity of the quantum system increases. Nevertheless, we have shown explicitly that such a parallel is feasible for a circular or linear array of identical monomers. This is realized by a corresponding array of coupled pendula. Similarly, the excitation transfer on a molecular dimer, or similar coupled two-level quantum systems, can be faithfully simulated by the beating motion of a pair of classical oscillators, either mechanical or electrical. Such experiments are carried out routinely in undergraduate physics laboratories.

Realistically, the multi-oscillator couplings necessary to construct p -and- q -coupled oscillators could be attained more easily with miniature LC oscillators than with mechanical ones. However, we have shown also that the complexity of p -and- q -coupled oscillators can be circumvented by using only q -coupled oscillators in the RCA. This explains the success of previous comparisons of quantum time-dependent motion with that of physical q -coupled oscillator systems.

Some comments are in order regarding the extent to which the coherent quantum states discussed here are entangled or, correspondingly, to what extent the classical motion reproduces the ostensibly solely quantum attribute of entanglement.

We make a separation of the total Hilbert space, for example, in the simplest case $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and consider entanglement between the two subspaces. In our $N = 2$ example we take two monomers. Each monomer has a ground state that we denote by $|0\rangle$ and an excited state that we denote by $|1\rangle$; these two states constitute a qubit. The dimer states $|10\rangle$ and $|01\rangle$ are then identical to the states $|\pi_1\rangle$ and $|\pi_2\rangle$ of Eq. (38). The states $|\psi_{\pm}\rangle$ defined in this equation are two of the entangled ‘‘Bell’’ states of the two qubits [19]. These singly excited states are in one-to-one correspondence with the excited states of two classical oscillators. Note that in the subspace spanned by these one-exciton states, the concurrence, which in the quantum case is a measure of entanglement, is simply given by $\mathcal{C}(t) = 2|c_1^*c_2|$, where c_1 and c_2 are the time-dependent coefficients entering Eq. (41). Since the quantum coefficients c_1 and c_2 are in a one-to-one correspondence with the classical position q_1 and q_2 and momentum p_1 and p_2 [via Eq. (48)], the measure $\mathcal{C}(t)$ can be reproduced by the classical coefficients. However, the classical motions of the two pendula are correlated but not entangled. This can be seen, for example, by considering times after interaction between the subsystems (monomers and oscillators) has been turned off. The complex coefficients classically are properties of a single oscillator whereas quantumly they are joint products of the amplitudes of two monomers. Hence the classical motion only mimics the entangled quantum state to the extent that it simulates the time development of the complex amplitudes. What is also not reproduced classically is the full entanglement of the four-dimensional two-qubit space $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ from which the total of four entangled Bell states are formed (see, e.g., Ref. [19]). Since the full Hilbert space has dimension four, from the exact mapping, four oscillators are needed in general to mimic the full quantum dynamics.

Throughout we have neglected the effects of coupling of the quantum system or the classical oscillators to the environment. Of course, in real systems such decohering or dissipating couplings are omnipresent. In the quantum case these take the form of radiative or nonradiative decay of excited states or in eigenenergy shifts. In the classical case they take the form of frictional forces for pendula or the equivalent resistance in the electrical case. At the phenomenological level of simply assigning a complex energy to quantum levels or, equivalently, adding a term proportional to velocity to the classical equation of motion, it is easy to show that the quantum-classical equivalence is preserved (see Appendix C). We have undertaken a more detailed study of this equivalence in the framework of standard theories of open quantum systems [6]. Similarly, the correspondence between inducing radiative transitions in quantum systems and driving classical oscillators with external fields is under investigation.

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APPENDIX A: COMMENTARY ON REALISTIC-COUPPLING APPROXIMATION

In the foregoing we have derived the RCA by approximating the classical eigenmode frequencies. Further light can be shed on the nature of this approximation by looking at alternative derivations. A useful variation is to include explicitly the eigenvalue phases in the quantum time development of Eq. (28); that is, to write

$$\Psi(t) = \sum_n a_n(t) e^{-\frac{i}{\hbar} \epsilon_n t} |\pi_n\rangle. \quad (\text{A1})$$

