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Stochastic Transport through Complex Comb Structures

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Abstract—A unified rigorous approach is used to derive fractional differential equations describing subdiffusive transport through comb structures of various geometrical complexity. A general nontrivial effect of the initial particle distribution on the subsequent evolution is exposed. Solutions having qualitative features of practical importance are given for joined structures with widely different fractional exponents.

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1. INTRODUCTION

In the past years, considerable attention has been given to anomalous stochastic transport described by fractional differential equations. Mathematical and physical aspects of this phenomenon have been discussed in numerous review papers (e.g., see [1–5]) and original publications. A very convenient and widely used proving ground for analyzing characteristics and laws of fractional transport are comb structures, which provide simple and graphic explanations for deviations from classical diffusion and are amenable to theoretical treatment. They were among the first physical systems for which transport equations were rigorously derived rather than inferred from scaling laws for the mean square displacement $\langle x^2\rangle \propto t^\alpha$, where $\alpha \neq 1$ (see the pioneering study in [6] and derivation of a scaling law with $\alpha = 1/2$ in [7]). Furthermore, permanent interest in these systems (e.g., see [8–10]) is stimulated by their importance in physics of heterogeneous materials with inclusions of arbitrary geometry. The purpose of this paper is twofold. First, several generalized comb structures are introduced in order to diversify their observed properties. Second, they are used as illustrative examples for analyzing important problems involving mutual influence between fractional and classical diffusion. As a starting point, we use a comb structure to expose one rarely discussed qualitative aspect of fractional differential models used in physical applications, which was pointed out in [11].

2. STANDARD COMB STRUCTURE

We present a rigorous and systematic derivation of an effective transport equation for the system shown in Fig. 1. The simplest comb structure consists of an infinite cylinder of cross sectional area S_1 (backbone) centered on the x axis and an array of infinite cylinders (branches) of cross sectional area S_0 connected thereto with a spacing of l between them. Particle transport along each element of the structure is characterized by diffusion coefficient D; i.e., unusual behavior of the overall transport is entirely due to geometry rather than to microscopic particle dynamics.

The particle concentrations on the backbone and the branch emanating from a point x are denoted by $n_1(x, t)$ and $n_0(x, y, t)$. Since we analyze macroscopic behavior of particles, particle concentration is hereinafter interpreted as the concentration along the horizontal axis averaged over scales much larger than l. We also assume that the distribution of particle concentration is sufficiently smooth that its variations across the cylinders and along any particular element of the structure are negligible. This assumption is obviously valid at sufficiently long evolution times, when $Dt \gg S_1, S_0, l^2$. We write equations for the particle concentration defined as the total number of particles per unit length

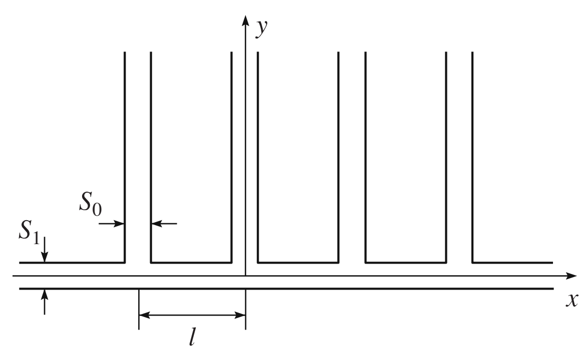

Fig. 1. Standard comb structure.

of the backbone,

$$N(x) = n_1(x) + \frac{1}{l}\frac{S_0}{S_1}\int_0^{\infty} n_0(x, y)dy. \quad (1)$$

This does not make our derivation essentially different from that given in [6], where evolution of n_1 was analyzed. However, we believe that the use of N is better suited for analyzing the effective transport along the x axis.

