Letter

Symmetry-induced decoherence-free subspaces

Jonathan Dubois, Ulf Saalmann, and Jan Michael Rost
Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Straße 38, 01187 Dresden, Germany

(Received 20 May 2022; revised 25 July 2022; accepted 21 December 2022; published 12 January 2023)

Preservation of coherence is a fundamental, yet subtle, phenomenon in open systems. We uncover its relation to symmetries respected by the system Hamiltonian and its coupling to the environment. We discriminate between local and global classes of decoherence-free subspaces for many-body systems through the introduction of "ghost variables". The latter are orthogonal to the symmetry and the coupling to the environment depends solely on them. Constructing them is facilitated in classical phase space and can be transferred to quantum mechanics through the equivalent role that Poisson and Lie algebras play for symmetries in classical and quantum mechanics, respectively. Examples are given for an interacting spin system.

DOI: 10.1103/PhysRevResearch.5.L012003

Introduction. A physical system interacting with an environment [1–4] relaxes to an equilibrium or a nonequilibrium steady state on time scales much longer than the relaxation time [5]. In the steady state the system no longer evolves in time due to its coupling to the environment and *all* associated physical observables reach a constant value. This characteristic is inherent to classical and quantum open systems.

Quantum mechanically, relaxation is accompanied by decoherence, which remains a major obstacle in putting physical devices to work for quantum computation [6–8]. Therefore, decoherence-free subspaces [9–12] (DFS), which are protected against decoherence effects [13], play a crucial role in realizing quantum computing [6,14,15]. Understanding and formulating the general conditions underlying DFS is the prerequisite to tame decoherence and to establish open systems, which exhibit nonstationary long-time dynamics (NLD), that is, remain out of equilibrium despite their interaction with an environment [12,16,17].

In closed systems, it is well known that symmetries, i.e., constants of motion, are crucial to characterize their dynamics. Here, we will show that this is also the case for open systems: Symmetries can be used to formulate simple conditions for DFS in terms of the dependence of the system and the coupling to the environment on these symmetries. Moreover, these conditions can be identified and formulated in an intuitive way via classical dynamics since exact symmetries hold quantum mechanically as well as classically through their equivalent formulation in terms of commutators and Poisson brackets, respectively. The classical phase-space perspective will also allow us to introduce naturally new classes of DFS where the coupling to the environment depends only on so called "ghost variables", orthogonal to the symmetries. First, we will present the quantized conditions for the DFS in terms of eigenstates of the symmetry operator.

Our starting point is the Lindblad master equation [4] $\dot{\rho} = -\mathrm{i}\mathscr{L}(\mathbb{H},\mathbb{L})\rho$ for the systems's density matrix ρ with Hamiltonian $\mathbb{H}$ coupled to the the environment in Markov approximation through the Lindblad operators $\mathbb{L}_\alpha$. In the framework of the master equation DFS exist if and only if the Lindbladian $\mathscr{L}$ has nonzero real eigenvalues [10,16,18,19]. Here, we will work with the adjoint Lindbladian $\mathscr{L}^\dagger(\mathbb{H},\mathbb{L})$, which has identical spectral properties, since we are interested in operators, most prominently a symmetry $\mathbb{J}$. As a constant of motion for the open system its dynamics is governed in the Heisenberg picture by $\mathscr{L}^\dagger$ through

$$0 = \frac{d\mathbb{J}}{dt} = \mathrm{i}\mathscr{L}^\dagger\mathbb{J}, \tag{1}$$

with $\mathscr{L}^\dagger = \mathscr{L}^\dagger_{\mathbb{H}} + \mathscr{L}^\dagger_{\mathbb{L}}$, where

$$\mathscr{L}^\dagger_{\mathbb{H}}\mathbb{J} = \frac{1}{\hbar}[\mathbb{H},\mathbb{J}], \tag{1a}$$

$$\mathscr{L}^\dagger_{\mathbb{L}}\mathbb{J} = \frac{\mathrm{i}}{\hbar}\sum_\alpha\left(\mathbb{L}^\dagger_\alpha[\mathbb{J},\mathbb{L}_\alpha] + [\mathbb{L}^\dagger_\alpha,\mathbb{J}]\mathbb{L}_\alpha\right). \tag{1b}$$

For a constant of motion $\mathbb{J}$, with $[\mathbb{H},\mathbb{J}]=[\mathbb{L}_\alpha,\mathbb{J}]=0$, we find that DFS exist if there exist two eigenspaces $\{|n\rangle\}$ and $\{|m\rangle\}$ of $\mathbb{J}$ with eigenvalues J_n and J_m, respectively, such that

$$\Delta\mathbb{H}_{nm} \neq 0 \quad \text{and} \quad \Delta\mathbb{L}_{\alpha nm} = 0, \tag{2}$$

for all α, with the explicit forms of $\Delta\mathbb{H}_{nm}$ and $\Delta\mathbb{L}_{\alpha nm}$ given below in Eq. (4).

While the condition on $\mathbb{H}$ is necessary for (oscillatory) nonstationary dynamics in the first place, the condition on $\mathbb{L}_\alpha$ makes sure that this dynamics is preserved for long times and therefore establishes NLD. Interestingly, since $\mathbb{L}_\alpha$ is in principle a function of all quantum operators $\mathbb{J}$ and $\{\mathbb{Z}_i\}$, the *ghost operators*, being orthogonal to $\mathbb{J}$, the condition $\Delta\mathbb{L}_{\alpha nm}=0$ can be achieved in two qualitatively different ways:

(i) the $\mathbb{L}_\alpha$ depend explicitly on $\mathbb{J}$ (local realization) and

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Open access publication funded by the Max Planck Society.

(ii) the $\mathbb{L}_\alpha$ do not depend on $\mathbb{J}$ but can depend on all the operators $\{\mathbb{Z}_i\}$ (global realization).

We have a set of commuting operators: $\mathbb{J}$, its conjugate, and the ghost operators $\{\mathbb{Z}_i\}$. Together, they span the Hilbert space for the open-system dynamics. The Hamilton operator $\mathbb{H}$ (and similarly all Lindblad operators $\mathbb{L}_\alpha$) with the symmetry $\mathbb{J}$ can be written as a direct sum of irreducible (square) matrices

$$\mathbb{H} = \bigoplus_n \mathbb{H}_n, \quad \mathbb{L}_\alpha = \bigoplus_n \mathbb{L}_{\alpha n}, \tag{3}$$

where $\mathbb{X}_n$ is associated with the eigenspace of J_n, an eigenvalue of $\mathbb{J}$. They are defined as $\mathbb{X}_n = \mathrm{tr}_n \mathbb{P}_n \mathbb{X} \mathbb{P}_n$ where $\mathbb{P}_n = \prod_{j(J_j \neq J_n)} (\mathbb{J} - J_j \mathbb{1})/(J_n - J_j)$ are projectors [20] rendering all subspaces to null operators but the one associated with the eigenvalue J_n of $\mathbb{J}$. tr_n, finally, takes the partial trace over all complementary subspaces, reducing the dimension to the irreducible one of J_n. Corresponding to the multiplicity η_n of the eigenvalue J_n, the $\mathbb{H}_n$ and $\mathbb{L}_{\alpha n}$ are $\eta_n \times \eta_n$ matrices. Due to the block-diagonal form (3), the superoperator $\mathscr{L}^\dagger$ can be written as blocks associated with a pair of eigenvalues (J_n, J_m) such that $\mathscr{L}^\dagger = \bigoplus_{n,m} \mathscr{L}^\dagger_{nm}$. Each block of the superoperator

$$\mathscr{L}^\dagger_{nm} = \frac{1}{\hbar} \Delta\mathbb{H}_{nm} + \frac{\mathrm{i}}{\hbar} \sum_\alpha \big((\mathbb{1}_n \otimes \mathbb{L}^\dagger_{\alpha m}) \Delta\mathbb{L}_{\alpha nm} - (\mathbb{L}^\top_{\alpha n} \otimes \mathbb{1}_m) \Delta\mathbb{L}^\dagger_{\alpha nm} \big) \tag{4}$$

is of size $(\eta_n \times \eta_m) \times (\eta_n \times \eta_m)$ with [21]

$$\Delta\mathbb{H}_{nm} \equiv \mathbb{H}_n \otimes \mathbb{1}_m - \mathbb{1}_n \otimes \mathbb{H}^\top_m, \tag{4a}$$

$$\Delta\mathbb{L}_{\alpha nm} \equiv \mathbb{L}_{\alpha n} \otimes \mathbb{1}_m - \mathbb{1}_n \otimes \mathbb{L}^\top_{\alpha m}. \tag{4b}$$

