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TRANSPORT PROCESSES

Theory of Heat Transport in a Magnetized High-Temperature Plasma

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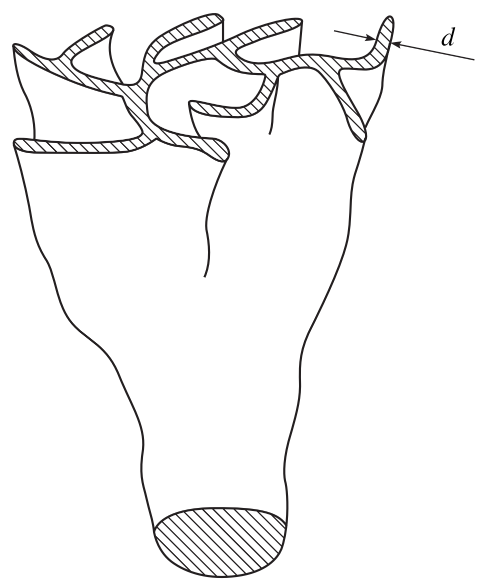
Abstract—The transport of charged particles across a strong magnetic field with a small random component is studied in the double diffusion approximation. It is shown that the density of the particles whose initial distribution is stretched along the field satisfies a subdiffusion equation with fractional derivatives. A more general initial particle distribution is also considered, and the applicability of the solutions obtained is discussed.

1. In this paper, we consider the diffusion of magnetized charged particles in a strong time-independent longitudinal magnetic field with a small random transverse component. Such a situation often occurs in systems in which there is a preferential direction of the magnetic field, e.g., in tokamaks, open magnetic traps, and other types of magnetic confinement systems. A stochastic magnetic field can be described in the simplest diffusion approximation [1–3]. Although this well-known approach has a long history and is widely used in studying the problems of heat transport in plasma, it is expedient to briefly outline its main aspects in order to provide a better insight into the phenomenon in question. The geometry of the problem is as follows. The magnetic field **B** points preferentially in the z direction. This indicates that the component of the magnetic field that is parallel to the z axis is much stronger than its random transverse component δB, $B_{\parallel} \gg \delta B$. An important point is that the magnetic field is nondivergent, $\nabla \cdot \mathbf{B} = 0$. A flux tube of such a field is shown in the figure, which is borrowed from the excellent review by Isichenko [4]. In a certain plane $z = z_0$, we choose a contour that encloses a bundle of magnetic field lines. In moving in the longitudinal (positive or negative) direction, we see that individual magnetic field lines move away from one another and the contour is deformed: it becomes more and more curved, but the area enclosed by it is conserved because of the conservation of the magnetic flux. As a result, the distance d between the walls of the magnetic flux tube decreases exponentially. When moving away from the $z = z_0$ plane, we see that the contour fills the perpendicular plane more and more uniformly. After averaging over the tube cross-sectional area, we can say that the averaged density b of the magnetic field lines decreases so as to satisfy the diffusion equation in which the role of time is played by the z coordinate (or, in a more general case, the superdiffusion equation in which the role of the Laplacian is played by its fractional power $\Delta_{\perp}^{\beta}$, with $\beta < 1$),

$$\frac{\partial b}{\partial z} = D_B \Delta_{\perp} b. \tag{1}$$

Here, D_B is the effective diffusion coefficient [5] and z is the absolute distance from the initial position of the contour. Let us briefly comment on this equation. In the problem as formulated, the mean magnetic field is generally the same over the entire space and has a certain constant strength B_0. However, if we wish to trace the behavior of the density of a bundle of magnetic field lines marked by the particles moving along them, then we see that, because of the magnetic field fluctuations,



Magnetic flux tube in a magnetic field with a random component.

this density behaves according to a diffusion law, as is implied by Eq. (1). In other words, by the quantity b satisfying Eq. (1), we mean the density of the marked magnetic field lines. As was said above, this approach implies that, before applying the averaging procedure, we must choose a certain contour enclosing magnetic field lines in order to trace its deformation and expansion in moving along the z axis. At this point, it is useful to mention other approaches to describing a stochastic magnetic field. The equation for an individual magnetic field line can be written as

$$\frac{d\mathbf{r}'}{dz} = \frac{\delta\mathbf{B}_\perp}{B_0}, \quad (2)$$

where $\mathbf{r}'$ is the coordinate in the plane perpendicular to the z axis. The corresponding averaging procedure, which inevitably involves certain assumptions about the behavior of the random magnetic field component $\delta\mathbf{B}_\perp$, reduces Eq. (2) to Eq. (1). The form of Eq. (2) clearly points to the analogy with the problem of a random two-dimensional incompressible flow with a time-dependent velocity field and with the problem of Hamiltonian chaos [6]. An analogous result on the diffusion of magnetic field lines can also be obtained in terms of quasilinear theory (see, e.g., [7]). The above behavior of the magnetic field may stem from different reasons, primarily from various plasma instabilities (see the papers cited above and also [8]).