To describe the variation of particle concentration on the branches, we write the classical diffusion equation

$$\frac{\partial n_0}{\partial t} = D\frac{\partial^2 n_0}{\partial y^2}, \quad (2)$$
$$n_0|_{t=0} = n_{00}(y), \quad n_0|_{y=0} = n_1(x, t),$$

where $n_{00}(y)$ is the initial concentration on a branch, which is essential for modifying the formulas of [6] so as to expose the qualitative aspect mentioned in the Introduction. Note that the role of a boundary condition for transport in a branch is played by the corresponding concentration on the backbone. To find a solution, we consider the Laplace transform of Eq. (2),

$$pn_{0p} = D\frac{d^2 n_{0p}}{dy^2} + n_{00}, \quad n_{0p}|_{y=0} = n_{1p}(x), \quad (3)$$

which yields

$$n_{0p}(y) = n_{1p}\exp\left(-\sqrt{\frac{p}{D}}y\right) + \int_0^{\infty} G(y, y')n_{00}(y')dy', \quad (4)$$

where the Green's function of Eq. (3) is

$$G(y, y') = \frac{1}{\sqrt{pD}} \times \begin{cases} \exp\left(-\sqrt{\frac{p}{D}}y'\right)\sinh\left(\sqrt{\frac{p}{D}}y\right), & y < y' \\ \exp\left(-\sqrt{\frac{p}{D}}y\right)\sinh\left(\sqrt{\frac{p}{D}}y'\right), & y > y'. \end{cases}$$

Substituting (4) into (1), we obtain an expression for the total concentration required for further analysis:

$$N_p = n_{1p}\left(1 + \frac{S_0}{S_1}\frac{1}{\sqrt{p\tau}}\right) + \frac{S_0}{S_1}\frac{1}{l} \times \int_0^{\infty} \frac{1 - \exp(-\sqrt{p\tau}y/l)}{p} n_{00}(y)dy, \quad \tau = \frac{l^2}{D}. \quad (5)$$

Since the evolution of the total particle concentration along x is completely determined by diffusion on the backbone, we have the time-domain and Laplace-domain equations

$$\frac{\partial N}{\partial t} = D\frac{\partial^2 n_1}{\partial x^2} \longrightarrow pN_p = D\frac{d^2 n_{1p}}{dx^2} + N_0. \quad (6)$$

Of particular interest is the long-time behavior at $t \gg \tau S_1/S_0$, when transport is strongly modified by particle diffusion into the branches. In the Laplace representation, we have the dual condition $p\tau S_1/S_0 \ll 1$. By using the inverted relation (5) between N and n_1, evolution equation (6) in this limit is rewritten as

$$\sqrt{p\tau}N_p = l^2\frac{S_1}{S_0}\frac{d^2 N_p}{dx^2} + N_0\sqrt{\frac{\tau}{p}} - l\int_0^{\infty} \frac{1 - \exp(-\sqrt{p\tau}y/l)}{p}\frac{d^2 n_{00}(x, y)}{dx^2}dy. \quad (7)$$

According to the standard interpretation of the fractional power of the Laplace variable, this is equivalent to the time-domain equation

$$\sqrt{\tau}\frac{\partial^{1/2} N}{\partial t^{1/2}} = l^2\frac{S_1}{S_0}\frac{\partial^2 N}{\partial x^2} + \frac{N_0}{\sqrt{\pi}}\sqrt{\frac{\tau}{t}} - l\int_0^{\infty}\left[1 - \mathrm{erfc}\left(\frac{y}{2l}\sqrt{\frac{\tau}{t}}\right)\right]\frac{\partial^2 n_{00}(x, y)}{\partial x^2}dy \quad (8)$$

($\mathrm{erfc}x = (2/\sqrt{\pi})\int_x^{\infty}\exp(-t^2)dt$). This equation agrees with that derived in [6] except for the terms containing the initial concentrations. The difference between the coefficient of N_0 and the factor that multiplies n_{10} in the equation derived for n_1 is insignificant, being a mere consequence of the scaling law $N_p \propto n_{1p}/\sqrt{p}$ in (5). However, the complicated integral term containing n_{00} (generally ignored in previous studies) is of primary importance, as mentioned above.

