Core breakup of spiral waves caused by radial
dynamics: Eckhaus and finite wavenumber
instabilities

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Abstract. In this article, we link numerical observations of spiral breakup to a
stability analysis of simple rotating spirals. We review the phenomenology of spiral
breakup, important applications in pattern formation and the state of the art in
numerical stability analysis of spirals. A strategy for the latter procedure is suggested.
Phenomenologically, spiral breakup can occur near the center of rotation („core
breakup“) or far away from it („far-field breakup“). It may be accompanied by
instabilities of the spiral core in particular spiral meandering that affect also the
stability of waves in the far-field, because an unstable core acts as a moving source
and introduces a (nonlinear) Doppler effect. Here, we report two cases of core breakup
which are caused in essence by radial instabilities of periodic waves far away from the
spiral core. Meandering may add to the instability, but turns out to be not crucial
for the issue, whether the spiral breaks near the core or in the far-field. In general,
spiral breakup is often related to an instability of planar wave train with the same
wave number. To simplify the stability problem, we consider a one-dimensional source
(„1D spirals“) with a fixed core position in simulations and compare the results to a
stability analysis of planar wave trains on a ring. These 1D spirals approximate the
radial dynamics of a nonmeandering two dimensional spiral well in the cases studied
here. To fully account for instabilities of 1D spirals, it is necessary to compute the
direction of propagation of the unstable modes of the wave trains. We carry out this
program for the case of core breakup in an excitable reaction-diffusion system, the
modified Barkley model. Our analysis yields that core breakup can result from a novel
finite wavenumber instability of the radial dynamics where the critical perturbations
are transported towards the core or alternatively from an absolute Eckhaus instability,
where the perturbations travel away from the core with only a very small group velocity.
From these results we conjecture that a simple spiral breaks up if the wave train in the
far-field is unstable to modes propagating in the direction opposite to the wave train,
i. e. towards the core region of the spiral.

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1. Introduction

Rotating spiral waves are frequently observed in homogeneously and heterogeneously catalyzed chemical reactions [1, 2] and various biological systems, namely slime mold aggregation [3], cardiac tissue [4, 5] and calcium waves in frog eggs [6]. In the dissipative (quasi two-dimensional) chemical systems the rotating spirals appear as iso-concentration contours of the spatial distribution of the reactants. The temporal evolution of the concentration patterns has been modelled by partial differential equations of reaction-diffusion type. Such models include oscillatory, excitable or bistable systems with either none, one or two linearly stable homogeneous states [7, 8].

Patterns form either due to the instability of a steady state in oscillatory media or due to suprathreshold, finite amplitude perturbations of homogeneous stable steady states in bistable and excitable media. In the latter case, they have a stable rest state and respond to a suprathreshold perturbation with an amplification followed by saturation and recovery. The existence of an excitation threshold is a generic feature. A popular model for oscillatory systems is the complex Ginzburg-Landau equation (CGLE) [9] that is suited to describe media near a supercritical Hopf bifurcation [10]. So called activator-inhibitor systems are widely used to model excitable media [11].

In this paper, we analyze instabilities of rotating spirals that lead to spatiotemporally chaotic dynamics [12] via spiral breakup. The problem has received considerable attention for various reasons. For instance, the observation of “defect-mediated” turbulence in numerical simulations of the CGLE has been attributed to the breakup of spirals [13, 14, 15, 16, 17, 18]. Until the early nineties, it was unclear if spatiotemporal chaos or irregular activity is possible in homogeneous two-dimensional (2D) continuous excitable media. The first observation of chaotic wave patterns were reported in discrete models [19, 20]. The availability of faster computers led to the discovery that various models exhibit spatiotemporal chaos. Among them are models of cardiac tissue [4, 21, 22, 23, 24] and activator-inhibitor models of FitzHugh-Nagumo type designed to capture essential aspects of pattern formation [25, 26, 27, 28].

An important motivation for the study of excitable media has been the quest for the cause of irregular electrical activity in cardiac muscle [29]. Experiments in thin sheets of heart tissue displayed only stable spirals in contrast to the irregular activity seen in experiments with whole hearts [30]. Consequently, it has been suggested that irregular activity in the heart might be a genuinely three-dimensional phenomenon [31]. Thus more realistic three-dimensional, anisotropic models of the heart and excitable media have been investigated and revealed various sources of irregular activity on the surface including intricate dynamics of scroll waves [32, 33] and the analogue of breakup in three dimensions [34, 35]. The reason for the onset of ventricular fibrillation as well as possible treatments still remains a subject of intense experimental and theoretical research [36]. In pattern forming chemical reactions, progress in the design of open reactors has finally also yielded experimental results that demonstrate a controlled transition to spatiotemporal chaos via spiral breakup in the Belousov-Zhabotinsky reaction [37].
Additional examples of transitions from spiral patterns to irregular spatial organization have been also reported in catalytic surface reactions [38].