We thus have determined how the magnetic field should be described in the model developed here. The next step is to describe the behavior of charged particles. It is well known that the squared ratio of the particle gyrofrequency to the collision frequency determines the ratio between the longitudinal and transverse transport coefficients in a magnetic field. We assume that the magnetic field is strong and, accordingly, that the particles are magnetized, $(\omega_B/\nu)^2 \propto D_{n\parallel}/D_{n\perp} \gg 1$. In the limit in which this ratio tends to infinity, the particles move exactly along the magnetic field lines and do not jump from one line to another (the questions of the transverse transport and about the applicability limit of this approximation will be discussed in more detail below). The density distribution of the particles along a magnetic field line is determined by collisions among them and by their collisions with other plasma particles. This distribution is also described by a diffusion equation,

$$\frac{\partial n_b}{\partial t} = D_n\frac{\partial^2 n_b}{\partial l^2}, \quad (3)$$

where l is the coordinate along the magnetic field line and the longitudinal diffusion coefficient D_n is assumed to be constant and to be the same for all particles (hereafter, we omit the subscript indicating that the diffusion is in the longitudinal direction). Since the magnetic field fluctuations are small, we can set $l \simeq z$.

Hence, we have described the model with which we will study the transport of charged particles in a stochastic magnetic field. All the simplifying assumptions of the model are well known and seem to be suitable for providing an adequate description of the physical picture of heat transport. Note again that, in the above simplified model, the magnetic field is time-independent and the magnetized particles take a random walk exactly along the magnetic field lines, without jumping from one line to another. Thus, if a randomly moving particle returns to its initial position, its transverse displacement is zero. The task now is to give a rigorous derivation of the equations that describe transverse diffusion in the problem as formulated.

2. Although the diffusion equations describing the magnetic field evolution and the evolution of the particles are simple and well studied, their simultaneous solution in the model under consideration is a nontrivial task. From the formal (mathematical) point of view, Eqs. (1) and (3) are not coupled to one another: Eq. (3) describes the particle transport as a function of time and the z coordinate, while Eq. (1) describes the expansion of the magnetic field lines in the transverse direction as a function of the same z coordinate. It is easy to see, however, that, when the stochastic behavior of a magnetic field line and the particle diffusion along it are taken into account simultaneously, the result is effective transport in the transverse direction (for simplicity, one can consider ballistic motion of the particles along the field; in this case, the transverse particle transport will occur in accordance with the diffusion of the magnetic field lines). Simple scaling estimates based on the set of Eqs. (1) and (3) yield the following self-similar relationship between the variables in the problem, or equivalently, the following new relationship between the spatial and the time scales: $\bar{r} \propto t^{1/4}$. The most likely candidates that possess this self-similarity property and that can be used to describe the diffusion of passive particles (i.e., the particles that have no effect on the medium) are, e.g., equations with the squared Laplacian on the right-hand side and equations with a time-dependent diffusion coefficient. However, equations of the first type represent an unphysical situation because their Green's function is not of fixed sign, whereas equations of the second type imply that the problem is spatially inhomogeneous. These preliminary considerations show that the answer should be sought in another class of equations. It should be emphasized that, in what follows, the desired equations will be rigorously derived based on a precise formulation of the problem.

In constructing the most general solution for the particle distribution step by step, we consider a simple problem in which the particles are initially localized at the point $(\mathbf{r}_0, \zeta)$, i.e., $n_0(\mathbf{r}) = n_0\delta(\mathbf{r}' - \mathbf{r}_0)\delta(z - \zeta)$. The radius vector $\mathbf{r}$ is defined by the pair $(\mathbf{r}', z)$, where $\mathbf{r}'$ is the position vector in the plane that is perpendicular to the z axis and passes through the point z. In this case, the solution is quite evident—it is simply a product of the solutions to Eqs. (1) and (3) with the initial conditions B_0 and n_0/B_0, respectively:

$$n(\mathbf{r}, t) = n_0 \frac{e^{-\frac{(z-\zeta)^2}{4D_n t}}}{\sqrt{4\pi D_n t}} \frac{e^{-\frac{(\mathbf{r}'-\mathbf{r}_0)^2}{4D_B|z-\zeta|}}}{4\pi D_B|z-\zeta|}. \quad (4)$$