This term is responsible for an extremely long-lasting effect of the particles that have diffused into the branches on the system's evolution. The particles that have diffused sufficiently far away from the backbone can contribute to active transport after an arbitrarily long delay. Under an appropriate choice of $n_{00}(y)$, this contribution can modify the evolution of N in an almost arbitrary manner. Fully fractional transport through a comb structure, characterized by a scaling exponent $\alpha = 1/2$ (see below), can develop only when the initial concentration on the branches is negligible. Otherwise, macroscopic transport cannot be described by any equation written in terms of N only. Furthermore, an appropriate treatment of the initial distribution eliminates the physically questionable "intransitivity" of evolution equations with fractional time derivatives that makes it impossible to use a profile $N(x)$ that has already evolved over time as an initial condition for Eq. (7) with $n_{00} = 0$ (routinely discussed in the literature), because its further evolution will be distorted. Our systematic derivation makes it clear that diffusion into initially empty parts of a comb structure must be taken into account. Similar argumentation regarding a different physical model was given in [11], where the onset of time-domain fractional transport was attributed to microscopic particle dynamics. The new treatment exposes its universal origin (the lack of a macroscopic equation) and concomitant circumstances (the possibility of arbitrary modification of macroscopic evolution).

Having pointed out a fact of importance for transport in geometrically complex structures, we will not expand on it any further. Hereinafter, the focus is on conventional description of fractional transport; i.e., we assume that $n_{00} \equiv 0$. Under this assumption, Eqs. (7) and (8) reduce to the standard equation with time derivative of order 1/2 for a subdiffusive stochastic process, which implies a scaling law with $\alpha = 1/2$.

3. RAMIFIED COMB STRUCTURE

We now consider possible generalizations of comb geometry. First, a hierarchical comb structure with ramifying branches can be constructed. Replacing each branch with the structure discussed above and repeating this step k times (see Fig. 2), we obtain the kth-order ramified structure whose branches are structures of order $k - 1$ (the simple comb structure is of order $k = 1$). To simplify calculations, we assume throughout this section that all backbones and branches have equal cross-sectional areas ($S_0 = S_1$ in the formulas above).

In the ith-order structure ($0 < i < k$), the total particle concentration is determined by the concentrations on its backbone and branches (cf. expression (1)):

$$N_i = n_i + \frac{1}{l}\int_0^{\infty} N_{i-1}(x_{i-1})dx_{i-1},$$

where x_i is the coordinate along corresponding backbone axis. Accordingly, its evolution is governed by the equation (cf. Eq. (6))

$$p\tau N_{ip} = l^2\frac{d^2 n_{ip}}{dx_i^2}, \quad n_{ip}|_{x_i = 0} = n_{i+1,\,p}.$$

At $t = 0$, the particles are confined to the backbone of the kth-order structure. If $N_{ip} = f_i(p\tau)n_{ip}$, then the formulas above yield the simple recursion relation

$$f_i = 1 + \sqrt{\frac{f_{i-1}}{p\tau}}.$$

Using the obvious fact that $f_0 = 1$ (the lowest order branches do not ramify) and taking the asymptotic limit of $p\tau \longrightarrow 0$ (requiring that particles have penetrated the

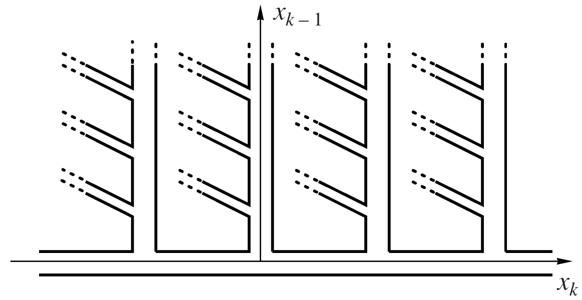

Fig. 2. Ramified comb structure.

structure to the lowest order branches), we obtain $f_k = (p\tau)^{\alpha_k - 1}$ with $\alpha_k = 1/2^k$. This leads to the desired equation of particle transport in a ramified structure:

$$\begin{aligned} (p\tau)^{\alpha_k}N_p &= l^2\frac{d^2N_p}{dx^2} + \frac{N_0}{p}(p\tau)^{\alpha_k} \\ \longrightarrow \tau^{\alpha_k}\frac{\partial^{\alpha_k}N}{\partial t^{\alpha_k}} &= l^2\frac{\partial^2 N}{\partial x^2} + \frac{N_0}{\Gamma(1-\alpha_k)}\left(\frac{\tau}{t}\right)^{\alpha_k}, \end{aligned} \tag{9}$$

where the subscript k of N is dropped and subscript 0 refers to the initial condition.