For a graphical representation of the Lindbladian structure discussed here, see the Supplemental Material [22].

Coherent oscillations of a beating operator $\mathbb{A}_{nm}$, which is an eigenoperator of $\mathscr{L}^\dagger_{nm}$ with nonzero real eigenvalues λ, occur for those $\Delta\mathbb{H}_{nm}$ and $\Delta\mathbb{L}_{\alpha nm}$ that fulfill condition (2). The eigenoperators $\mathbb{A}_{nm}$ are matrices with an $\eta_n \times \eta_m$ nonvanishing part such that $\mathbb{A}_{nm} = \mathbb{P}_n \mathbb{A}_{nm} \mathbb{P}_m$ and can have multiple beating frequencies. Furthermore, the $\mathbb{A}_{nm}$ are also eigenoperators of the Lindbladian since under condition (2), with (4b), $[\mathbb{L}_{\alpha n}, \mathbb{L}^\dagger_{\alpha n}] = [\mathbb{L}_{\alpha m}, \mathbb{L}^\dagger_{\alpha m}] = 0$. Since symmetries are respected equivalently by a quantum system and its classical counterpart (here, via Poisson brackets), we can give an intuitive account of conditions (2) in classical phase space [23,24]. There, local versus global realization of these conditions become clearer.

Phase-space perspective of symmetry-induced DFS. For a constant of motion J with $\{H, J\} = \{L_\alpha, J\} = 0$, we find that semiclassically DFS exist if there is at least one layer $J = J_0$ on which

$$\omega(J_0) \equiv \partial H/\partial J\big|_{J_0} \neq 0, \quad \ell_\alpha(J_0) \equiv \partial L_\alpha/\partial J\big|_{J_0} = 0, \tag{5}$$

where $\omega(J)$ is the Hamiltonian frequency and $\ell_\alpha(J)$ can be interpreted as the decay rate induced by the environment. Equation (5) is the classical analog of the quantum condition (2) and can be obtained by taking the limit $\hbar$ very small or derived semiclassically from dynamical conditions as is shown below. Note that we denote quantum operators with $\mathbb{X}$ and analogous (classical) functions with X.

We consider phase-space variables $(J, \theta, \mathbf{Z})$ with J conserved, $\{J, H\} = \{J, L_\alpha\} = 0$ for all α. The angle θ is canonically conjugate to the action-like J such that $\{\theta, J\} = 1$. Then, the phase space is foliated by manifolds on which J is conserved. The $\mathbf{Z}$ are a set of conjugate *ghost variables* such that $\{\mathbf{Z}, J\} = \{\mathbf{Z}, \theta\} = \mathbf{0}$, whose relevance for DFS will become clear below.

From the action-angle variables we can construct "beating variables"

$$A = f(J)\exp(-\mathrm{i}\theta) \quad \text{and} \quad A^* = f(J)\exp(\mathrm{i}\theta), \tag{6}$$

with $A^*A = f^2(J)$ [22,25]. The beating variables describe dynamics along a path on which J is conserved. In the semiclassical limit of the adjoint Lindbladian [22–24] the dynamics of A on the manifold of constant J is given by $\dot{A} = \mathrm{i}\mathscr{L}^* A$ with

$$\mathscr{L}^* A = -\left(\partial_J H + \sum_\alpha \big(2\mathrm{Im}\,(L_\alpha \partial_J L^*_\alpha) - \mathrm{i}\hbar |\partial_J L_\alpha|^2 \big) \right) A, \tag{7}$$

which follows from the Poisson brackets $\{A, H\} = -\mathrm{i}(\partial_J H)A$ and $\{A, L_\alpha\} = -\mathrm{i}(\partial_J L_\alpha)A$, where $\partial_J H$ is real and $\partial_J L_\alpha$ can be complex (since L_α can be complex). Hence, A evolves in time with an oscillatory and a decaying part, regardless of the dynamics of the ghost variables $\mathbf{Z}$. The oscillatory part comes from a combination of the Hamiltonian and the dissipative components, while the decay part is a consequence of dissipation only. Note, that the decaying part is of order $\hbar$ and is a consequence of diffusion or quantum noise.

If $\partial_J L_\alpha = 0$ the decaying part in (7) vanishes and the beating variable A exhibits NLD, provided that $\partial_J H \neq 0$, which leads to the conditions (5) specifying the existence of DFS. If the Lindblad function depends explicitly on the conserved quantity J and there exists a manifold $J = J_0$ on which $\ell_\alpha(J_0, \mathbf{Z}) = 0$, the environment couples to the degree of freedom associated with J, but not on the subspace $J = J_0$. Although J is conserved, NLD is restricted to the manifold $J = J_0$, and therefore we call this realization (i) of DFS *local*. One also gets $\ell_\alpha = 0$, if L_α does not depend explicitly on J at all. In this case, the environment does not couple to the degree of freedom associated with J. These conditions apply to all manifolds labeled by J, and therefore this constitutes a *global* realization (ii) of DFS.

In both cases, the environment does not affect the oscillatory dynamics of the beating variable A characterized by the (nonzero) real eigenvalues $\omega(J, \mathbf{Z})$ of the Lindbladian. If the Hamiltonian frequency ω depends explicitly on the ghost variables $\mathbf{Z}$, this oscillatory dynamics can be very complicated. If $\omega = \omega(J)$ only, one can directly solve $\dot{A} = \mathrm{i}\mathscr{L}^* A$ to obtain $A(t) = \exp(-\mathrm{i}\omega(J)t)A(0)$. In case (i) the environment extinguishes for long times all oscillatory motion linked to J but one with frequency $\omega(J_0)$ on the manifold J_0.

