

## Random walks with random velocities

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We consider a random walk model that takes into account the velocity distribution of random walkers. Random motion with alternating velocities is inherent to various physical and biological systems. Moreover, the velocity distribution is often the first characteristic that is experimentally accessible. Here, we derive transport equations describing the dispersal process in the model and solve them analytically. The asymptotic properties of solutions are presented in the form of a phase diagram that shows all possible scaling regimes, including superdiffusive, ballistic, and superballistic motion. The theoretical results of this work are in excellent agreement with accompanying numerical simulations.

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### I. INTRODUCTION

What is common in the motion of charged particles in plasmas, of grains in inelastic gases, of tracers in turbulent flows, of locomoting animals, and of diffusing cells? The answer is that they perform a nonstop random motion with random velocities. Furthermore, in many cases their velocity distributions are qualitatively different from a conventional Maxwellian and typically have pronounced power law asymptotics. The transport processes in these systems can be anomalously fast. We propose here a random walk model with special emphasis on a nontrivial velocity distribution of moving entities that is able to describe such anomalous behavior.

In astrophysical plasmas, planetary magnetospheres, and solar wind, particles generally possess a non-Maxwellian high-energy tail [1]. The “Lorentzian plasma” is characterized by the generalized Lorentzian ( $\kappa$ ) velocity distribution  $h(\mathbf{v}) \propto |\mathbf{v}|^{-2\kappa-2}$ . In homogeneous granular gases, stationary solutions of the kinetic equations for the velocity distribution have power law tails  $h(v) \propto |v|^{-\sigma}$  [2]. The exponent  $\sigma$  depends on the dimension of the problem and properties of inelastic collisions between grains. For Maxwellian molecules in one dimension,  $\sigma=2$ . There is also experimental evidence for anomalously slow-decaying velocity distributions in dense granular systems [3]. In turbulent flows, tracer particles are carried by a random velocity field and separate from each other. The relative distance between the particles grows linearly with time,  $|r(t)| \propto t$ , on time scales below the classical correlation time [4], and at larger time scales, the separation behaves even superballistically,  $|r(t)| \propto t^{3/2}$  [5–7]. In two-dimensional turbulence the distribution of relative velocities is Lorentzian [8]. A large variety of living organisms spanning the full hierarchy of body sizes move randomly with varying velocities and exhibit anomalous diffusion. Spider monkeys [9], zooplankton [10], bacteria [11], and cells [12,13] are just a few of many examples sharing similar movement patterns. For completeness we also mention the

related topics of nonextensive thermodynamics [14], soft-mode turbulence in electroconvection [15], and some more astrophysical models [16].

The standard continuous time random walk model (CTRW) [17,18] may be viewed as a general concept suitable for describing the phenomena mentioned above. In this approach a random walker makes instantaneous jumps of varying lengths intermitted by random waiting times. Although successful for many applications this method possesses some unphysical features. For example, in the parameter range corresponding to superdiffusion the mean square displacement might become infinite [19]. Different coupling models were introduced to overcome this problem, e.g., long jumps of particles were penalized by corresponding long waiting times. With the correct behavior of moments and proper scaling, these models still have a conceptual drawback. They consider a particle sitting at a point and making an instantaneous jump to the destination point, whereas in real systems particles move with a certain velocity. This means that at the moment of measurement, the real particle is neither at the starting nor at the destination point—it is somewhere in between. At first glance this is not a significant difference. However, for the regimes of anomalous diffusion it drastically influences the shape of the density profile (see below). Therefore, giving the particle a finite velocity during its flight brings the random walk model closer to real physical systems. The Lévy walk with a constant velocity is the first example of such a model [19].

However, as we stressed above, in many situations the velocity of random walkers is far from being constant and is rather sampled from a broad distribution. On the other hand, the velocity distribution is often the first and only distribution that can be measured experimentally. Therefore, in the present paper we make the velocity distribution the key part of the model and generalize the CTRW approach to incorporate it.