In reaction-diffusion media, two fundamentally different mechanisms of spiral breakup have been observed. Spirals can break because the waves emitted from the spiral’s center (core) are either destabilized by transverse perturbations that appear only for fast inhibitor diffusion [39, 40, 41, 42] or by unstable modes in the radial direction [4, 21, 22, 23, 25, 26, 27, 28, 43, 44, 45]. A third possibility can arise from the destabilization of the core’s location (meandering). In what follows we shall concentrate only on destabilization against modes in the radial direction.

It is important to note that in all mentioned examples of radial spiral breakup in models and experiments two different scenarios are observed – spirals may break first close to their core or alternatively far away from the core [43]. Breakup near the core is found in simulations in excitable media [22, 25, 27] and in experiments with a chemical reaction [46], while breakup far away from the core is typically seen under oscillatory conditions both in chemical experiments [37] and in simulations of the CGLE [16, 17]. One exception from this rule of thumb is provided by recent studies of a model of intracellular calcium waves where breakup far away from the core is observed under excitable conditions [47].

Here, we will present results on a simple activator-inhibitor model that exhibits both types of breakup depending on the chosen control parameters [27, 43]. After we have discussed the phenomenology of breakup in this model we will turn to a stability analysis of periodic waves in two variants of the model. The rationale of the treatment of periodic waves is as follows: stable rotating spirals do emit periodic waves, and far away from the core the concentration patterns resemble planar periodic wavetrains. If these wavetrains are unstable, then the spirals may also disappear (breakup).

For the CGLE it is possible to compute the stability of periodic wave solution analytically, because the waves and the corresponding eigenfunctions have the form of Fourier-modes [15]. For given parameters, periodic waves in the CGLE become unstable below a certain wavelength through an Eckhaus instability. If the spatially homogeneous oscillations becomes unstable to spatial perturbations, one uses the term Benjamin-Feir instability. Eckhaus instabilities first appear through long wavelength perturbations that lead to spatially slowly varying modulations of the periodic waves. In the nonlinear stage, Eckhaus unstable wavetrains may evolve to the modulated amplitude waves (MAWs). In the CGLE, the Eckhaus instability is of convective nature and several groups have pointed out that spiral breakup requires an absolute instability of the emitted wave train [15, 16, 17].

In excitable media, stability analysis has been originally restricted to the kinematic treatment of the phenomenon of alternans [44, 45]. Alternans is a period doubling instability of periodic waves, wherein the width of excitation pulses oscillates between two different values. It has been identified to play an important role in the process of breakup in models of cardiac tissue like the Beeler-Reuter model and some simplifications [4, 22]. Analytical approaches are also successful in the case of very sharp interfaces resp.
large separation of time-scales of the activator and inhibitor [48]. When an analytic stability analysis is not feasible, then the numerical analysis of the one-dimensional asymptotic wavetrain provides important insight into the basic mechanism of spiral breakup. It has been shown that the stability properties of 2D spirals and periodic 1D waves are identical up to a transformation of the imaginary parts of the eigenvalues [49].

Potential discrepancies may arise from nonlinear behaviour not captured by the linear stability analysis or ingredients of the 2D dynamics like spiral meandering [8, 50] that do not have a counterpart in one dimension. Numerical stability analysis has been crucial in the understanding of the meander instability of rotating spirals. It was identified as a Hopf bifurcation that introduces a second frequency apart from the rotation frequency in the spiral movement [51]. The spiral tip no longer follows a circular trajectory during spiral meandering, but instead gives rise to flowerlike trajectories. However, numerical stability analysis in two dimensions is computationally expensive and in practice often restricted to rather small domain sizes (typically up to system sizes of a few spiral wavelength) as well as to the iterative computation of the largest eigenvalues.

The paper is organized as follows. In Section 2, we introduce a modified version of the Barkley model which exhibits spiral breakup and describe the different phenomenologies of spiral breakup arising therein. The main distinction between far-field and core breakup is also reproduced in a 1D sources with stationary „core“ position. In Section 3, we perform numerical stability analysis of planar wave trains and compare the results to the observations in simulations of 1D sources. Two main reasons for the spiral instability are found: The absolute Eckhaus instability where the perturbations travel away from the spiral core and a novel finite wavelength instability where perturbations travel toward the core. The latter instability causes spiral breakup near the core, while the absolute Eckhaus instability produces far-field breakup as well as core breakup depending on the magnitude of the group velocity of the outward propagating fastest growing modes. Finally, we summarize and discuss the results in Section 4.