From the semiclassical perspective as developed above we can draw further conclusions. The conditions (5) do not require integrability (or near integrability) of the system. Therefore, also classically chaotic systems can have DFS, which we demonstrate here explicitly with the Heisenberg spin model. Furthermore, if DFS exist due to a symmetry, small perturbations of the system will not destroy them. This follows from the KAM theorem [26], which ensures the

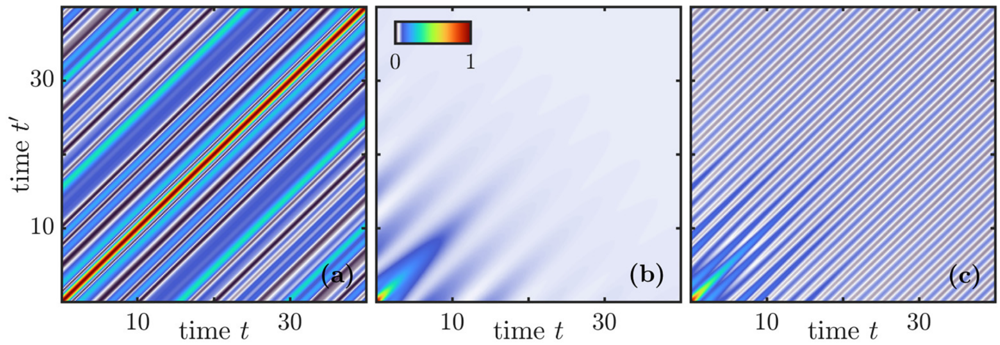

FIG. 1. Autocorrelation function $|\mathrm{tr}\,\rho(t)\rho(t')|$ for Hamiltonian (8) with N=6. Parameters ω_i, δ_{ij}, and Δ_{ij} are given within the Supplemental Material [22]. (a) No dissipation, $\mathscr{L}_{\mathbb{L}} = 0$. (b) Dissipation with Lindblad operators $\mathbb{L}_\alpha = \gamma\,\mathbb{S}^z_\alpha$, $\alpha = 1\ldots N$. (c) Dissipation with ghost operators such that $\mathbb{L}_\alpha = \gamma\,\widetilde{\mathbb{S}}^z_\alpha$, $\alpha = 1\ldots N{-}1$. In both cases γ=1/3.

existence of modified action-angle variables for open systems subjected to perturbations. Therefore, we construct now explicitly a coupling to the environment according to (ii) for a chaotic Hamiltonian.

Application to the Heisenberg XXZ spin model. Since systems, which we classify as local realizations (i) have been discussed [16,17], we focus on global realizations (ii), which require the Lindblad operators to be independent of the symmetry $\mathbb{J}$. Therefore, only many-body systems render this case nontrivial.

We consider the Heisenberg XXZ spin model [27–30]

$$\mathbb{H} = \sum_i \omega_i \mathbb{S}^z_i + \sum_{ij}\left[\sigma_{ij}(\mathbb{S}^+_i\mathbb{S}^-_j + \mathbb{S}^-_i\mathbb{S}^+_j) + \Delta_{ij}\mathbb{S}^z_i\mathbb{S}^z_j\right], \quad (8)$$

where the $\mathbb{S}_i$=$(\mathbb{S}^x_i, \mathbb{S}^y_i, \mathbb{S}^z_i)$ are spin-1/2 operators on site i with Pauli matrices, and $\mathbb{S}^\pm_i = \mathbb{S}^x_i \pm \mathrm{i}\,\mathbb{S}^y_i$ being ladder operators, with $i = 1\ldots N$ and N the number of spins. Combinations $\mathbb{S}^\pm_i\mathbb{S}^\mp_j$ evoke spin flips between site i and j. The total spin $\mathbb{J} = \mathbb{S}^z = \sum_{k=1}^N \mathbb{S}^z_k/N$ is conserved in time as $\mathscr{L}^\dagger\mathbb{J} = 0$. Thus one can construct Lindblad operators that generate NLD, as shown in Fig. 1. For $\mathbb{H}$ of Eq. (8) with the specific parameters given within the Supplemental Material [22], we compare pure Hamiltonian dynamics, i.e., $\mathscr{L}_{\mathbb{L}} = 0$ [Fig. 1(a)], with an "arbitrary" dissipative part [Fig. 1(b)]

$$\mathbb{L}_\alpha = \gamma\mathbb{S}^z_\alpha \quad \alpha = 1\ldots N, \quad (9a)$$

and with a dissipative part constructed from ghost operators only [Fig. 1(c)]

$$\mathbb{L}_\alpha = \gamma\widetilde{\mathbb{S}}^z_\alpha \quad \alpha = 1\ldots N{-}1. \quad (9b)$$

While the autocorrelation function decays for (9a), it leads to pertinent oscillations for (9b), since the $\mathbb{L}_\alpha$ commute with $\mathbb{J} = \mathbb{S}^z$. In order to construct the ghost operators $\widetilde{\mathbb{S}}^z_\alpha$, see Eq. (11) below for their explicit form, we resort to classical phase space.

To this end, we consider the classical description of the Heisenberg XXZ spin model [22,27–30] for spin variables $\mathbf{S}_i = (S^x_i, S^y_i, S^z_i)$ where $i = 1\ldots N$. The conserved quantity is the total spin $J = S^z = \sum_i S^z_i/N$. We derive the associated classical ghost variables using polar coordinates $\mathbf{S}_i = [S_i^2 - (S^z_i)^2]^{1/2}(\mathbf{e}_x\cos\theta_i + \mathbf{e}_y\sin\theta_i) + \mathbf{e}_z S^z_i$ for the spin on site i, where the azimuthal angles θ_i are canonically conjugate to the S^z_i and S_i^2=$|\mathbf{S}_i|^2$. Akin to relative center-of-mass motion for massive particles we take advantage of this fact by introducing relative angles $\Delta\theta_k = \theta_{k+1} - \theta_k$ for $k = 1\ldots N{-}1$, and the total angle $\theta = \sum_i \theta_i$. They are related to the Jacobi coordinates for celestial many-body systems [31].

The canonical transformation from phase-space variables (θ_i, S^z_i) to phase-space variables $(J, \theta, \mathbf{Z})$, with ghost variables $\mathbf{Z} = (\Delta\theta_k, \widetilde{S}_k)$ for $k = 1\ldots N{-}1$, and its inverse are given within the Supplemental Material [22]. The transformation is obtained by using the canonical rules [32] with a suitable generating function associated with the center of mass and the relative angles. Besides $\Delta\theta_k$, we obtain as ghost variables (see Supplemental Material [22])

$$\widetilde{S}^z_k = \frac{k}{N}\sum_{i=1}^N S^z_i - \sum_{i=1}^k S^z_i, \quad (10)$$

which are linear functions of the spin variables. Note that using canonical transformations ensures that the new set of variables spans the entire phase space. If the interaction with the environment is any analytic function of these variables or operators (i.e., if it does not depend explicitly on θ and J) DFS exist and the open system exhibits NLD. The ghost variables or operators do not affect the oscillatory dynamics associated with the conserved quantity J. Therefore, coherent oscillations persist on long-timescales and the spectrum of the Lindbladian contains nonzero real eigenvalues λ, cf. Fig. 2. The beating variables associated with the conserved quantity, as given in Eq. (6), are obtained in terms of spin variables by performing a noncanonical change of coordinates [22] leading to $A \propto \prod_{i=1}^N S^-_i$ and $A^* \propto \prod_{i=1}^N S^+_i$. The beating variables oscillate at a frequency $\Omega = \sum_{i=1}^N \omega_i$, corresponding to the absolute value of the nonzero real eigenvalues and are not affected by dissipation.