We show that the model of random walks with random velocities is analytically solvable and that it has clear and tractable asymptotic properties. By using scaling arguments, we construct the full phase diagram for possible regimes of transport with its important superdiffusive, ballistic, and superballistic domains. It was shown that for the generalized Lorentzian velocity distribution, its knowledge alone is suf-

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ficient to describe the transport completely. All our analytical results are confirmed by numerical simulations.

## II. THE MODEL

We start by introducing our model. Consider for simplicity the one-dimensional case. A particle moves for a random time  $\tau$  (flight time) with a certain velocity  $v$  that can have positive as well as negative values to include the direction of motion. Then, it instantaneously changes the direction and magnitude of its velocity to another random value and continues the flight for another random time. Flight time and velocity are the two basic and independent random variables of the model with probability density functions (PDFs)  $f(\tau)$  and  $h(v)$ , respectively (instead of the flight time one can equally consider the length of the flight). They are normalized to 1,  $\int_{-\infty}^{+\infty} h(v)dv=1$ ,  $\int_0^{+\infty} f(\tau)d\tau=1$ , and the velocity distribution is symmetric,  $h(v)=h(-v)$ , so that there is no bias in the system. Already 20 years ago the model of random walks with a finite velocity of walking particles was suggested [19–21] and later elaborated in Refs. [22]. There, a walking particle with or without stops moves with a *constant* magnitude of velocity and randomly changes the direction of motion. From a conceptual point of view, this is the closest “relative” of the model considered here, and we shall obtain it as a particular case.

For given PDFs  $h$  and  $f$ , and the initial distribution of particles  $n_0(x)$ , we would like to know the evolution of the density of particles  $n(x,t)$ . There is an additional quantity whose dynamics helps to determine the density profile. We introduce the probability density function  $\nu(x,t)$  that a particle changes its velocity at the location  $[x, x+dx]$  in the time interval  $[t, t+dt]$ , and refer to it as the frequency of velocity changes. The equation governing the dynamics of  $\nu$  is very similar to the standard CTRW transport equation [18] (see also [23]):

$$\nu(x,t) = \int_{-\infty}^{+\infty} dv \int_0^t \nu(x-v\tau, t-\tau)h(v)f(\tau)d\tau + n_0(x)\delta(t). \quad (1)$$

A particle changes its velocity at the point  $(x,t)$ , when it has already changed its velocity to the value  $v$  at a time  $t-\tau$  and position  $x-v\tau$ , where  $\tau$  is the time of flight. The first term on the right-hand side of Eq. (1) integrates over all these events, taking into account that  $h(v)f(\tau)$  is the probability for a certain velocity  $v$  and a flight time  $\tau$  to occur. The last term of Eq. (1) assumes that there was an initial distribution of particles  $n(x,t=0)=n_0(x)$ , and that they immediately changed their velocities at  $t=0$ , thus starting the whole evolution.

Now we express the density of particles  $n(x,t)$  with the help of the frequency of velocity changes:

$$n(x,t) = \int_{-\infty}^{+\infty} dv \int_0^t \nu(x-v\tau, t-\tau)h(v)F(\tau)d\tau, \quad (2)$$

where  $F(\tau)$  is the probability *not* to change the velocity until the time  $\tau$ :  $F(\tau)=1-\int_0^\tau f(\tau')d\tau'$ . The density of particles at a given point  $(x,t)$  is a result of the velocity changes in all

other points in the past,  $\nu(x-v\tau, t-\tau)$ .  $F(\tau)$  assures that the particles do not choose another velocity before they pass the point  $(x,t)$ . The integration over all possible velocities and flight times gives (2).

Equations (1) and (2) fully describe the dynamics of the system with a given initial density of particles and the two PDFs for the flight times and velocities. Moreover, as we proceed to show, these equations can be solved analytically.