Thus, particle transport in a hierarchically ramified comb structure is also a subdiffusive process governed by a fractional differential equation with exponent $\alpha = 1/2^k$ progressively decreasing with increasing order of ramification. In the ramified structure formally iterated ad infinitum, there is no transport along the backbone and the initial distribution is weakly modified (over the time required for the system to evolve into a fully developed regime). Accordingly, Eq. (9) becomes

$$\begin{aligned} N_p - \frac{N_0}{p} &= l^2\frac{d^2N_p}{dx^2} \longrightarrow N(x,t) - N_0(x) \\ &= l^2\frac{\partial^2 N(x,t)}{\partial x^2}, \end{aligned} \tag{10}$$

where the right-hand side is small by virtue of assumptions made above. The obvious reason is that the particle concentration on the "active" part (the backbone) tends to zero as all particles diffuse into the branches. Since similar behavior is exhibited by different generalizations considered below, we note that the result obtained here corresponds to a finite velocity of absorption of particles with concentration n into the branches. Indeed, if Eq. (2) for the structure in Fig. 1 is replaced with

$$\frac{\partial n_0}{\partial t} + v\frac{\partial n_0}{\partial y} = 0, \quad n_0(t)|_{y=0} = n_1(t),$$

while diffusion along the backbone is retained, then Eq. (8) is rewritten as

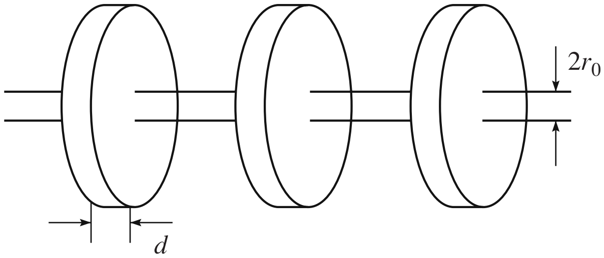

Fig. 3. Disk garland.

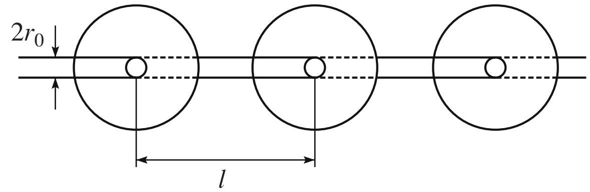

Fig. 4. Ball garland.

$$\left(p+\frac{v}{l}\right)N_p = D\frac{d^2N_p}{dx^2}+N_0\left(1+\frac{v}{lp}\right),$$

which reduces to (10) as $p \longrightarrow 0$ and $D/v \longrightarrow l$.

4. GARLANDS

We can also consider structures where the backbone is furnished with multidimensional objects, such as disks or balls making up garlands similar to those used to decorate the Christmas tree. Since "branches" of this kind absorb particles more easily, the evolution of N along the axis should be slower than in finite hierarchies. These unusually shaped "branches" can serve as models of real (e.g., fractal) structures that have sufficient capacity to absorb particles, as suggested by the limit case considered at the end of the previous section.

We begin with the structure where disks of infinite radius and thickness d are "spitted" on a backbone of radius r_0 (see Fig. 3). As in the classical case, diffusion with coefficient D is assumed to take place both along the backbone and in the disks. Then, analysis reduces to technically simple modifications of the formulas obtained for the standard comb structure: Eq. (1) is replaced by

$$N = n_1+\frac{2d}{lr_0^2}\int_{r_0}^{\infty}n_0(r)rdr; \tag{11}$$

Eq. (3), by

$$pn_{0p} = D\frac{1}{r}\frac{d}{dr}\left(r\frac{dn_{0p}}{dr}\right), \quad n_{0p}|_{r=r_0} = n_{1p} \tag{12}$$