Now, the *ghost operators* as the quantum analogs of the *ghost variables* follow simply from replacing the Cartesian spin variables in the ghost variable expressions by the corresponding Cartesian spin operators. Therefore, the ghost operators are functions of the spin operators (Pauli matrices) and read

$$\widetilde{\mathbb{S}}^z_k = \frac{k}{N}\sum_{i=1}^N \mathbb{S}^z_i - \sum_{i=1}^k \mathbb{S}^z_i, \quad (11a)$$

$$\widetilde{\mathbb{S}}^\pm_k = \mathbb{D}_k\,\mathbb{S}^\pm_{k+1}\,\mathbb{S}^\mp_k, \quad (11b)$$

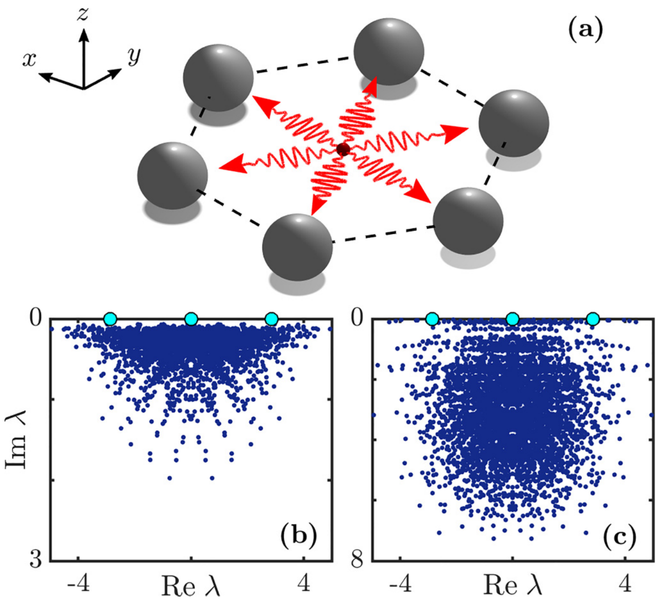

FIG. 2. (a) Schematic of the spin system for $N = 6$ governed by Hamiltonian (8) as used before in Fig. 1. The red lines illustrate the dissipations induced by the ghost operators. [(b),(c)] Spectrum of the adjoint of the Lindbladian $\mathscr{L}^\dagger$ for Lindblad operators as a function of the ghost operators (11) such that $\mathbb{L}_\alpha = \gamma\,\widetilde{\mathbb{S}}^z_\alpha$ and $\mathbb{L}_\alpha = \gamma\,\widetilde{\mathbb{S}}^+_\alpha$, respectively. In both cases $\alpha = 1\ldots N-1$ and γ=1/3. The cyan circles are the purely real eigenvalues.

with $k = 1\ldots N-1$ and a diagonal matrix $\mathbb{D}_k$ [22]. By construction, all combinations of ghost operators are also ghost operators. Any Lindblad operator, which is an analytic function of these ghost operators leads to DFS.

In Fig. 2(b), we have used $N-1$ Lindblad operators as a linear combination of the ghost operators (11a) such that $\mathbb{L}_\alpha = \gamma\widetilde{\mathbb{S}}^z_\alpha$ with $\gamma = 1/3$ for $\alpha = 1\ldots N-1$. For Fig. 2(c), we have used $N-1$ Lindblad operators of the ghost operators (11b) such that $\mathbb{L}_\alpha = \gamma\widetilde{\mathbb{S}}^+_\alpha$ with γ=1/3 and $\alpha = 1\ldots N-1$. In both cases, the coherent oscillations associated with the symmetry $\mathbb{J}$ are not affected, neither are the beating operators $\mathbb{A} = \prod_{i=1}^N \mathbb{S}_i^-$ and $\mathbb{A}^\dagger = \prod_{i=1}^N \mathbb{S}_i^+$. Note also that the (real) eigenvalues (cyan circles) are the same for the cases of Figs. 2(b) and 2(c), since they come from the Hamiltonian part of the Lindbladian $\mathscr{L}^\dagger_{\mathbb{H}}$, which we have not changed. We find $\mathscr{L}^\dagger_{\mathbb{H}}\mathbb{A} = \Omega\mathbb{A}$ and $\mathscr{L}^\dagger_{\mathbb{L}}\mathbb{A} = 0$ corresponding to the cyan circles for positive real eigenvalues (the negative one corresponds to the one associated with $\mathbb{A}^\dagger$) in Figs. 2(b) and 2(c).

Summary. This completes our classical, semiclassical, and quantum treatment of symmetry-induced DFS. It clearly reveals that the notion of nonstationary (oscillating) behavior and its equivalence to real eigenvalues of the Lindbladian applies equally to classical, semiclassical, and quantum systems, which establishes DFS for classical and semiclassical systems. For the latter, coherent oscillations in classical systems can still occur in the presence of dissipation (through the second term in (7), see [33–35] for examples), but diffusion can inhibit long-time oscillatory nonstationary motion (through the third term in (7) being of order $\hbar$). The eigenoperator of the adjoint Lindbladian corresponding to the real nonzero eigenvalues can be constructed classically, see Eq. (6), or quantum mechanically ($\mathbb{A}_{nm}$) from a symmetry. Playing the role of a beating operator A or $\mathbb{A}_{mn}$ connect different elements of the constant of motion J or $\mathbb{J}$ in form of its layers in phase space or its irreducible representations in Hilbert space, respectively. Elements $\mathbb{J}_n$ of the symmetry itself can be expressed in terms of the beating operator as $\mathbb{J}_n = \mathbb{A}^\dagger_{mn}\mathbb{A}_{mn}$. This irreducible representation also plays a crucial role in the context of breaking the ergodicity thermalization hypothesis in Hamiltonian systems [36], which is related to NLD in the absence of dissipation. For DFS, the Lindblad operators must be degenerate with respect to at least two different irreducible subspaces $n \neq m$ of the symmetry, while the hamiltonian must not be degenerate, see (2). This gives a clear picture how DFS emerge.

Guided by this insight, we have identified a new class of couplings to the environment leading to DFS, namely when the Lindblad operators do only depend on ghost operators, which span the Hilbert space but commute with the symmetry $\mathbb{J}$, as explicitly demonstrated with the Heisenberg XXZ spin model. This global realization of DFS requires Lindblad operators only linearly dependent on the spin operators, which can be achieved in cavity QED experiments much easier than what has been proposed so far theoretically based on local realizations [16,17].

In terms of the eigenoperator, DFS exist if $[\mathbb{L}_\alpha, \mathbb{A}_{nm}] = [\mathbb{L}^\dagger_\alpha, \mathbb{A}_{nm}] = 0$ [16]. Note, however, that the latter condition is not necessary since for the adjoint Lindbladian to have real eigenvalues, it follows from (1b) that only the sum $\mathbb{L}^\dagger_\alpha[\mathbb{A}, \mathbb{L}_\alpha] + [\mathbb{L}^\dagger_\alpha, \mathbb{A}]\mathbb{L}_\alpha = 0$ must vanish. How this extends the classes of possible interactions with the environment leading to DFS even further beyond those depending only on ghost operators will be a subject of further research [37].

Acknowledgments. J.D. acknowledges Andreas Buchleitner, Charlie Duclut, Christian Johansen, and Francesco Piazza for helpful discussions.

[1] C. W. Gardiner, *A Handbook of Stochastic Methods* (Springer, Berlin, 1983).

[2] H. Risken, *The Fokker-Planck equation* (Springer, Berlin, 1984).

[3] W. Gotze and L. Sjogren, Relaxation processes in supercooled liquids, Rep. Prog. Phys. **55**, 241 (1992).

[4] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).

[5] H. Qian, Open-system nonequilibrium steady state: Statistical thermodynamics, fluctuations, and chemical oscillations, J. Phys. Chem. B **110**, 15063 (2006).

[6] L. K. Grover, Quantum Mechanics Helps in Searching for a Needle in a Haystack, Phys. Rev. Lett. **79**, 325 (1997).

[7] T. D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe, and J. L. O'Brien, Quantum computers, Nature (London) **464**, 45 (2010).