First, we determine the frequency of velocity changes  $\nu(x,t)$  and then substitute it in the equation for the particle density (2). We apply the Fourier transform with respect to the spatial coordinate in (1). Due to the shift property of the Fourier transform, an additional exponential factor  $e^{-ikv\tau}$  appears under the integral. Integration with respect to  $v$  gives the Fourier transform of  $h(v)$  with a reciprocal velocity  $k\tau$ . The Fourier transform of Eq. (1) reads

$$\nu_k(t) = \int_0^t \nu_k(t-\tau)h_{k\tau}f(\tau)d\tau + n_{0,k}\delta(t), \quad (3)$$

where the indices  $k$  and  $k\tau$  denote the Fourier components. Next, we apply the Laplace transform with respect to time and use its convolution property to obtain

$$\nu_{k,p} = \nu_{k,p}[h_{k\tau}f(\tau)]_p + n_{0,k}, \quad (4)$$

where the index  $p$  corresponds to the Laplace component. Then, the final expression for the frequency of velocity changes in the Fourier-Laplace domain,  $\nu_{k,p}$ , follows immediately:

$$\nu_{k,p} = \frac{n_{0,k}}{1 - [h_{k\tau}f(\tau)]_p}. \quad (5)$$

Similarly, the Fourier-Laplace transform of (2) together with (5) gives

$$n_{k,p} = \frac{[F(\tau)h_{k\tau}]_p n_{0,k}}{1 - [h_{k\tau}f(\tau)]_p}. \quad (6)$$

This is our analytical expression for the density of particles in the model with random velocities in the Fourier-Laplace representation (cf. [18,23]).

## III. ASYMPTOTIC PROPERTIES

For the analysis of the asymptotic properties of the density profile, we consider large spatial and temporal scales:  $x, t \gg 1$ . In the Fourier-Laplace space this corresponds to the limit  $k, p \rightarrow 0$ . Hence, instead of taking the full transforms of the functions involved, we take only the first terms in their expansions with respect to small  $k$  and  $p$ . This is a common step in the asymptotic treatment of random walk models. As compared to the standard CTRW, there is an additional technical difficulty hidden in the coupling of the velocity and the flight time distributions. Before we systematically analyze the possible regimes of transport, let us consider two concrete examples. In the first one, the velocity has a constant value, and therefore it reproduces the results of the Lévy walk model [19–22]. In the second example, we take the generalized Lorentzian velocity distribution typical for several systems mentioned in the Introduction.

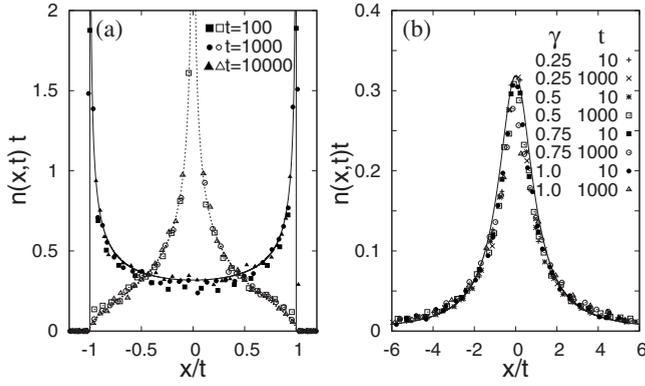


FIG. 1. Rescaled density of particles at different times: (a) the finite velocity problem ( $v_0=1$ ,  $\gamma=1/2$ ) (solid line and symbols) and coupled CTRW model (dashed line and open symbols); (b) Lorentzian profile given by (8) with  $d=1$ . Lines and symbols are theoretical and numerical results, respectively.

First assume that the velocity  $v$  can take only two values  $\pm v_0$ ; therefore  $h(v)=[\delta(v-v_0)+\delta(v+v_0)]/2$ . This corresponds to the Lévy walk model. For exponentially distributed flight times,  $f(\tau)=(1/\tau_0)e^{-\tau/\tau_0}$ , the asymptotic expansion of (6) with respect to small  $k$  and  $p$  is straightforward and reads

$$n_{k,p} = \frac{n_{0,k}}{p + k^2 \tau_0 v_0^2}.$$

In real space and time coordinates, it gives the classical diffusion equation with a diffusion constant  $D=v_0^2 \tau_0$ , which is the natural answer for a random walk with a finite average flight time  $\tau_0$  and constant velocity  $v_0$ .