The same solution can be obtained in a mathematically more precise way. For simplicity, we consider a planar problem in which the position vector $\mathbf{r}'$ has only one component, $\mathbf{r}' \longrightarrow x$ (with this simplification, we will arrive at the same final result but after far more illustrative manipulations). The probability density for the occurrence of a particle at a given point is equal to

$$\delta[\mathbf{r} - \mathbf{r}(t)] = \delta[\mathbf{r}' - \mathbf{r}'(t)]\delta[z - z(t)].$$

From the equation of particle motion (see Eq. (2)), we have

$$d\mathbf{r}/dt = V(t)\mathbf{h}, \quad \mathbf{h} = (\mathbf{B}_0 + \delta\mathbf{B}_\perp)/|\mathbf{B}_0 + \delta\mathbf{B}_\perp|,$$

$$z(t) = z_0 + \tilde{z}(t) = z_0 + \int_0^t V(t')dt'.$$

Here, $V(t)$ is a random Gaussian variable with a zero mean, which describes the collision-induced random variations in the velocity of a particle moving along a magnetic field line. The quantity $\tilde{z}(t)$ is the sum of independent random quantities and is therefore a random Gaussian quantity having the probability density function

$$\frac{1}{\sqrt{4\pi D_n t}} e^{-\tilde{z}^2/(4D_n t)}.$$

Analogously, we have

$$x(t) = x_0 + \tilde{x}(t) = x_0 + \int_0^t V(t') \frac{\delta B(z(t))}{B_0} dt'$$

$$= \int_0^{z(t)} \frac{\delta B(z')}{B_0} dz',$$

where $\tilde{x}(t)$, too, is a random Gaussian quantity with the probability density function

$$\frac{1}{\sqrt{4\pi D_B|z(t)|}} e^{-\tilde{x}^2/(4D_B|z(t)|)}.$$

Consequently, the sought-for probability density function is equal to

$$n(x, z, x_0, z_0, t) = \iint \delta(x - x_0 - x')\delta(z - z_0 - z')$$

$$\times \frac{1}{\sqrt{4\pi D_B|z'|}} e^{-x'^2/(4D_B|z'|)} \frac{1}{\sqrt{4\pi D_n t}} e^{-z'^2/(4D_n t)} dz' dx'.$$

Taking the integrals in this function yields formula (4).

The above mathematical procedure corresponds to a random time replacement or to the substitution of one random process for the argument (carrier) of another random process. The resulting random process will be nondiffusive, and, moreover, it will be non-Markovian. In what follows, it will be shown that the random-walk process just considered, namely, the one whose argument is also a random-walk process, obeys a subdiffusion scaling.

To do this, we consider an initial particle distribution that is uniform along the z axis. In experiments, such a distribution can be produced, e.g., with the help of a laser pulse.

It should be kept in mind, however, that, in this situation, particles with different coordinates ζ can occur at the same magnetic field line; in this case, the above initial condition in the form of a delta function for the equation describing the diffusion along this field line is incorrect. However, for a two-dimensional random-walk process, the probability that the trajectory will return to its initial point is zero. This question, which is important for achieving a realistic formulation of the problem of the transverse evolution of particles distributed initially around the z axis in such a manner that the transverse size of the distribution is much less than its longitudinal size, will be discussed below.

In the model formulation of the problem, the solution is obtained by integrating formula (4) over ζ and by setting $\mathbf{r}_0 = 0$:

$$n(\mathbf{r}', t) = 2\int_{-\infty}^{z} n_0 \frac{e^{-\frac{(z-\zeta)^2}{4D_n t}}}{\sqrt{4\pi D_n t}} \frac{e^{-\frac{\mathbf{r}'^2}{4D_B|z-\zeta|}}}{4\pi D_B|z-\zeta|} d\zeta$$
$$= \frac{1}{8\pi^2 D_B\sqrt{D_n t}} G_{03}^{30}\left(\frac{\mathbf{r}'^4}{256 D_B^2 D_n t}\bigg|_{0,\ 0,\ 1/2}\right). \quad (5)$$