whose solution is (cf. expression (4))

$$n_{0p}(r) = n_{1p}\frac{K_0(\sqrt{p/D}r)}{K_0(\sqrt{p/D}r_0)}, \tag{13}$$

where K_0 is the MacDonald function; and Eq. (6) holds. As a result, relation (5) becomes

$$N_p = n_{1p}\left[1+\frac{2d}{r_0}\frac{1}{\sqrt{p\tau}}\frac{K_1(\sqrt{p\tau}r_0/l)}{K_0(\sqrt{p\tau}r_0/l)}\right], \tag{14}$$

leading to the following evolution equation in the asymptotic limit of $p\tau\ln(p\tau) \ll dl/r_0^2$:

$$-\frac{N_p}{\ln p} = \frac{r_0^2l}{2d}\frac{d^2N_p}{dx^2}-\frac{N_0}{p\ln p}. \tag{15}$$

Note that it does not contain the diffusion coefficient.

The operator $1/\ln p$ in Eq. (15) can be interpreted as the logarithm of the time derivative. It should be used with care because, unlike p^α, it changes sign at $p = 1$, implying an unphysical high-frequency instability in (15). Even though the trivial regularization $-\ln p \longrightarrow \ln(1 + 1/p)$ [12] does eliminate the problem (recall that long-time regime of slow diffusion is of actual interest), the result is hardly tractable since this function does not have an analytical Laplace inverse. Note also that the model at hand admits the subtle use of an asymptotic limit only for $K_1(\xi) \approx 1/\xi$ when $\xi \ll 1$, whereas the function $K_0(\xi)$ associated with $\ln\xi$ is preserved (it has the correct sign on the entire half-line $(0, \infty)$). The resulting time-domain equation is

$$\begin{aligned}N(x,t) &= N_0(x)+\frac{r_0^2l}{4d}\frac{\partial^2}{\partial x^2}\\ &\times\int_0^t\frac{N(x,t-t')}{t'}\exp\left(-\frac{r_0^2}{4Dt'}\right)dt'.\end{aligned} \tag{16}$$

Formally, Eq. (16) corresponds to the zeroth order of the fractional derivative, but the fact that it differs from Eq. (10) implies nontrivial behavior in the limit of $\alpha \longrightarrow 0$. This behavior is intermediate in the sense that the concentration continues to evolve (in contrast to the case of ball garland discussed below), but the evolution is infinitely slower than predicted by any power law.

The transport of particles with concentration N in a ball garland (see Fig. 4) is described by the equations

$$N = n_1+\frac{4}{lr_0^2}\int_{r_0}^{\infty}n_0(r)r^2dr, \tag{17}$$

$$pn_{0p} = D\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dn_{0p}}{dr}\right), \quad n_{0p}|_{r=r_0} = n_{1p} \tag{18}$$

and the ubiquitous Eq. (6). Equation (18) yields

$$n_{0p}(r) = n_{1p}\frac{r_0}{r}\exp\left[-\sqrt{\frac{p}{D}}(r-r_0)\right]. \tag{19}$$

Therefore, Eq. (17) can be rewritten as

$$N_p = n_{1p}\left[1+4\left(\frac{l}{r_0}\frac{1}{p\tau}+\frac{1}{\sqrt{p\tau}}\right)\right]. \tag{20}$$

Substituting this expression into (6) and taking the limit as $p \longrightarrow 0$, we obtain the time-independent (degenerate) evolution equation

$$\begin{aligned} N_p - \frac{N_0}{p} &= \frac{r_0 l}{4}\frac{d^2 N_p}{dx^2} \longrightarrow N(x,t) - N_0(x) \\ &= \frac{r_0 l}{4}\frac{\partial^2 N(x,t)}{\partial x^2}. \end{aligned} \tag{21}$$