[8] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2010).
[9] P. Zanardi and M. Rasetti, Noiseless Quantum Codes, Phys. Rev. Lett. **79**, 3306 (1997).
[10] D. A. Lidar, I. L. Chuang, and K. B. Whaley, Decoherence-Free Subspaces for Quantum Computation, Phys. Rev. Lett. **81**, 2594 (1998).
[11] V. V. Albert, B. Bradlyn, M. Fraas, and L. Jiang, Geometry and Response of Lindbladians, Phys. Rev. X **6**, 041031 (2016).
[12] W. Tarnowski, I. Yusipov, T. Laptyeva, S. Denisov, D. Chruściński, and K. Życzkowski, Random generators of Markovian evolution: A quantum-classical transition by superdecoherence, Phys. Rev. E **104**, 034118 (2021).
[13] S. Habib, K. Shizume, and W. H. Zurek, Decoherence, Chaos, and the Correspondence Principle, Phys. Rev. Lett. **80**, 4361 (1998).
[14] M. Mohseni, J. S. Lundeen, K. J. Resch, and A. M. Steinberg, Experimental Application of Decoherence-Free Subspaces in an Optical Quantum-Computing Algorithm, Phys. Rev. Lett. **91**, 187903 (2003).
[15] M. Friesen, J. Ghosh, M. A. Eriksson, and S. N. Coppersmith, A decoherence-free subspace in a charge quadrupole qubit, Nat. Commun. **8**, 15923 (2017).
[16] B. Buča, J. Tindall, and D. Jaksch, Non-stationary coherent quantum many-body dynamics through dissipation, Nat. Commun. **10**, 1730 (2019).
[17] J. Tindall, S. C. Muñoz, B. Buča, and D. Jaksch, Quantum synchronisation enabled by dynamical symmetries and dissipation, New J. Phys. **22**, 013026 (2020).
[18] B. Baumgartner and H. Narnhofer, Analysis of quantum semigroups with GKS-Lindblad generators: II. General, J. Phys. A: Math. Theor. **41**, 395303 (2008).
[19] V. V. Albert and L. Jiang, Symmetries and conserved quantities in Lindblad master equations, Phys. Rev. A **89**, 022118 (2014).
[20] P.-O. Löwdin, The normal constants of motion in quantum mechanics treated by projection technique, Rev. Mod. Phys. **34**, 520 (1962).
[21] The unit operators in Eqs. (4a) and (4b) follow the definition for $\mathbb{H}_n$ and $\mathbb{L}_{\alpha n}$. Thus $\mathbb{1}_n = \mathrm{tr}_n \mathbb{P}_n \mathbb{1} \mathbb{P}_n = \mathrm{tr}_n \mathbb{P}_n$ is a unit operator of dimension η_n.
[22] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevResearch.5.L012003 for the classical and semiclassical limit of the Lindbladian and its adjoint, the derivation of the ghost variables and the ghost operators for the Heisenberg spin model, details on the numerical calculations, and a graphical representation of the Lindbladian matrix structure.
[23] W. Strunz and I. C. Percival, Classical mechanics from quantum state diffusion—A phase-space approach, J. Phys. A: Math. Gen. **31**, 1801 (1998).
[24] J. Dubois, U. Saalmann, and J. M. Rost, Semi-classical Lindblad master equation for spin dynamics, J. Phys. A: Math. Theor. **54**, 235201 (2021).
[25] R. Campoamor-Stursberg, M. Gadella, S. Kuru, and J. Negro, Action-angle variables, ladder operators and coherent states, Phys. Lett. A **376**, 2515 (2012).
[26] R. de la Llave, A. González, A. Jorba, and J. Villanueva, KAM theory without action-angle variables, Nonlinearity **18**, 855 (2005).
[27] T. Prosen, Exact Nonequilibrium Steady State of a Strongly Driven Open XXZ Chain, Phys. Rev. Lett. **107**, 137201 (2011).
[28] M. Ganahl, E. Rabel, F. H. L. Essler, and H. G. Evertz, Observation of Complex Bound States in the Spin-1/2 Heisenberg XXZ Chain Using Local Quantum Quenches, Phys. Rev. Lett. **108**, 077206 (2012).
[29] W. Liu and N. Andrei, Quench Dynamics of the Anisotropic Heisenberg Model, Phys. Rev. Lett. **112**, 257204 (2014).
[30] L. Sá, P. Ribeiro, and T. Prosen, Complex Spacing Ratios: A Signature of Dissipative Quantum Chaos, Phys. Rev. X **10**, 021019 (2020).
[31] H. Cabral and F. Diacu, *Classical and Celestial Mechanics* (Princeton University Press, Princeton, 2002).
[32] H. Goldstein, *Classical Mechanics* (Addison-Wesley, London, 1980).
[33] J. Keeling, M. J. Bhaseen, and B. D. Simons, Collective Dynamics of Bose-Einstein Condensates in Optical Cavities, Phys. Rev. Lett. **105**, 043001 (2010).
[34] M. J. Bhaseen, J. Mayoh, B. D. Simons, and J. Keeling, Dynamics of nonequilibrium Dicke models, Phys. Rev. A **85**, 013817 (2012).
[35] C. S. Muñoz, B. Buča, J. Tindall, A. González-Tudela, D. Jaksch, and D. Porras, Symmetries and conservation laws in quantum trajectories: Dissipative freezing, Phys. Rev. A **100**, 042113 (2019).
[36] P. Sala, T. Rakovszky, R. Verresen, M. Knap, and F. Pollmann, Ergodicity Breaking Arising from Hilbert Space Fragmentation in Dipole-Conserving Hamiltonians, Phys. Rev. X **10**, 011047 (2020).
[37] J. Dubois, F. Piazza, U. Saalmann, and J. M. Rost (unpublished).

Symmetry-induced decoherence-free subspaces
– Supplemental material –

Jonathan Dubois, Ulf Saalmann and Jan Michael Rost

1 Lindbladian operator, its adjoint operator and its semiclassical limit

In the quantum mechanical context, the inner product between two observables $\mathbb{F}$ and $\mathbb{G}$ in the Hilbert space is given by $\langle \mathbb{F}, \mathbb{G}\rangle = \mathrm{tr}\,(\mathbb{F}^\dagger \mathbb{G})$ where $\mathbb{F}^\dagger$ denotes the complex-conjugate transpose of $\mathbb{F}$. The adjoint $\mathscr{L}^\dagger$ of the Lindbladian is defined through $\langle \mathscr{L}^\dagger \mathbb{F}, \mathbb{G}\rangle = \langle \mathbb{F}, \mathscr{L}\mathbb{G}\rangle$. The Lindbladian and its adjoint are given by [1]

$$\mathscr{L}\rho = \frac{1}{\hbar}[\mathbb{H},\rho] + \frac{\mathrm{i}}{\hbar}\sum_\alpha \big([\mathbb{L}_\alpha, \rho\mathbb{L}_\alpha^\dagger] + [\mathbb{L}_\alpha\rho, \mathbb{L}_\alpha^\dagger]\big), \tag{S1a}$$

$$\mathscr{L}^\dagger\mathbb{F} = \frac{1}{\hbar}[\mathbb{H},\mathbb{F}] - \frac{\mathrm{i}}{\hbar}\sum_\alpha \big(\mathbb{L}_\alpha^\dagger[\mathbb{F}, \mathbb{L}_\alpha] + [\mathbb{L}_\alpha^\dagger, \mathbb{F}]\mathbb{L}_\alpha\big), \tag{S1b}$$

respectively, such that we have $\dot{\rho} = -\mathrm{i}\mathscr{L}\rho$ and $\dot{\mathbb{F}} = \mathrm{i}\mathscr{L}^\dagger\mathbb{F}$. In the limit $\mathbb{L}_\alpha$=0 it is $\mathscr{L}$=$\mathscr{L}^\dagger$. In the semiclassical limit, the scalar product in the Hilbert space with phase-space variables $\mathbf{z}$ is given by $\langle F, G\rangle = \int \mathrm{d}^n_{}z\, F^*(\mathbf{z})G(\mathbf{z})$ for an n-dimensional phase space. The semiclassical limit of the Lindbladian and its adjoint are given by [2]

