Antispiral waves in reaction-diffusion systems

In a recent Letter, Gong and Christini [1] provide evidence for the emergence of inwardly rotating spiral waves (antispiral waves) in simulations of oscillatory reaction-diffusion (RD) models. They present a theoretical argument within the complex Ginzburg-Landau equation (CGLE)

\[ \partial_t W = W - (1 + \alpha \beta) W W^2 + (1 + i \beta \gamma) \nabla^2 W, \]

that is valid near a supercritical Hopf bifurcation [2]. Therein, it is stated that antispiral waves (AS) occur for \( \alpha \beta > 0 \) in the parameter regions \( \alpha > \beta > 0 \) and \( 0 \geq \beta > \alpha \) of the CGLE. The authors of [1] suggest to derive the CGLE for a given RD model and use the latter criteria to decide whether spiral or antispiral waves may appear. This Comment clarifies two crucial points not treated in [1]: (i) AS are characterized by opposite signs of the radial phase and group velocities; Starting from this property, it follows that their rotation frequency is smaller than the bulk oscillation frequency. (ii) Regions for AS in the CGLE and in the corresponding RD model are different; the correct CGLE criterion for occurrence of AS in a RD model is \( \alpha < \beta \) in contrast to the prediction in [1].

Consider a general RD model in two dimensions (2D)

\[ \partial_t \mathbf{u} = \tilde{f}(\mathbf{u}, \mu) + D \nabla^2 \mathbf{u}, \]

where \( \mathbf{u}(\mathbf{x}, \tau) \) is a vector of space- and time-dependent concentrations and \( \mu \) a control parameter. Near a supercritical Hopf bifurcation of a uniform state \( \mathbf{u}_0 \), defined by \( \tilde{f}(\mathbf{u}_0) = 0 \) and with critical frequency \( \Omega \) and eigenvector \( \mathbf{u}_1 \) the vector of concentrations \( \mathbf{u} \) may be decomposed as \( \mathbf{u}(\mathbf{x}, \tau) = \mathbf{u}_0 + \mathbf{u}_1 A(\mathbf{x}, t) e^{i \Omega \tau} + \mathbf{u}_1^* A^*(\mathbf{x}, t) e^{-i \Omega \tau} \). The CGLE Eq. (1) describes the evolution of slow modulations \( A(\mathbf{x}, t) = \sqrt{\Omega} A(\mathbf{x}, t) e^{i \Omega \tau} \) of a homogeneous oscillation; its coefficients have to be derived from Eq. (2). Here, \( \epsilon = (\mu - \mu_c)/\mu_c \ll 1 \) measures the distance from threshold and \( \epsilon_0 \) an overall frequency shift [2]. The CGLE coordinates are \( \mathbf{x} = \sqrt{\epsilon} \mathbf{x} \) and \( \tau = \epsilon \tau \). Note, that frequencies \( \omega \) in the CGLE result only in a small correction of order \( \epsilon \omega \) to the original frequency \( \Omega \).

Spiral waves of the 2D CGLE have the form \( W(\mathbf{r}, \theta, t) = F(r) \exp(i(\theta + f(r, t))) \) in polar coordinates \( (r, \theta) \). For \( r \to \infty \) the radial dynamics follow \( F'(r) = \sqrt{1 - k_S^2} \) and \( f(r, t) \to k_S t - \omega_S t = k_S (r - v_{ph} t) \) with a selected wavenumber \( k_S \) uniquely determined by \( \alpha, \beta \) and a frequency \( \omega_S = \alpha + (\beta - \alpha) k_S^2 \). Hagan constructed a nonlinear eigenvalue problem for \( k_S(\alpha, \beta) \) [3], which can be solved numerically. Kramer and coworkers showed that the \( k_S \) found analytically by Hagan in 1D agrees well with the \( k_S \) selected by spirals in numerical simulations of the 2D CGLE [4]. The results of [3, 4] indicate that \( k_S = 0 \) if \( \alpha = \beta \) and \( k_S > 0 (k_S < 0) \) if \( \alpha < \beta (\alpha > \beta) \). The group velocity \( v_{gr} = d\omega_S/dk_S = 2(\beta - \alpha) k_S \) is thus always non-negative. The phase velocity \( v_{ph} = \omega_S/k_S \) can change sign, if either \( k_S \) or \( \omega_S \) change sign. Phase waves travel outward (inward) corresponding to \( v_{ph} > 0 \) (\( v_{ph} < 0 \)) in the CGLE. Consequently, \( v_{ph} > 0 \) corresponds to spirals and \( v_{ph} < 0 \) to AS in the CGLE. The two curves \( \omega_S = 0 \) and \( k_S = 0 \) separate regions where antispirals and spirals appear (see Fig. 1). \( v_{ph} \) diverge for \( k_S \to 0 \), while \( v_{ph} \to 0 \) for \( \omega_S = 0 \). Fig. 1 recovers the CGLE results of [1] extend them to regions with \( \alpha \beta < 0 \).

In the original RD model, the group velocity \( \bar{v}_{gr} = \sqrt{\epsilon} v_{gr} \geq 0 \). For the phase velocity of waves in the coordinate \( \bar{r} = v_{ph} \bar{r} \), one obtains \( \bar{v}_{ph} = (-\Omega + \epsilon(\omega_S - \epsilon_0))/k_S \approx -\Omega/(\sqrt{\epsilon} k_S) \). Hence, \( \bar{v}_{ph} \) changes sign only where \( k_S \) changes sign, i.e. \( \epsilon \). Any \( \omega_S \) in the CGLE is compensated by the fast frequency \( \Omega \); thus, an AS in the CGLE can represent a spiral in the corresponding RD model and vice versa (see Fig. 1). Altogether, AS occur in RD systems for positive \( k_S \), i.e. for \( \alpha < \beta \). The frequency of AS becomes \( \Omega_{AS} = \Omega_0 - k_S^2 (\beta - \alpha) \) and is smaller than the bulk frequency \( \Omega_0 \) as has been first noted empirically in experiments exhibiting AS in a chemical reaction [5].

![FIG. 1: Parameter space (\( \alpha, \beta \)) of the CGLE is separated by the curves \( \omega_S = 0 \) and \( k_S = 0 \) in four subdomains. Bold (italic) text indicates behavior in the RD model (corresponding CGLE).](https://example.com/fig1.png)

Lutz Brusch, Ernesto M. Nicola, and Markus Bär
Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Str. 38, 01187 Dresden, Germany