Forming the equivalent classical Hamiltonian as before we obtain

$$\mathcal{H}(t) = \sum_{nm} H_{nm} a_n^\dagger(t) e^{i\omega_n t} a_m(t) e^{-i\omega_m t}, \quad (\text{A2})$$

where we have set $\omega_n = \epsilon_n/\hbar$. Note that, if one proceeds to second quantization and elevates the coefficients a_n to being annihilation operators, then this equation is just that of the quantum Hamiltonian in the Heisenberg representation and the time-dependent phase factors are just the time propagators of the creation and annihilation operators. In particular, for fixed n and m the off-diagonal coupling terms are of the form, for H Hermitian and real,

$$\mathcal{H}_{nm} = H_{nm} [a_n^\dagger a_m e^{i(\omega_n - \omega_m)t} + a_m^\dagger a_n e^{-i(\omega_n - \omega_m)t}]. \quad (\text{A3})$$

If one now sets

$$a_n e^{i\omega_n t} \equiv \frac{1}{\sqrt{2}} [q_n(t) + i p_n(t)], \quad (\text{A4})$$

corresponding to classical amplitudes $q_n(t)$ and $p_n(t)$ or equivalently Heisenberg operators, one obtains coupling elements

$$\mathcal{H}_{nm} = \frac{1}{2} H_{nm} (q_n q_m + p_n p_m), \quad (\text{A5})$$

which are those of Eq. (10). By contrast, if one restricts the coupling to terms $q_n q_m$ as in the q -coupled case, one has

$$\mathcal{H}_{nm} = H_{nm} [a_n^\dagger a_m e^{i(\omega_n - \omega_m)t} + a_m^\dagger a_n e^{-i(\omega_n - \omega_m)t} + a_n^\dagger a_m^\dagger e^{i(\omega_n + \omega_m)t} + a_n a_m e^{-i(\omega_n + \omega_m)t}]. \quad (\text{A6})$$

Then one sees that, to obtain the Hamiltonian of Eq. (A3), which maps exactly to the Schrödinger equation, it is necessary to neglect the ‘‘nonrotating-wave’’ terms involving the rapidly oscillating phase factors $e^{\pm i(\omega_n + \omega_m)t}$, which contribute weakly to transition probabilities. More exactly, let us take K to be the value of the largest of the elements H_{nm} (i.e., $H_{nm} = K \gamma_{nm}$ where all γ_{nm} are less than unity). Then we can define a dimensionless time as $\tau = Kt$. The rotating-wave factors are then of the order of $\exp[i(\Delta\omega/K)\tau]$, where $\Delta\omega$ is the mean frequency difference and the nonrotating-wave terms are of the order of $\exp[i(\omega/K)\tau]$, where ω is the mean frequency. Now if $\omega/K \gg 1$ and $\Delta\omega/K \approx 1$, which is the RCA, again we see that classical and quantum couplings will be the same in that the nonrotating-wave terms can be neglected.

A rather different view of the RCA is obtained from the Hamilton equations. The resulting q -coupled Newton

equations are

$$\ddot{q}_n + \omega_n^2 q_n = -\omega_n \sum_{m \neq n} H_{nm} q_m, \quad (\text{A7})$$

which are identical in form to Eq. (33) for standard classical oscillators. By contrast, for the p -and- q coupling one has

$$\ddot{q}_n + \omega_n^2 q_n = -\omega_n \sum_{m \neq n} H_{nm} q_m - \sum_{m \neq n} H_{nm} \sum_{m' \neq m} H_{mm'} q_{m'}, \quad (\text{A8})$$

which differ from the Newton equations by the last term. However, again scaling as above we have

$$\ddot{q}_n + \left(\frac{\omega_n}{K}\right)^2 q_n = -\frac{\omega_n}{K} \sum_{m \neq n} \gamma_{nm} q_m - \sum_{m \neq n} \gamma_{nm} \sum_{m' \neq m} \gamma_{mm'} q_{m'}. \quad (\text{A9})$$

Since in the RCA we have $\frac{\omega_n}{K} \gg 1$ we can neglect the second-order term so that the p -and- q -coupled Hamiltonian gives the same second-order classical equations of motion as the q -coupled Hamiltonian.

APPENDIX B: COUPLED ELECTRICAL OSCILLATORS

As above, we ignore for the moment friction of resistance and consider an LC oscillator circuit. For a single oscillator the balance of emf from inductance and capacitor is expressed by

$$-L \frac{dI}{dt} + \frac{q}{C} = 0, \quad (\text{B1})$$

where q is the charge and I is the current related by

$$I = -\frac{dq}{dt}. \quad (\text{B2})$$

Then we have the harmonic oscillator equation

$$L\ddot{q} + \frac{q}{C} = 0, \quad (\text{B3})$$

or

$$\ddot{q} + \omega^2 q = 0, \quad (\text{B4})$$

where the frequency is $\omega = \sqrt{1/(LC)}$. Note that, compared to a mechanical oscillator, L plays the role of mass, $1/C$ that of coupling constant, and $-LI$ corresponds to momentum.