$$\mathscr{L}\rho = \mathrm{i}\{H,\rho\} - \sum_\alpha \Big(\big(\{L_\alpha, \rho L_\alpha^*\} + \{L_\alpha\rho, L_\alpha^*\}\big) - \frac{\mathrm{i}\hbar}{2}\big(\{\{L_\alpha,\rho\}, L_\alpha^*\} + \{L_\alpha, \{\rho, L_\alpha^*\}\}\big)\Big), \tag{S2a}$$

$$\mathscr{L}^*F = \mathrm{i}\{H,F\} + \sum_\alpha \Big(\big(L_\alpha^*\{F, L_\alpha\} + \{L_\alpha^*, F\}L_\alpha\big) + \frac{\mathrm{i}\hbar}{2}\big(\{\{L_\alpha,F\}, L_\alpha^*\} + \{L_\alpha, \{F, L_\alpha^*\}\}\big)\Big). \tag{S2b}$$

The Lindbladian can always be cast into a Fokker-Planck equation [2, 3].

2 Heisenberg spin model

We provide details of the classical construction of ghost variables in order to define environments that allow for decoherence-free subspaces for the Heisenberg model presented in the text.

Classical spin algebra

We consider a chain of N spins with in the classical limit is described by the spin variables $\mathbf{S}_i = S_i^x\mathbf{e}_x + S_i^y\mathbf{e}_y + S_i^z\mathbf{e}_z$ with $i = 1\ldots N$ and the spin algebra

$$\{S_i^\alpha, S_j^\beta\} = \delta_{ij}\varepsilon_{\alpha\beta\gamma}S_i^\gamma \tag{S3}$$

with $\varepsilon_{\alpha\beta\gamma}$ the Levi-Civita symbol. The corresponding non-canonical Poisson bracket is given by

$$\{F,G\} = \sum_{i=1}^N \mathbf{S}_i \cdot \frac{\partial F}{\partial \mathbf{S}_i} \times \frac{\partial G}{\partial \mathbf{S}_i}. \tag{S4a}$$

Equivalently, we can describe the dynamics with the beating variables $S_i^\pm = S_i^x \pm \mathrm{i}S_i^y$, for which the Poisson bracket reads

$$\{F,G\} = \sum_{i=1}^N \Big(-2\mathrm{i}S_i^z\Big(\frac{\partial F}{\partial S_i^+}\frac{\partial G}{\partial S_i^-} - \frac{\partial F}{\partial S_i^-}\frac{\partial G}{\partial S_i^+}\Big) + \mathrm{i}S_i^+\Big(\frac{\partial F}{\partial S_i^+}\frac{\partial G}{\partial S_i^z} - \frac{\partial F}{\partial S_i^z}\frac{\partial G}{\partial S_i^+}\Big) - \mathrm{i}S_i^-\Big(\frac{\partial F}{\partial S_i^-}\frac{\partial G}{\partial S_i^z} - \frac{\partial F}{\partial S_i^z}\frac{\partial G}{\partial S_i^-}\Big)\Big), \tag{S4b}$$

with the fundamental spin algebra $\{S_i^z, S_i^\pm\} = \mp \mathrm{i} S_i^\pm$ and $\{S_i^+, S_i^-\} = -2\mathrm{i} S_i^z$. Note that in both cases, the classical Poisson bracket is non-canonical and is related to the quantum Lie bracket by $[\mathbb{F}, \mathbb{G}] \equiv \mathrm{i}\hbar\{F, G\}$.

Cylindrical coordinates and conserved quantity

In cylindrical coordinates (θ_i, S_i^z) it is $\mathbf{S}_i = \sqrt{S_i{}^2 - (S_i^z)^2}\,(\mathbf{e}_x \cos\theta_i + \mathbf{e}_y \sin\theta_i) + S_i^z \mathbf{e}_z$ or equivalently $S_i^\pm = \sqrt{S_i{}^2 - (S_i^z)^2}\,\exp(\pm \mathrm{i}\theta_i)$ with $S_i{}^2 = |\mathbf{S}_i|^2$ a Casimir invariant. Therewith the Hamiltonian (in the semiclassical limit) reads

$$H(\theta_i, S_i^z) = \sum_{i=1}^{N} \left(\omega_i S_i^z + \sum_{j=1}^{N} \left(2\sigma_{ij} \sqrt{[S_i^2 - (S_i^z)^2][S_j^2 - (S_j^z)^2]} \cos\left(\theta_i - \theta_j\right) + \Delta_{ij} S_i^z S_j^z \right) \right), \tag{S5}$$

and the Poisson bracket becomes

$$\{F, G\} = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial \theta_i} \frac{\partial G}{\partial S_i^z} - \frac{\partial F}{\partial S_i^z} \frac{\partial G}{\partial \theta_i} \right). \tag{S6}$$

From the Hamiltonian (S5), it is clear that

$$J = \frac{1}{N} \sum_{i=1}^{N} S_i^z, \tag{S7}$$

is a conserved quantity due to the invariance under a rotation around $\mathbf{e}_z$. The variable canonically conjugate to J such that $\{\theta, J\} = 1$ is given by

$$\theta = \sum_{i=1}^{N} \theta_i, \tag{S8}$$

corresponding to the total angle. There is some freedom for choosing the set of canonically-conjugate variables from there and therefore the *ghost variables*. Indeed, note that one can transform one set of ghost variables into another one by canonical transformations. The most natural starting point are the Jacobi coordinates for the celestial many-body problem $\Delta\theta_i = \theta_{i+1} - \theta_i$. It is clear that $\{\Delta\theta_i, J\} = \{\Delta\theta_i, \theta\} = 0$ for $i = 1 \dots N-1$. The goal is now to find the set of variables $\widetilde{S}_i^z$ canonically conjugate to $\Delta\theta_i$, i.e., $\{\Delta\theta_i, \widetilde{S}_j^z\} = \delta_{ij}$ with δ_{ij} the Kronecker delta.

Ghost variables using canonical transformations

For doing so, we use the F_2 generating function [4]. We recall that for a set of canonically conjugate variables $(\boldsymbol{\theta}, \mathbf{S}^z)$ the new set of canonically conjugate variables $(\widetilde{\boldsymbol{\theta}}, \widetilde{\mathbf{S}}^z)$ can be found by using transformations, such that $F_2(\boldsymbol{\theta}, \widetilde{\mathbf{S}}^z)$, such that

$$\mathbf{S}^z = \frac{\partial F_2}{\partial \boldsymbol{\theta}}, \qquad \widetilde{\boldsymbol{\theta}} = \frac{\partial F_2}{\partial \widetilde{\mathbf{S}}^z}. \tag{S9}$$

Here, the old variables are given by $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N)$ and $\mathbf{S}^z = (S_1^z, \dots, S_N^z)$. The new variables are $\widetilde{\boldsymbol{\theta}} = (\Delta\theta_1, \dots, \Delta\theta_{N-1}, \theta)$ and $\widetilde{\mathbf{S}}^z = (\widetilde{S}_1^z, \dots, \widetilde{S}_{N-1}^z, J)$. Given the form of the new variables $\widetilde{\boldsymbol{\theta}}$ which we have imposed, the generating function is given by

$$F_2(\boldsymbol{\theta}, \widetilde{\mathbf{S}}^z) = J \sum_{i=1}^{N} \theta_i + \sum_{i=1}^{N-1} \widetilde{S}_i^z (\theta_{i+1} - \theta_i). \tag{S10}$$