The integral in formula (5) is expressed through the Meyer's G function, which is defined in terms of a fairly involved contour integral containing Euler's gamma function [9]. The solution obtained, however, can be investigated without reference to the asymptotic expressions of this complicated special function. Let us try to find out what equation function (5) would satisfy. Taking the Fourier transformation of function (5) in the variable $\mathbf{r}'$ and then the Laplace transformation of the resulting function in the time t, we can easily calculate the integral to obtain

$$n_{p,\mathbf{k}} = \frac{1}{\sqrt{pD_n}} \frac{n_0}{\sqrt{p/D_n} + D_B\mathbf{k}^2}. \quad (6)$$

Here, the subscripts p and $\mathbf{k}$ refer to the Laplace and Fourier components of the function and the quantities p and $\mathbf{k}$ themselves are the variables in the Laplace and Fourier representations, respectively. We transform expression (6) by multiplying it by the denominator of its right-hand side and by applying the inverse Fourier transformation in the coordinate:

$$\frac{n_p\sqrt{p}}{\sqrt{D_n}} = D_B\Delta_\perp n_p + \frac{n_0\delta(\mathbf{r}')}{\sqrt{D_n p}}.$$

To within a factor of $\sqrt{\pi}$, the expression on the left-hand side is nothing more than the fractional time derivative of order 1/2. Finally, in conventional coordinates, the equation satisfied by function (5) is rewritten as

$$\frac{\partial^{1/2} n}{\partial t^{1/2}} = D_B\sqrt{\pi D_n}\Delta_\perp n + \frac{n_0\delta(\mathbf{r}')}{\sqrt{t}}. \quad (7)$$

In order for Eq. (7) to be capable of describing particles that obey a uniform distribution in the longitudinal direction and a given distribution in a direction transverse to the magnetic field, it is sufficient to make the replacement $n_0 \longrightarrow n_0(\mathbf{r}')$. We thus have shown that the expansion of an initial particle distribution stretched along the z axis is described by a subdiffusion equation.

The possibility of using a fractional-derivative subdiffusion equation with an appropriate self-similarity property in analogous problems was pointed out by Balescu [10]. In that paper, the parameters of the equation were chosen to satisfy the dimensional estimates of the characteristic spatial and time scales of the problem and also to be consistent with the results that were obtained for the moments of the distribution function in other models. It was not, however, clearly formulated to what extent this equation is applicable to the physical problem. In a subsequent paper [11] (see also [12]), Vanden Eijnden and Balescu used a hybrid kinetic equation in order to derive an asymptotic expression for the particle density that was analogous to the expression obtained from an equation with fractional derivatives and that provided exactly the same behavior of the moments of the distribution function. An important advantage of [11] is that Vanden Eijnden and Balescu considered the possible mechanisms for collisional transverse transport; this problem, however, is more complicated and goes beyond the scope of the present paper (see [13]). In what follows, a simple and rigorous derivation of an effective transport equation that is valid on arbitrary time scales will be proposed that does not require any additional model assumptions. Moreover, solution (4) applies to any localized initial particle distribution. The solution method proposed here also helps to demonstrate a relationship with the model of random-walk processes in continuous time and to analyze memory effects that are exhibited by subdiffusion equations (see below) and are often fall out of consideration and thereby are not discussed in the literature. It should be noted that there are alternative approaches to solving the problem under consideration, e.g., the approach developed by Kota and Jokipii [14], who used the Kubo formalism, which is based on an analysis of the velocity correlation functions and yields a subdiffusion scaling, too.

At this point, it is also expedient to mention a paper by Zybin and Istomin [15], who studied particle transport in a random magnetic field and considered an analogous model called the “second-order diffusion” model. They asserted that, in such a formulation of the problem, the transverse transport is purely diffusive and the assumption of a subdiffusion scaling is erroneous. Note that the subdiffusion regime in the double diffusion model was proposed as early as 1962 by Getmantsev [16] (see also [17]). A more detailed discussion of the history of this issue, as well as of the relevant numerical and theoretical results, can be found in [4], in a recent review by Bakunin [18], and in a paper by Kota and Jokipii [14]. It is of interest to note that the true derivation of the subdiffusion scaling at the beginning of a paper by Zybin and Istomin [15] was subsequently declared invalid. The main error made by the authors of [15] in considering their model example was that they estimated the rate of diffusion of particles along the magnetic field lines by the particle characteristic velocity. However, this estimate in fact corresponds to switching from the original diffusive motion of the particles to their ballistic unidirectional motion and leads naturally to a diffusion scaling. The use of such an approach can stem from the assumption that the magnetic field is unsteady or from the assumption that the particles can jump from one magnetic field line to another due to collisions. Averaging over the ensemble of realizations of the magnetic field, i.e., switching from a fixed random magnetic field configuration to an averaged configuration, also can yield analogous results, which are erroneous in the model under consideration.