Again, it does not contain D (cf. Eq. (15)), being fully equivalent to (10). This result, valid at $t \gg \tau$, is obviously explained by the fact that the concentration flux corresponding to three-dimensional diffusion under a finite boundary condition is finite rather than monotonically decreasing (as in fewer dimensions), which leads to an exponential decrease in particle concentration on the backbone in this model. Since the balls act as powerful "pumps," steady diffusive influx into the balls requires substantial gradients in n_1 between the ball inlet points and the midpoints between balls. This requirement is inconsistent with the assumed slow variation of $n_1(x)$: the actual boundary condition for the diffusive influx is a quantity proportional to, but lower than, $\langle n_1 \rangle$. Similar effects should obviously be observed in the models discussed at the end of the previous section. An analysis of diffusion between a branch segment of length l and a ball connected thereto shows that the corresponding coefficient of proportionality is $r_0/2l$ if $r_0 \ll l$.

To wrap up this section, we discuss yet another issue. The linear array of densely packed balls would seem structureless since particles can diffuse in any direction. Yet there is diffusive transport along the x axis even when backbone is embedded in this seemingly structureless medium (e.g., see [13]). The paradox is resolved by noting that, since the backbone is connected to the medium only via widely spaced small segments and direct particle transfer between balls is impossible, a particle diffusing from a ball into its neighbor along the x axis must find its way to the point where the former ball is connected to the backbone.

5. TYPICAL CONTACT PROBLEMS

In practice, one frequently has to deal with situations where media having different transport properties are in direct contact, as in the case of diffusive mass or heat transfer through a heterogeneous material consisting of regions with different D. This motivates theoretical analysis of effects of local inclusions on global transport. Whereas simple problems of classical diffusion have long since been solved and included in textbooks on mathematical physics, composite problems of greater complexity are still the subject of ongoing studies [13]. However, analysis of the simplest problems in fractional transport is far from complete, even though it is a popular research area. We believe that the tractability of transport on comb structures makes them a good proving ground for this line of research.

In this paper, we consider several problems of this type that are analogous to classical diffusion ones. The difference in transport properties lies in the scaling exponent (the order of the fractional time derivative) rather than in the second-order spatial differential term (as in classical diffusion). In all generalizations discussed above, the exponents are lower than 1/2. However, it is well known that structures where individual branches have different lengths and the average branch is infinitely long may serve as models with larger values of α, depending on the order of divergence of the average length [8]. Analogous modifications of complex structures and garlands will lead to similar results. Now, suppose that two comb structures (hereinafter referred to as α and β according to the corresponding scaling exponents), are joined along their backbones; i.e., different branches "grow" from the same backbone. In what follows, we analyze their mutual influence, using transport equations written in dimensionless form for simplicity.

5.1. Interaction between Two Finite Structures

Suppose that structures α and β of equal length L are joined at $x = 0$ and the initial particle concentration on their common backbone is constant. The respective evolution equations are

$$\begin{aligned} p^{\alpha}N_p &= \frac{d^2N_p}{dx^2} + N_0 p^{\alpha-1}, \quad x \in [-L, 0), \\ p^{\beta}N_p &= \frac{d^2N_p}{dx^2} + N_0 p^{\beta-1}, \quad x \in (0, L]. \end{aligned} \tag{22}$$

The boundary conditions are written in terms of n_1, being determined by the following physical requirements: both concentrations are equal on the backbone, and so are the corresponding fluxes, while the ends of the heterogeneous structure are impermeable (cf. classical diffusion):

$$\begin{aligned} n_1\big|_{x=-0} = n_1\big|_{x=+0}, \quad \frac{\partial n_1}{\partial x}\bigg|_{x=-0} &= \frac{\partial n_1}{\partial x}\bigg|_{x=+0}, \\ \frac{\partial n_1}{\partial x}\bigg|_{x=-L} = \frac{\partial n_1}{\partial x}\bigg|_{x=L} &= 0. \end{aligned} \tag{23}$$

It follows from the discussion above that the relation between N and n_1 is nonlocal in time and differs between the regions $x < 0$ and $x > 0$: $N_p = p^{\alpha-1}n_{1p}$ and