For flight time distributions with a power law tail,  $f(\tau) = \gamma/(1+t)^{1+\gamma}$  with  $0 < \gamma < 1$ , we recover the results for Lévy walks (cf. [19,21,22]):

$$n_{k,p} = \frac{\cos[\varphi(\gamma-1)]}{|p^2 + k^2 v_0^2|^{1/2} \cos(\varphi\gamma)}, \quad (7)$$

$$\cos \varphi = \frac{p}{\sqrt{p^2 + k^2 v_0^2}}.$$

To illustrate such a solution, we calculate the inverse Laplace-Fourier transform of (7) for the case  $\gamma=1/2$ :  $n(x,t) = \pi^{-1} \theta(v_0 t - |x|) (t^2 v_0^2 - x^2)^{-1/2}$ . In Fig. 1(a), we plot it in rescaled coordinates together with the results of numerical computations, where we directly simulate the paths of an ensemble of random walkers. For comparison the result for a coupled CTRW model is shown where a particle waits for a random time  $\tau$  and then makes an instantaneous jump of a length  $|x|=v\tau$  (dashed line and open symbols). Note the qualitative difference of the density profiles.

Let us consider for a moment space of arbitrary dimension  $d \geq 1$ . All the above formulas are still valid if the quantities  $x$ ,  $v$ , and  $k$  are considered as  $d$ -dimensional vectors. Take the velocity distribution in generalized Lorentzian (or Cauchy) form:  $h(v) \propto 1/(1+v^2)^{(d+1)/2}$ . Independent of the choice of the flight time distribution, we obtain a surprisingly

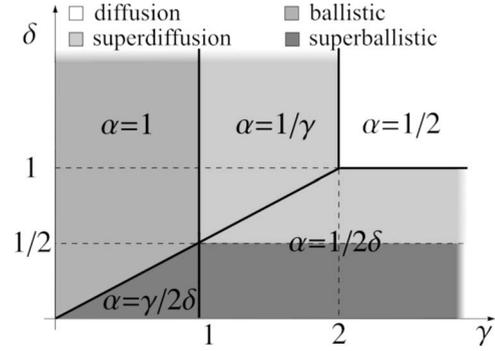


FIG. 2. Phase diagram of possible regimes of transport in the model of random walks with random velocities, where  $\delta$  and  $\gamma$  are the exponents in the power law tails of velocity and flight time distributions, respectively. The resulting scaling is given by  $\alpha: x \propto t^\alpha$ .

simple answer for the density of particles in real space and time coordinates:

$$n(x,t) = \frac{\Gamma\left(\frac{d+1}{2}\right)t}{[\pi(t^2 + x^2)]^{(d+1)/2}}, \quad (8)$$

which is also a generalized Lorentzian (one of the Lévy-stable distributions). For  $d=1$  the density profile (8) is presented in Fig. 1(b). This is a very remarkable result since it demonstrates that a (generalized) Lorentzian velocity distribution always leads to the (generalized) Lorentzian density profile for *any* distribution of flight times or jump lengths. We note here that such a result is very unlikely to be recovered in any other CTRW model. Furthermore, a Lorentzian velocity profile appears in real physical phenomena such as two-dimensional turbulence [8] and it is also one of the model distributions of kinetic theory [2].

Results of numerical simulations with various exponents  $\gamma$  excellently collapse on the theoretical curve and confirm the independence of the density profile of the flight time distribution.