3. Subdiffusion equations have been known for a fairly long time: rich experience has been gained in using them, and their properties, solutions, and asymptotics have been examined in detail [19–21]. However, not all of the papers considering subdiffusion equations were based on a reasonable physical model (or even any model at all) from which they were derived. This is why it is necessary to mention interesting papers [22–24] (see also [25]). It is expedient to point out only the main properties of Eq. (7) (see [26]) because our purpose here is not to consider the general features of subdiffusion equations. The Green’s function for Eq. (7) is a self-similar function of the form (see formula (5))

$$G(r, t) = \frac{1}{t^{1/2}}\Phi\left(\frac{r'}{t^{1/4}}\right). \quad (8)$$

In our case, the self-similarity property, which, as a rule, greatly simplifies the analysis of the equations, is attracting in character. This means that any initial particle distribution will asymptotically evolve to the Green’s function profile. Recall also that the subdiffusion regime corresponds to a slower expansion of a particle cloud than in the case of conventional diffusion: the characteristic cloud width $\bar{r}$ increases according to the law

$$\bar{r}^2 \propto D_B(D_n t)^{1/2}. \quad (9)$$

Note that this law determines the applicability limit of the model proposed here: on long time scales, the evolution of a particle cloud is governed by the competition between subdiffusion expansion (9) and the slow transverse diffusion $\bar{r}^2 \propto D^*t$, which was ignored up to this point because of the assumption of a strongly magnetized plasma (see above). The determination of the effective transverse diffusion coefficient D^* for particles in a stochastic magnetic field is a fairly complicated task. Following [2, 3], this coefficient can be estimated by the formula $D^* = (\delta B/B)(D_{n\perp}D_n)^{1/2}$; in this case, we have $D_n(\delta B/B)^2 > D_{n\perp}$ (of course, other estimates can also be used, see, e.g., [27]). A comparison of the above expansion rates yields the following estimate for the applicability limit of the model developed here: $t \ll t^* = (D_B^2/D_{n\perp})(B/\delta B)^2$. We see that, for a strongly magnetized plasma and for small magnetic field fluctuations, this time can be fairly long.

All the features mentioned above could also be derived directly from formula (5). For an asymptotic analysis, however, the Laplace–Fourier transformation method makes this derivation somewhat more illustrative. Moreover, in rare cases only, the Green's function can be expressed in terms of tabulated special functions in conventional coordinate space, as in the above analysis.

An important property of subdiffusion equations is that they exhibit memory effects, which were analyzed in [28]. Equation (7) does not possess the property of continuous evolution. In other words, if we consider a state to which the system has evolved by a certain time as a new initial condition, then the continuity of the evolution is violated. Through a special choice of the initial condition, it is also possible to affect the initial stage of the process. For subdiffusion equations of form (7), these effects manifest themselves on macroscopic time scales. In [28], it was shown that, in order to provide an adequate description of the situation, it is necessary to take into account the dependence on the microscopic details of the transport process, as well as of the initial distribution. Presumably, the reason why, in our case, the continuity of the evolution is violated is associated with the averaging of the magnetic field over a small cross-sectional area in formulating the initial condition in the diffusion approximation. This averaging corresponds to a redistribution of the particles over the magnetic field lines in such a way that no two of them have different coordinates and occur at the same field line. In order to preserve the continuity property, it is necessary to take into account the distribution of the particles over the magnetic field lines and to solve the diffusion equations with an initial condition that remembers all information about the prehistory of the evolution. In so doing, however, it is necessary to know the behavior of each magnetic field line and, consequently, to solve exact dynamic equation (2), which is a separate and complicated task. This is why, in what follows, we will again use the above averaged description of the magnetic field but will give an estimate of the extent to which this approach is realistic.