Indeed, from (S9) and (S10), we obtain that $\widetilde{\theta}_0 \equiv \theta = \sum_{i=1}^{N} \theta_i$ and $\widetilde{\theta}_i \equiv \Delta\theta_i = \theta_{i+1} - \theta_i$ for $i = 1 \ldots N{-}1$. The expression of the old momenta with respect to the new ones are found using (S9) as

$$S_1^z = J - \widetilde{S}_1^z, \qquad S_i^z = J + \widetilde{S}_{i-1}^z - \widetilde{S}_i^z, \qquad S_N^z = J + \widetilde{S}_{N-1}^z, \tag{S11}$$

for $i = 2 \ldots N{-}1$. We can easily check that (S7) is fulfilled. In order to invert this transformation and obtain the expression of the ghost variables with respect to S_i^z, a convenient way is to write it in a matrix form

$$\begin{bmatrix} S_1^z \\ S_2^z \\ S_3^z \\ \vdots \\ S_{N-2}^z \\ S_{N-1}^z \\ S_N^z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & \ldots & 0 & 0 & 1 \\ 1 & -1 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 1 & -1 & \ldots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 1 & -1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{S}_1^z \\ \widetilde{S}_2^z \\ \vdots \\ \widetilde{S}_{N-3}^z \\ \widetilde{S}_{N-2}^z \\ \widetilde{S}_{N-1}^z \\ J \end{bmatrix}. \tag{S12}$$

The inverse transformation is given by

$$\begin{bmatrix} \widetilde{S}_1^z \\ \widetilde{S}_2^z \\ \vdots \\ \widetilde{S}_{N-3}^z \\ \widetilde{S}_{N-2}^z \\ \widetilde{S}_{N-1}^z \\ J \end{bmatrix} = \frac{1}{N} \begin{bmatrix} -N{+}1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ -N{+}2 & -N{+}2 & 2 & \ldots & 2 & 2 & 2 \\ -N{+}3 & -N{+}3 & -N{+}3 & \ldots & 3 & 3 & 3 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ -2 & -2 & -2 & \ldots & -2 & N{-}2 & N{-}2 \\ -1 & -1 & -1 & \ldots & -1 & -1 & N{-}1 \\ 1 & 1 & 1 & \ldots & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} S_1^z \\ S_2^z \\ S_3^z \\ \vdots \\ S_{N-2}^z \\ S_{N-1}^z \\ S_N^z \end{bmatrix}. \tag{S13}$$

Therefore, the ghost variables are given by the Jacobi momentum coordinates

$$\widetilde{S}_k^z = \frac{k}{N} \sum_{i=1}^{N} S_i^z - \sum_{i=1}^{k} S_i^z, \qquad \Delta\theta_k = \theta_{k+1} - \theta_k, \qquad k = 1, \ldots, N{-}1 \tag{S14}$$

as given in the main text. One can easily check that the transformation is canonical with $\{\theta, J\} = 1$, $\{\Delta\theta_i, \widetilde{S}_j^z\} = \delta_{ij}$ and $\{\theta, \widetilde{S}_i^z\} = \{J, \widetilde{S}_i^z\} = \{\theta, \Delta\theta_i\} = \{J, \Delta\theta_i\} = \{\widetilde{S}_i^z, \widetilde{S}_j^z\} = \{\Delta\theta_i, \Delta\theta_j\} = 0$ for all i and j. In the same way, we find the change of coordinates for the angles and relative angles.

In this new set of variables, the Hamiltonian $H(J, \Delta\theta_i, \widetilde{S}_i^z)$ replacing the one in Eq. (S5) is independent of the angle θ.

Beating variables and transformation back to Cartesian spin algebra

As mentioned in the main text, from canonical variables associated to a conserved quantity (θ, J), one can construct beating observables $A = f(J)\exp(-\mathrm{i}\theta)$ and $A^* = f(J)\exp(\mathrm{i}\theta)$. One can always perform the inverse change of coordinates from (S6) to (S4) by going back to a spin algebra. For instance, using $f(J){=}1$ we find the beating variables

$$A = \prod_{i=1}^{N} \frac{S_i^-}{\sqrt{S_i{}^2 - (S_i^z)^2}}, \qquad A^* = \prod_{i=1}^{N} \frac{S_i^+}{\sqrt{S_i{}^2 - (S_i^z)^2}}. \tag{S15}$$

In the previous paragraph, we have described the canonical transformation from (θ_i, S_i^z) to variables $(\theta, J, \Delta\theta_i, \widetilde{S}_i^z)$. One can always go back to a spin representation by performing the inverse change of coordinates from (S6) to (S4). Here, this change of coordinates reads

$$\widetilde{S}_N^{\pm} = \sqrt{\widetilde{S}_N{}^2 - J^2}\exp(\pm\mathrm{i}\theta), \qquad \widetilde{S}_N^z = J, \tag{S16a}$$

$$\widetilde{S}_i^{\pm} = \sqrt{\widetilde{S}_i{}^2 - (\widetilde{S}_i^z)^2}\exp(\pm\mathrm{i}\Delta\theta_i), \qquad \widetilde{S}_i^z = \widetilde{S}_i^z, \qquad i = 1,\dots,N-1, \tag{S16b}$$

which allows us to obtain the spin algebra (S6). Note that $A \propto \widetilde{S}_N^-$ and $A^* \propto \widetilde{S}_N^+$ corresponding to the beating variables associated with the conserved quantity J introduced in the main text.

To summarize, we started with a Hamiltonian in terms of a spin algebra in the main text. Switching to canonical variables we obtained Hamiltonian (S5). Then, using a canonical transformation, we found the new coordinates which consist of the DOFs of the collective spin variables (θ, J) and the ghost variables $(\Delta\theta_i, \widetilde{S}_i^z)$. From (S16), the expression of the ghost variables with respect to the initial spin variables read by means of the abbreviations $S_i^\rho \equiv \sqrt{S_i^2 - (S_i^z)^2}$ and $\widetilde{S}_i^\rho \equiv \sqrt{\widetilde{S}_i^2 - (\widetilde{S}_i^z)^2}$ (with Casimir invariants $\widetilde{S}_i^2 = |\widetilde{\mathbf{S}}_i|^2$)

$$\widetilde{S}_N^z = \frac{1}{N}\sum_{i=1}^N S_i^z, \qquad \widetilde{S}_N^{\pm} = \frac{\widetilde{S}_N^\rho}{\prod_{i=1}^N S_i^\rho}\prod_{i=1}^N S_i^{\pm}, \tag{S17a}$$

$$\widetilde{S}_k^z = \frac{k}{N}\sum_{i=1}^N S_i^z - \sum_{i=1}^k S_i^z, \qquad \widetilde{S}_k^{\pm} = \frac{\widetilde{S}_k^\rho}{S_{k+1}^\rho S_k^\rho} S_{k+1}^{\pm} S_k^{\mp}, \qquad k = 1,\dots,N-1. \tag{S17b}$$

This transformation is canonical, in the sense that it preserves the form of the Poisson bracket (S3) for the spin algebra. The beating variables associated with the conserved quantity J are therefore $A = \widetilde{S}_N^-$. We go from $\{S_n^i, S_m^j\} = \varepsilon_{ijk} S_m^k \delta_{nm}$, to $\{\widetilde{S}_n^i, \widetilde{S}_m^j\} = \varepsilon_{ijk}\widetilde{S}_m^k\delta_{nm}$. Or equivalently, from $\{S_n^{\pm}, S_m^z\} = \pm\mathrm{i}S_n^{\pm}\delta_{nm}$ and $\{S_n^+, S_m^-\} = -2\mathrm{i}S_n^z\delta_{nm}$ to $\{\widetilde{S}_n^{\pm}, \widetilde{S}_m^z\} = \pm\mathrm{i}\widetilde{S}_n^{\pm}\delta_{nm}$ and $\{\widetilde{S}_n^+, \widetilde{S}_m^-\} = -2\mathrm{i}\widetilde{S}_n^z\delta_{nm}$. Therefore, the new spin variables $\widetilde{\mathbf{S}}_i$ for $i = 1,\dots,N-1$ are *ghost variables* which do not affect the DOF associated with the conserved quantity $\widetilde{\mathbf{S}}_N$.