$N_p = p^{\beta-1}n_{1p}$, respectively. In this dimensionless form, the solution to problem (22), (23) is

$$N_p(x) = \frac{N_0}{p}\begin{cases}\{1 + A\cosh[p^{\alpha/2}(x+L)]\}, & x \in [-L, 0)\\ \{1 + B\cosh[p^{\beta/2}(x-L)]\}, & x \in (0, L],\end{cases}$$

$$\left.\begin{matrix}A\\B\end{matrix}\right\}$$

$$= \frac{b^2 - a^2}{(a+b)\sinh(a+b) - (b-a)\sinh(b-a)}\begin{cases}-\dfrac{\sinh b}{b}\\ \dfrac{\sinh a}{a},\end{cases}$$

where $a = p^{\alpha/2}L$ and $b = p^{\beta/2}L$. For example, it can be used to calculate the asymptotic regime in which all particles are absorbed by the more ramified structure β (if $\beta < \alpha$): as $t \longrightarrow \infty$, performing the inverse Laplace transform for $x < 0$ yields

$$\frac{N}{N_0} \propto \frac{t^{-\gamma}}{\Gamma(1-\gamma)}, \quad \gamma = \frac{\alpha-\beta}{2}. \tag{24}$$

5.2. Effect of a Finite "Inclusion"

If an initially empty system is to be "soaked" with particles or heat by injection of a flux q into the backbone, then a very important characteristic is the degree of "screening" of inner regions by more ramified inclusions. Suppose that the structure β (the region $x > L$) is separated from the injection point $x = 0$ by a layer of structure α. Then, evolution of n_1 is conveniently described by the equations

$$\begin{aligned} p^{\alpha}n_{1p} &= \frac{d^2 n_{1p}}{dx^2}, \quad x \in [0, L),\\ p^{\beta}n_{1p} &= \frac{d^2 n_{1p}}{dx^2}, \quad x \in (L, \infty)\end{aligned} \tag{25}$$

subject to the boundary conditions

$$\begin{aligned} \left.\frac{\partial n_1}{\partial x}\right|_{x=0} &= -q\left(\left.\frac{dn_{1p}}{dx}\right|_{x=0} = -\frac{q}{p}\right),\\ n_{1p}|_{x=L-0} &= n_{1p}|_{x=L+0},\\ \left.\frac{\partial n_1}{\partial x}\right|_{x=L-0} &= \left.\frac{\partial n_1}{\partial x}\right|_{x=L+0}.\end{aligned} \tag{26}$$

The solution is

$$\begin{aligned} n_{1p}(x) &= \frac{q}{p}\frac{1}{p^{\alpha/2}}\\ &\times\begin{cases}A\exp a\cosh(p^{\alpha/2}x) + \exp(-p^{\alpha/2}x), & x \in [0, L)\\ B\exp(b-a)\exp(-p^{\beta/2}x), & x \in (L, \infty),\end{cases}\end{aligned} \tag{27}$$

where a and b are defined above and the coefficients are

$$A = \frac{p^{\alpha/2} - p^{\beta/2}}{p^{\alpha/2}\sinh a + p^{\beta/2}\cosh a},$$

$$B = \frac{p^{\alpha/2}\cosh a - p^{\beta/2}\sinh a}{p^{\alpha/2}\sinh a + p^{\beta/2}\cosh a}.$$

It follows that the flux injected into the structure β asymptotically tends to behave as

$$Q_p = -\left.\frac{dn_{1p}}{dx}\right|_{x=L} \sim \frac{q}{p}\frac{p^{\beta/2}}{p^{\beta/2} + p^{\alpha}L}.$$

Performing the inverse Laplace transform and taking the limit of $t \longrightarrow \infty$, we obtain

$$Q(t) \sim q\begin{cases}1 - \dfrac{t^{-\gamma}L}{\Gamma(1-\gamma)}, & \gamma = \alpha - \dfrac{\beta}{2} > 0\\ \dfrac{t^{\gamma}}{\Gamma(1+\gamma)L}, & \gamma < 0.\end{cases} \tag{28}$$

Thus, the flux Q transported through a relatively weakly ramified structure with $\alpha > \beta/2$ approaches the injected flux q with time elapsed. When the opposite inequality holds, particles are trapped in the more ramified structure α, and further transport vanishes: $Q \longrightarrow 0$ as $t \longrightarrow \infty$. This result can be interpreted as follows: if every branch of the structure characterized by the exponent α is more ramified than the structure with exponent β connected to the backbone, then all particles end up in these branches.