4. In order to demonstrate the consequences of the memory effects, which, in the case at hand, stem from the characteristic features of the initial particle distribution over the magnetic field lines, we consider a problem in which the particles are initially distributed over a cylindrical region Ω of a certain radius and of comparable height. In this problem, we are dealing with two possible situations. In the first situation, the particles are distributed in such a way that no two of them occur at the same magnetic field line and have different coordinates. In this situation, the solution for the particle distribution is obtained by simply integrating formula (4) (the case when the region Ω lies in a plane perpendicular to the z axis also presents no problem because, in this case, each of the particles occurs at its own magnetic field line):

$$n(\mathbf{r}, t) = \int_{\Omega} n_0(\mathbf{r}_0, \zeta)\frac{e^{-\frac{(z-\zeta)^2}{4D_n t}}}{\sqrt{4\pi D_n t}}\frac{e^{-\frac{(\mathbf{r}'-\mathbf{r}_0)^2}{4D_B|z-\zeta|}}}{4\pi D_B|z-\zeta|}d\mathbf{r}_0 d\zeta. \quad (10)$$

In the second situation, the particles are distributed over a cylindrical region Ω in such a way that they obey a certain given distribution along each of the magnetic field lines that cross the cylinder. What are the consequences of such a distribution? Let us consider a magnetic field line such that its portion inside the cylinder coincides with the cylinder axis and has a length a (with such a symmetric model condition, the final result will not change qualitatively, because we will be interested in the behavior of the particles on spatial scales much greater than the dimensions of the region Ω). Let the coordinate origin be at the center of one of the bases of the cylinder. In this case, the particle density is calculated by the formula ($z > a$)

$$n(\mathbf{r}, t) = B(\mathbf{r}', z-a)\int_0^a n_0(\zeta)G(z-\zeta, t)d\zeta \simeq A\, B(\mathbf{r}', z-a)G(z-z_0, t), \quad (11)$$

where $G(z, t)$ and $B(\mathbf{r}', z)$ are the Green's functions of Eqs. (1) and (3), respectively; $A = \int_0^a n_0(\zeta)d\zeta$; and $Az_0 = \int_0^a \zeta n_0(\zeta)d\zeta$. If we use approximation (10) for the chosen magnetic field line, then we arrive at a different solution,

$$n'(\mathbf{r}, t) = \int_0^a n_0(\zeta)G(z-\zeta, t)B(\mathbf{r}', z-\zeta)d\zeta. \quad (12)$$

The main difference between density distributions (11) and (12) is as follows. In formula (11), the particle evolution is described by a single diffusion equation with the initial condition $n_0(z)$ and the diffusive random walk of a given magnetic field line comes into play at the point $z = a$. In formula (12), we are dealing with point sources producing particles that diffuse along their own magnetic field lines, which, in turn, are distributed diffusively and originate from given points. By subtracting one integral from another, we can estimate the accuracy of the averaged approximate formula (10). We subtract formula (11) from formula (12) and take into account the condition $z \gg a$ to obtain

$$\Delta n_1 \equiv n - n' \simeq \frac{\partial B}{\partial z}(\mathbf{r}', z)\int_0^a n_0(\zeta)G(z-\zeta, t)(a-\zeta)d\zeta \simeq A z_0' \frac{\partial B}{\partial z}(\mathbf{r}', z)G(z, t), \quad (13)$$

where $Az_0' = \int_0^a (a - \zeta)n_0(\zeta)d\zeta$ is a certain longitudinal dimension of the region Ω that is averaged over the initial distribution. We thus see that, because of the diffusive character of the function B on long spatial scales $z \gg a$, the discrepancy Δn_1 is small in comparison to the mean value n of the particle density on such spatial scales. This indicates that the particle transport is asymptotically described by subdiffusion equation (7).

Let us now turn to Eq. (7), because it is the equation for which we wish to analyze the influence of the memory effects. As the initial condition, we choose a general particle density distribution whose asymptotic behavior is described by Eq. (7). Let the region Ω be stretched along the z axis and let the characteristic longitudinal dimension of this region be much greater than the spatial scale on which we will follow the evolution of the particle density and which, in turn, substantially exceeds the transverse dimension of the region, $l_\parallel \gg r' \gg l_\perp$. The mean length a' of the portion of the magnetic field line that is inside the region can be estimated from the diffusion scaling: $a' = l_\perp^2/D_B$. We are thus faced with the situation that was considered above. The only difference is that, in formula (13), we must replace the length a with the estimate a' and integrate over the cross-sectional area S of the region,

$$\Delta n_2 \simeq \int_S \frac{\partial B}{\partial z}(\mathbf{r}' - \mathbf{r}_0, z)G(z, t)\int_0^{a'} n_0(\mathbf{r}_0, \zeta)(a' - \zeta)d\zeta d\mathbf{r}_0$$

$$\simeq \frac{\partial B}{\partial z}(\mathbf{r}', z)G(z, t)\int_S\int_0^{a'} n_0(\mathbf{r}_0, \zeta)(a' - \zeta)d\zeta d\mathbf{r}_0.$$