Ghost and beating operators

From (S17a), we upgrade the beating and ghost variables to their quantum mechanical counterparts, the beating and ghost operators. To do so, we replace the variables by their associated operators (which are matrices) in the expressions (S17a). We first notice that the quantities $\mathbb{S}_i^\rho$, in the basis of the operators $\mathbb{S}_i^z$, are diagonal matrices since $\mathbb{S}_i$ is proportional to identity (Casimir invariant) and $\mathbb{S}_i^z$ is diagonal. This allows us to compute the square roots and inverse of the operators $\mathbb{S}_i^\rho$. In addition, these operators commute with each other. For the Heisenberg spin model, we find a simplified version of a beating operator

$$\mathbb{A} = \prod_{i=1}^N \mathbb{S}_i^-, \tag{S18}$$

since these operators are defined up to a factor. The expectation value of this operator oscillates in time. From the classical ghost variables (S17b), we obtain the ghost operators in the main text

$$\widetilde{\mathbb{S}}_k^z = \frac{k}{N}\sum_{i=1}^N \mathbb{S}_i^z - \sum_{i=1}^k \mathbb{S}_j^z \qquad \widetilde{\mathbb{S}}_k^{\pm} = \mathbb{D}_k\, \mathbb{S}_{k+1}^{\pm}\mathbb{S}_k^{\mp}, \qquad \mathbb{D}_k = \frac{\widetilde{\mathbb{S}}_k^\rho}{\mathbb{S}_{k+1}^\rho\,\mathbb{S}_k^\rho}. \tag{S19}$$

As mentioned above, all $\mathbb{S}_k^\rho$ are diagonal, which makes the expression for $\mathbb{D}_k$ unique.

Parameters for the Heisenberg model presented in the text

In Figs. 1 and 2, the self-energy ω_i of each spin site i has been generated randomly. The actual values and the matrices for nearest-neighbors interactions, cf. Eq. (6) in the text, are

$$[\omega_i] = \begin{bmatrix} 0.6358 \\ 0.9452 \\ 0.2089 \\ 0.7093 \\ 0.2362 \\ 0.1194 \end{bmatrix}, \quad [\sigma_{ij}] = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad [\Delta_{ij}] = \frac{1}{4}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{S20}$$

As initial condition we have taken $\rho(0) = |\psi\rangle\langle\psi|$ with

$$|\psi\rangle = \frac{\sum_{k_1\ldots k_6} |k_1 \ldots k_6\rangle\, C_{k_1\ldots k_6}}{\sqrt{\sum_{k_1\ldots k_6} C_{k_1\ldots k_6}{}^2}} \quad \text{where} \quad C_{k_1\ldots k_6} = \sum_{\xi=\pm 63} \exp\left(-\left[\textstyle\sum_j k_j\, 2^j - \xi\right]^2/128\right) \tag{S21}$$

and $k_j = \{-\frac{1}{2}, +\frac{1}{2}\}$ and thus 64 different states $|k_1 \ldots k_6\rangle$.

3 Lindbladian strcuture

In order to illustrate the block structure of the adjoint Lindbladian $\mathscr{L}^\dagger$ we show in Fig. S1 a graphical representation of it, with blueish/reddish colors for negative/positive matrix elements. As example we use the Heisenberg model discussed in the text, cf. Eq. (8), for $N=3$ with a supermatrix size of 64×64, which allows for a reasonable presentation.

The block structure according to the symmetry $\mathbb{J}$ is highlighted by black lines. As $\mathscr{L}^\dagger$ obeys this symmetry, all non-diagonal blocks do vanish. The square-shaped diagonal blocks are those from Eq. (3). The blocks, as defined in Eqs. (4a,b) and used in the conditions (2) for DFS, are marked by circles. Indeed, one sees that $\Delta\mathbb{H}_{nm} \neq 0$ and $\Delta\mathbb{L}_{anm} = 0$.

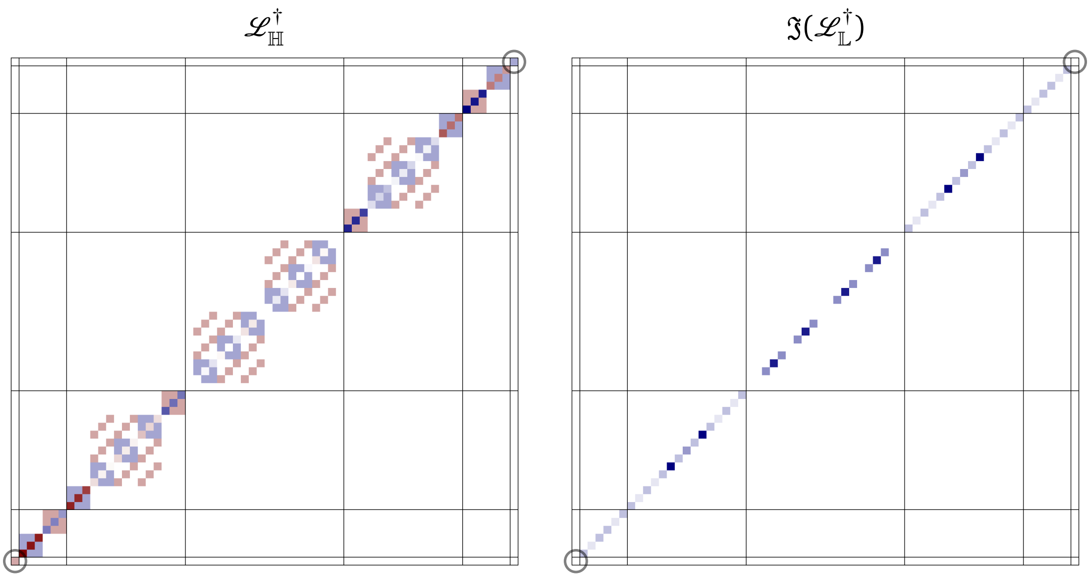

Figure S1: Graphical representation of the adjoint Lindbladian $\mathscr{L}^\dagger$ of the Heisenberg model for $N=3$ with Lindblad operators $\mathbb{L}_\alpha = \gamma\widetilde{\mathbb{S}}^z_\alpha$ for $\alpha = 1\ldots N$.

[1] V. V. Albert and L. Jiang, *Symmetries and conserved quantities in Lindblad master equations.* Phys. Rev. A **89**, 022118 (2014).

[2] J. Dubois, U. Saalmann, and J. M. Rost, *Semi-classical Lindblad master equation for spin dynamics.* J. Phys. A: Math. Theor. **54**, 235201 (2021).

[3] W. Strunz and I. C. Percival, *Classical mechanics from quantum state diffusion — a phase-space approach.* J. Phys. A: Math. Gen. **31**, 1820 (1998).

[4] H. Goldstein, *Classical mechanics* (Addison-Wesley, 1980).