In the intermediate case of $\alpha = \beta/2$, the structure on the right of the backbone can be viewed as just an additional branch of the structure on the left. It is obvious that the corresponding asymptotic distribution of the flux q between all equivalent branches is uniform:

$$Q(t) \sim \frac{q}{L+1}(1 - \mathrm{const}\,t^{-\alpha})$$

(in dimensionless representation, distance along the backbone is measured in the units of l). Even though the approximations used here formally rely on the condition that $L \gg 1$, the result obtained for $L \sim 1$ also seems quite reasonable.

5.3. Transport Capacity of Structures

Finally, we consider the effect of an "inclusion" on transport when the structure on the right adjoins a perfectly absorbing medium. Formally, this problem is equivalent to the previous one in the limit of $\beta \longrightarrow 0$. However, we analyze it separately to demonstrate how easily comb structures can be manipulated in test problems. The evolution equation on $[0, L]$ has the form of (25) with a single parameter, $0 < \alpha < 1$, while conditions (26) are rewritten as

$$\left.\frac{\partial n_1}{\partial x}\right|_{x=0} = -q\left(\left.\frac{dn_{1p}}{dx}\right|_{x=0} = -\frac{q}{p}\right),$$
$$n_1|_{x=L} = 0. \tag{29}$$

The concentration on the backbone increases as

$$n_{1p} = \frac{q}{p}\frac{1}{p^{\alpha/2}}\frac{\sinh[p^{\alpha/2}(L-x)]}{\cosh(p^{\alpha/2}x)}.$$

As $t \longrightarrow \infty$, it approaches the steady distribution

$$n_1 = q(L-x),$$

which is similar to the steady profile in the classical diffusion problem. The difference lies in the fact that it is approached by a power law (rather than exponentially) when $\alpha \neq 1$. The flux $-\partial n_1/\partial x|_{x=L}$ transported through the structure behaves as

$$Q_p \sim \frac{q}{p}\frac{1}{1+p^{\alpha}L^2/2} \longrightarrow Q(t) \sim q\left(1-\frac{t^{-\alpha}L^2}{2\Gamma(1-\alpha)}\right)$$

approaching q very slowly ($\int Q dt$ is divergent). The shortage is explained by the continuing absorption into the structure as a whole, where the concentration increases as

$$N = q(L-x)\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}. \tag{30}$$

6. CONCLUSIONS

The results obtained by systematic application of the fractional differential approach to stochastic transport through comb structures with branches of different types are summarized as follows.

1. Evolution equations are derived by rigorous treatment of initial conditions. For the simplest structure, an expression is written out that necessarily depends on microscopic details of the initial distribution of the total concentration on the branches, leading to an unconventional time dependence of the right-hand side of the macroscopic equation. This result demonstrates that the initial distribution must be taken into account in solving subdiffusion equations of any form.

2. It is shown that transport through ramified comb structures is also subdiffusive and is slower than that through simple comb structures. The order of the time derivative in the corresponding transport equation decreases to zero as $\alpha = 1/2^k$ with increasing order of ramification. In the limit of infinite ramification order, particles diffuse away from the backbone at a constant rate, which leads to vanishing transport through the structure.

3. Garland-like structures are proposed where evolution is slower and $\alpha = 0$. In the intermediate case of a disk garland, the particle transport along the axis does not vanish, but is slower than predicted by any power law. In a ball garland, the transport along the axis vanishes over the characteristic time of particle diffusion between balls, because the balls absorb particles at a constant rate.

4. Several general problems are stated and solved for subdiffusive fractional differential equations subject to boundary conditions that are nonlocal in time. When two structures characterized by different scaling exponents are joined together, absorption by the more ramified one removes particles almost completely from the less ramified one. A segment of the structure that has more ramified branches acts as an effective barrier that blocks particle transport, absorbing the injected flux.

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