Now we couple two identical LC oscillators by a capacitor C_K connected by leads in which current $J = dQ/dt$ flows. The equations of the coupled circuits are

$$\begin{aligned} -L \frac{d(I_1 - J)}{dt} + \frac{q_1}{C} &= 0, \\ -L \frac{d(I_2 + J)}{dt} + \frac{q_2}{C} &= 0, \end{aligned} \quad (\text{B5})$$

with the emf balance

$$\frac{q_1}{C} = \frac{q_2}{C} - \frac{Q}{C_K}. \quad (\text{B6})$$

Using $J = -dQ/dt$ one can eliminate J to obtain

$$\begin{aligned} (1 + K)\ddot{q}_1 + \omega^2 q_1 - K\ddot{q}_2 &= 0, \\ (1 + K)\ddot{q}_2 + \omega^2 q_2 - K\ddot{q}_1 &= 0, \end{aligned} \quad (\text{B7})$$

where the dimensionless ratio $K \equiv C_K/C$ is defined. These equations can be put also in the form of Eqs. (51):

$$\begin{aligned} (1 + 2K)\ddot{q}_1 + (1 + K)\omega^2 q_1 + K\omega^2 q_2 &= 0, \\ (1 + 2K)\ddot{q}_2 + (1 + K)\omega^2 q_2 + K\omega^2 q_1 &= 0. \end{aligned} \quad (\text{B8})$$

Clearly, when $K \rightarrow 0$ the equations become those of uncoupled oscillators. Adding and subtracting either of these two sets of equations leads to uncoupled equations in the new variables $q_{\pm} = q_1 \pm q_2$; that is,

$$\ddot{q}_+ + \Omega_+^2 q_+ = 0, \quad (\text{B9})$$

$$\ddot{q}_- + \Omega_-^2 q_- = 0, \quad (\text{B10})$$

which are identical to Eqs. (45) except that now $\Omega_+ = \omega = \sqrt{1/(LC)}$ and $\Omega_- = \omega/\sqrt{1 + 2K}$. For coupled LC oscillators the RCA corresponds to $K \ll 1$ to give

$$\Omega_- \sim \omega - K\omega. \quad (\text{B11})$$

Comparison with Eq. (54) shows that, in this case, we make the identification $2V = -K\omega$. Then the coupled electric oscillator equations giving $q_1(t)$ and $q_2(t)$ are identical to the q -coupled pendula equations and, as we have shown, in the RCA reproduce the complex amplitudes of the quantum dimer.

APPENDIX C: INCLUSION OF DAMPING

To include damping phenomenologically in the classical case we add a velocity-dependent term to the oscillator equations (33), ignoring the corresponding fluctuations; namely,

$$\ddot{x}_n + 2\Gamma_n \dot{x}_n + \omega_n^2 x_n = -\sum_m K_{nm} x_m. \quad (\text{C1})$$

Then the coupled equations (34) become

$$(\Omega_k^2 - \omega_n^2 + 2i\Omega_k \Gamma_n) C_{kn} = \sum_m K_{nm} C_{km}. \quad (\text{C2})$$

This equation is written in the form

$$\left(\Omega_k - \omega_n + 2i \frac{\Omega_k}{(\Omega_k + \omega_n)} \Gamma_n \right) C_{kn} = \frac{1}{(\Omega_k + \omega_n)} \sum_m K_{nm} C_{km}. \quad (\text{C3})$$

Again, in the RCA, we consider that the spread in eigenfrequencies Ω_k and ω_n is small compared to the mean natural frequency ω to approximate the above equations by

$$(\Omega_k - \omega_n + i\Gamma_n) C_{kn} = \frac{1}{2\omega} \sum_m K_{mn} C_{km}. \quad (\text{C4})$$

If we assign each quantum level a width γ_n then the quantum coupled equations become,

$$(E_k - \epsilon_n + i\gamma_n) B_{kn} = \sum_m \mathcal{V}_{nm} B_{km} \quad (\text{C5})$$

which are identical in form to the classical equations. In Ref. [6] it is shown how pure dephasing can be realized in a system of coupled oscillators which reproduces the quantum results.

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