We denote by $A_0 = \text{const}$ the integral over the region on the right-hand side of this formula. For $n_0 = \text{const}$, the integral is approximately equal to $-Sa'^2 n_0/2 \sim -l_\perp^2 a'^2 n_0$, which will be used below for estimates. In order to calculate the accuracy of this integral estimate at the point $(\mathbf{r}', z)$, we must take the sum of the contributions from the regions of length on the order of a' and switch from the sum to an integral over ζ with a weighting function of $1/a'$,

$$\Delta n \simeq \frac{2A_0}{a'}\int_{-\infty}^{z} \frac{\partial B}{\partial z}(\mathbf{r}', z-\zeta)G(z-\zeta, t)d\zeta.$$

Since the magnetic field satisfies Eq. (1), we can replace the derivative with respect to the z coordinate with the transverse Laplacian operator. Taking into account that the Green's function for the equation describing the particle density is independent of $\mathbf{r}'$, we can factor it out of the integral sign. As a result, we arrive at the following final estimate of the order of smallness of the approximation accuracy (see formula (4)):

$$\Delta n \simeq \frac{2A_0 D_B}{a'}\Delta_\perp \int_{-\infty}^{z} B(\mathbf{r}', z-\zeta)G(z-\zeta, t)d\zeta \sim \frac{l_\perp^2}{2}\Delta_\perp n(\mathbf{r}', t) = O\left(\frac{1}{t}\right), \quad (14)$$

where the particle density $n(\mathbf{r}', t)$ satisfies subdiffusion equation (7) and is a self-similar function of form (8).

A more sophisticated problem is that in which the magnetic field lines cross the initial region many times. According to the theory of Brownian motion, the probability that the magnetic field line passes through the initial region in a finite time (a finite value of the z coordinate) is equal to unity. Since the diffusion equation automatically takes into account the contribution of such trajectories, it can be stated that the above diffusion approximation adequately describes the situation under analysis. Let us discuss this problem in more detail. We choose a certain point with the coordinates $(\mathbf{r}_1', z_1)$ in the region Ω. With a probability determined by diffusion, a certain number of the magnetic field lines that pass through the vicinity of this point also pass through the vicinity of a point $(\mathbf{r}_2', z_2)$ lying in the region Ω (for definiteness, we set $z_2 > z_1$). By virtue of the symmetry of the problem (see the comments on Eq. (1) and formula (4)), we can reverse the direction of motion[1] and choose a bundle of magnetic field lines passing through the vicinity of the point $(\mathbf{r}_2', z_2)$ to see that the same number of them should pass through the vicinity of the point $(\mathbf{r}_1', z_1)$. In other words, any two points in the initial region are connected by magnetic field lines whose density depends on the relative positions of the points. The contributions of these points to the density of the magnetic field lines at a certain spatial

[1] A similar effect underlies the mechanism of enhanced diffusion in a stochastic magnetic field [2, 3]: a particle that starts from a certain initial point will execute a random walk along a magnetic field line and then, because of the slow transverse diffusion, it will occur at a nearby magnetic field line and will move along it but in the opposite direction; as a result, the particle, on average, moves away over a long distance from the initial point.

point ($\mathbf{r}'_3$, z_3) should be calculated by summing them with the corresponding diffusive weighting functions $B(|\mathbf{r}'_1 - \mathbf{r}'_3|, |z_1 - z_3|)$ and $B(|\mathbf{r}'_2 - \mathbf{r}'_3|, |z_2 - z_3|)$. By doing this, we automatically and correctly take into account the trajectories that pass through all three of these points (it is obvious that such trajectories always exist). We thereby have shown that the above two approaches to calculating the particle density at a given point are completely equivalent to one another, specifically, the approach based on a correct total initial condition that is formulated for the common magnetic field lines and involves nontrivial density values $n_0(\mathbf{r}'_1, z_1)$ and $n_0(\mathbf{r}'_2, z_2)$, which should be obtained by calculating the probability for the trajectory to pass through three fixed points (or even through more points, if account is taken of the repeated returns of the trajectory), and the approach based on formula (4) with the same initial conditions on the particle distributions in the form of delta functions but with allowance for the diffusive expansion of the magnetic field lines. Consequently, for an initial particle distribution over a region of finite transverse dimension, the deviation of the evolution of the particle density in the initial stage from that predicted by Eq. (7) can be attributed to the particles moving along the portions of the magnetic field lines that are inside the initial region and have the mean length $a' = l_\perp^2/D_B$. By virtue of estimate (14), the contribution of these particles is small.

