# Supplementary Material for 'Perturbation spreading in many-particle systems: a random walk approach'

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PACS numbers: 05.40.Fb, 05.45.Jn, 45.50.Jf

## I. TRANSPORT EQUATIONS FOR THE MODEL PROPAGATOR

In order to gain analytical insight into the generalized LW dynamics, we proceed along known reasoning, see in [1, 2], and derive the transport equation for the probability distribution function (PDF) $P(x,t)$. We first introduce the probability distribution of the end points of flights, or the turning points, $\nu(x,t)$, where a particle chooses its new velocity. Its evolution is governed by the following balance equation:

$$\nu(x,t) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} dvdw \int\limits_{0}^{t} \nu(x-v\tau-w,t-\tau)\psi(\tau)h(v)p(w,\tau)d\tau + \varphi(t)\int\limits_{-\infty}^{\infty} h(v)p(x-vt,t)dv\ . \tag{1}$$

Equation (1) shows that a particle changes its velocity at the point $(x,t)$ if it was the end point of the preceding step. That previous step had some flight time $\tau$ and occurred with some velocity $v$. Therefore the step originated in the point $x-v\tau-w$, where $w$ takes into account the accumulation of velocity fluctuations during a single flight. Next we have to integrate over all possible flight times, velocities and fluctuations with the corresponding probability densities. This is how the first term on the right side of Eq. (1) is obtained. The second term reflects the influence of initial conditions. In this work we use so-called equilibrated initial conditions [3] which assumes that walkers evolved for an infinitely long time when the observation started. Different initial setups affect only the probability of when a flying particle experiences the first turn after the start of observation. For a system evolving for an infinite time the PDF of the first turn after the observation has been started is given by $\varphi(t) = \langle\tau\rangle^{-1}\int_0^\infty \psi(t+\tau)d\tau$ [3]. If a particle starts at $x=0$, the spatial position of where the first turn occurs is influenced by the velocity fluctuations and is given by: $\int_\infty^\infty \delta(x-vt-w)p(w,t)dw = p(x-vt,t)$.

We next evaluate the actual density of particles $P(x,t)$, to obtain:

$$P(x,t) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} dvdw \int\limits_{0}^{t} \nu(x-v\tau-w,t-\tau)\Psi(\tau)h(v)p(w,\tau)d\tau + \Phi(t)\int\limits_{-\infty}^{\infty} h(v)p(x-vt,t)dv\ . \tag{2}$$

The particle currently finds itself in the point $(x,t)$, if it previously changed the direction of flight at the point $x-v\tau-w$, and keeps flying afterwards for the time $\tau$ with the probability $\Psi(\tau) = 1-\int_0^\tau \psi(t)dt$. Similarly, the probability to continue the first flight reads $\Phi(t) = \langle\tau\rangle^{-1}\int_0^\infty \Psi(t+\tau)d\tau$ [2].

Formally, the above two equations can be solved with a help of combined Fourier- and Laplace-transform, thereby turning all convolution integrals into products and thus rendering the integral equations algebraic. The answer for the density of particles in Fourier/Laplace space, $\widetilde{P}_{k,s}$ is given by (we use tilde-notation to denote the Fourier/Laplace transform):

$$\widetilde{P}_{k,s} = \frac{\left[\Psi(\tau)\widetilde{h}_{k\tau}\widetilde{p}_k(\tau)\right]_s\left[\varphi(\tau)\widetilde{h}_{k\tau}\widetilde{p}_k(\tau)\right]_s}{1-[\widetilde{h}_{k\tau}\widetilde{p}_k(\tau)\psi(\tau)]_s} + \left[\widetilde{h}_{kt}\widetilde{p}_k(\tau)\Phi(t)\right]_s\ . \tag{3}$$

This exact analytical expression serves as the starting point for the asymptotic analysis for large spatial and temporal scales, that corresponds to small $k,s$ coordinates in Fourier/Laplace-space. It is possible to show that for the case of Gaussian fluctuations the central part of the profile can be described by the Lévy distribution, which is the solution of the fractional diffusion equation

[2, 4]:

$$\widetilde{P}_{k,s} \simeq \frac{1}{s + \tau_0^{\gamma-1} v_0^{\gamma} (\gamma-1)\Gamma[1-\gamma] k^{\gamma} \sin(\pi\gamma/2)} \quad .$$

In original coordinate space and original time this expression delivers the following scaling relation for the central part of the density profile:

$$P(x,t') \simeq \frac{1}{Ku^{1/\gamma}} P\left(\frac{x}{Ku^{1/\gamma}}, t\right), \qquad |x| \ll vt \ , \qquad (4)$$

where $K \propto \tau_0^{1-1/\gamma} v_0$ and $u = t'/t$, see Fig. 2.

To describe the density of particles in the ballistic humps of the profile we use the second term on the right hand side of Eq. (2):

$$P_{\rm hump}(x,t) = \Phi(t) \int\limits_{-\infty}^{\infty} h(v) p(x - vt, t) dv \ , \qquad (5)$$

where $\Phi(t) = \langle \tau \rangle^{-1} \int_0^{\infty} \Psi(t+\tau) d\tau$. For the chosen flight time distribution it scales as $\Phi(t) \propto (t/\tau_0)^{1-\gamma}$.

## II. NUMERICS

For the integration of the FPU $\beta$ chain's equations of motion we used the symplectic $SABA_2C$ scheme [5, 6], with the integration time step $dt = 0.01 \div 0.02$. Calculations have been performed on a Tesla S1070 supercomputer, with 960 CPU's on board.


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