#### IV. SCALING

Now we would like to give a more general prediction for possible regimes of transport in the model returning to  $d=1$ . Instead of determining all details of the density profile, we only investigate its scaling properties. To do so, we employ the results for the standard CTRW model [18]. There, it is shown that the density profile has the self-similar form  $n(x,t) = (1/t)^\alpha \Phi(x/t^\alpha)$ . If the lengths of jumps of particles are distributed as a power law  $g(|x|) \propto |x|^{-1-2\beta}$  and the waiting times as well have a power law tail  $\psi(\tau) \propto \tau^{-1-\gamma}$ , the exponent  $\alpha$  in the scaling function  $\Phi$  depends on  $\beta$  and  $\gamma$  in the following way [18]:

$$\alpha = \begin{cases} \gamma/2\beta, & 0 < \beta < 1, \quad 0 < \gamma < 1, \\ 1/2\beta, & 0 < \beta < 1, \quad \gamma > 1, \\ \gamma/2, & \beta > 1, \quad 0 < \gamma < 1. \end{cases} \quad (9)$$

The scaling of different transport regimes is determined by the argument of the function  $\Phi$ . It shows how the spatial

expansion of the cloud of random walkers scales with time:  $x \propto t^\alpha$ . To derive the scaling relation in our model, we construct a pair of functions “equivalent” to  $g(x)$  and  $\psi(\tau)$  of the standard model and use the exponents given in (9). Since  $\psi(\tau)$  is a waiting time distribution, it represents the time “cost” of a single jump. Hence we can directly identify it with our PDF  $f$  for the flight times. The distribution of jump lengths is calculated from

$$g(x) = \int_{-\infty}^{+\infty} dv \int_0^{+\infty} \delta(x - v\tau) h(v) f(\tau) d\tau, \quad (10)$$

where for the velocity distribution we employ a power law of the general form  $h(v) \propto |v|^{-1-2\delta}$ . Evaluation of the integrals in Eq. (10) gives quite an involved expression. However, we are interested only in the asymptotic properties of  $g(x)$ , namely, the exponent  $\beta(\gamma, \delta)$  of its power law tail. This exponent is then substituted into Eq. (9) to obtain the correct value for  $\alpha$ . We present the results for the scaling exponent in the form of a phase diagram in Fig. 2. Solid lines border the regions with different dependencies of  $\alpha$  on the parameters  $\delta$  and  $\gamma$ . The color coding from white to dark gray classifies the different regimes of transport: diffusion, superdiffusion, ballistic, and superballistic. Vertical and horizontal solid and dashed lines mark critical values of  $\delta$  and  $\gamma$  at which different moments of  $h$  and  $f$  start to diverge. The line  $\delta = \gamma/2$  separates the regions where the effective jump length distribution [Eq. (10)] is more strongly influenced by the velocity or flight time distribution, respectively.

Let us discuss in brief the main features of this phase diagram. All examples considered so far fit into this diagram. Note that, besides the classical diffusive transport, superdiffusive, ballistic, and superballistic scalings are possible. In

the latter case, the mean velocity has to be infinite ( $\delta < 1/2$ ). An important difference of our model as compared to the standard CTRW is the absence of the subdiffusion regime, which has a clear physical reason. With nonzero random velocities there is no possibility to trap a particle for a long time. We have checked the phase diagram with numerical simulations, by computing density profiles and observing their collapse after the corresponding rescaling.

## V. CONCLUSION

Concerning the applications of the model, we note that any real physical or biological process may involve other typical random walk features, such as persistence, waiting times, multidimensionality, etc. They can easily be added to our model; however, they would significantly increase the parameter space and mask the effects of the velocity distribution. Moreover, a correlation between velocities and flight times could also be included [24,25].

Summarizing our results, we suggest a model that is able to describe the random motion of particles with nontrivial velocity distributions. It is analytically solvable and allows for a careful and complete asymptotic analysis. It embraces a wide spectrum of stochastic transport regimes from classical diffusion to superballistic dispersal. Therefore, it is an indispensable tool for describing random motion in physical and biological systems with inherent velocity distributions.

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