5. Thus, in considering the problem of the particle transport in a strong time-independent longitudinal magnetic field with a small random transverse component, a simple method for calculating the particle density has been proposed and the criteria for its applicability have been given. With this method, it has been demonstrated that the evolution of an initial particle distribution stretched along the magnetic field is described by a subdiffusion equation with fractional derivatives that has a self-similar solution consistent with the well-known scaling $\bar{r} \propto t^{1/4}$. It should be noted that the problem considered above constitutes one of the few examples of the rigorous derivation of an equation with fractional derivatives and thereby shows the naturalness and importance of this approach to describing stochastic processes in which the subdiffusive behavior of the particles is an inherent feature of the physical phenomenon.

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REFERENCES

1. T. Stix, Phys. Rev. Lett. **30**, 833 (1973).
2. B. B. Kadomtsev and O. P. Pogutse, in *Proceedings of the 7th International Conference on Plasma Physics and Controlled Nuclear Fusion Research, Innsbruck, 1979*, Vol. I, p. 949.
3. A. B. Rechester and M. N. Rosenbluth, Phys. Rev. Lett. **42**, 1247 (1979).
4. M. B. Isichenko, Rev. Mod. Phys. **64**, 961 (1992).
5. M. N. Rosenbluth, R. Z. Sagdeev, J. B. Taylor, and G. M. Zaslavski, Nucl. Fusion **6**, 297 (1966).
6. B. V. Chirikov, Phys. Rep. **52**, 263 (1979).
7. L. A. Artsimovich and R. Z. Sagdeev, *Plasma Physics for Physicists* (Atomizdat, Moscow, 1979) [in Russian].
8. A. A. Galeev and L. M. Zeleny, *Theoretical and Computational Plasma Physics* (IAEA, Vienna, 1978), p. 93.
9. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marychev, *Integrals and Series* (Nauka, Moscow, 1986; Gordon & Breach, New York, 1989), Vol. 3.
10. R. Balescu, Phys. Rev. E **51**, 4807 (1995).
11. E. Vanden Eijnden and R. Balescu, Phys. Plasmas **3**, 874 (1996).
12. R. Balescu, Plasma Phys. Controlled Fusion **42**, B1 (2000).
13. M. Vlad, F. Spineanu, J. H. Misguich, and R. Balescu, Phys. Rev. E **61**, 3023 (2000).
14. J. Kota and J. R. Jokipii, Astrophys. J. **531**, 1067 (2000).
15. K. P. Zybin and Ya. N. Istomin, Zh. Éksp. Teor. Fiz. **89**, 836 (1985) [Sov. Phys. JETP **62**, 479 (1985)].
16. G. G. Getmantsev, Astron. Zh. **39**, 607 (1962) [Sov. Astron. **6**, 477 (1962)].
17. J. R. Jokipii, Astrophys. J. **146**, 480 (1966).
18. O. G. Bakuninm, Rep. Prog. Phys. **67**, 965 (2004).
19. E. W. Montroll and M. F. Schlesinger, in *Studies in Statistical Mechanics*, Ed. by J. Leibowitz and E. W. Montroll (North-Holland, Amsterdam, 1984), Vol. 2, p. 1.
20. J.-P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
21. J. Klafter and R. Metzler, Phys. Rep. **339**, 1 (2000).
22. S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987).
23. L. A. Bol'shov, A. M. Dykhne, and P. S. Kondratenko, Pis'ma Zh. Éksp. Teor. Fiz. **75**, 291 (2002) [JETP Lett. **75**, 246 (2002)].
24. I. A. Lubashevskiĭ and A. A. Zemlyanov, Zh. Éksp. Teor. Fiz. **114**, 1284 (1998) [JETP **87**, 700 (1998)].
25. K. V. Chukbar, Zh. Éksp. Teor. Fiz. **109**, 1335 (1996) [JETP **82**, 719 (1996)].
26. K. V. Chukbar, Zh. Éksp. Teor. Fiz. **108**, 1875 (1995) [JETP **81**, 1025 (1995)].
27. A. A. Galeev and L. M. Zeleniy, Physica D (Amsterdam) **2**, 90 (1981).
28. V. Yu. Zaburdaev and K. V. Chukbar, Pis'ma Zh. Éksp. Teor. Fiz. **77**, 654 (2003) [JETP Lett. **77**, 551 (2003)].

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