2. Phenomenologies of Spiral Breakup

To study spiral breakup, we use here a reaction-diffusion model that was originally introduced by Barkley [52] but with modified reaction kinetics [53]. It describes the interaction of a fast activator \( u \) and a slow inhibitor \( v \) variable:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= - \frac{1}{\epsilon} u(u-1) (u - \frac{b+v}{a}) + \Delta u, \\
\frac{\partial v}{\partial t} &= h(u) - v, \\
\end{align*}
\]

\[
h(u) = \begin{cases} 
0, & 0 \leq u < 1/3 \\
1 - 6.75 u(u-1)^2, & 1/3 \leq u \leq 1 \\
1, & 1 < u 
\end{cases}
\]
The form of $h(u)$ describes a delayed production of the inhibitor and the equations have been used to model patterns in a catalytic surface reaction [53]. The change of $h(u)$ from the standard choice $h(u) = u$ [52] lead to the possibility of spatiotemporal chaos in 2D due to spiral breakup for $\epsilon > \epsilon_{BU}$ [27, 54]. For smaller $\epsilon$, the system settles into stable rotating resp. traveling waves.

The parameter choice $a < 1$ yields excitable (oscillatory) behaviour for $b > 0$ ($b < 0$) and $0 < \epsilon \ll 1$. In both cases, an unstable focus exists with $(u, v) = (u_0, v_0)$ in the local dynamics of equations (1). In the excitable case, two more fixed points appear: $(u, v) = (0, 0)$ is the stable rest state and $(u, v) = (b/a, 0)$ is a saddle that marks the threshold of the excitable medium. Throughout this study, $a$ is fixed to 0.84 and $b$ and $\epsilon$ are varied.

Zero-flux boundary conditions have been employed for simulations in 2D. We complement these by simulations of 1D sources where a fixed boundary condition (Dirichlet) is chosen such that it emits a wave of identical wavelength as that measured in 2D simulations. This intermediate step between simulations in 2D and stability analysis of periodic waves in 1D enables us to discriminate between contributions from the spiral’s core (cause discrepancies between 1D and 2D simulations) and its radial dynamics (agreement of 1D and 2D simulations). Furthermore, the influence of secondary spirals is suppressed, which in 2D are created in the breakup process of the initial main spiral. The simulations of 1D sources may also reveal possible nonlinear effects as discrepancies to the results of the linear stability analysis. Moreover, the linear stability analysis is performed in 1D systems with periodic boundary conditions. We perform these three steps of our strategy for a large number of values of the control parameters.

In the following, results of numerical simulations for the model are described. We have investigated the dynamics of spiral waves as a function of the excitation threshold parameter $b$ and the ratio $\epsilon$ of the timescales of the local dynamics. Almost independent of the choice of $b$, we find a transition from rotating spiral waves to spatio-temporal chaos via spiral breakup at $\epsilon$-values between 0.07 and 0.08 in both the excitable and oscillatory regimes [27, 43]. The simulations are usually started with a spiral initial condition that has been obtained at nearby values of $\epsilon$. The first run has been performed at $\epsilon = 0.02$ and then $\epsilon$ is increased in small steps ($\Delta \epsilon = 0.0025$) until destabilization occurs. In the vicinity of the spiral breakup we have used much finer “resolution” in $\epsilon$. The destabilization of spirals just above the critical $\epsilon$ however depends on the choice of $b$.

For the given model we find two different scenarios that are exemplified in figure 1. For excitable conditions, spiral breakup appears first close to the center and the irregular pattern spreads then outward (cf. figure 1(a)). For oscillatory conditions, we observe a different behaviour. Spirals first break up far away from the center and eventually relatively large spiral fragments surrounded by a “turbulent” bath remain (cf. figure 1(b)). The size of the surviving part of the spiral shrinks if $\epsilon$ is further increased until finally no trace of the initial spiral persists in the long run.

At closer inspection we observe that, even inside the surviving part of the
initial spiral, modulations in wavelength and period appear. The amplitude of these modulation grows with growing distance to the core. With increasing $\epsilon$ the modulation gets stronger and the critical modulation amplitude necessary to cause breakup is reached at distances closer to the center. In addition, the breakup in the excitable regime is usually preceded by a meander instability of the rotating spirals, while for oscillatory conditions breakup bounds a region of stable spiral rotation [27]. The breakup far away from the core has been also observed in experiments in the Belousov-Zhabotinsky reaction, [37] and in numerical simulations of the CGLE [16, 17, 37]. More recent studies provide also an experimental verification of the core breakup scenario, see [55].

As discussed in the introduction, further insight on the nature of the spiral
instability may be obtained by a reduction to one spatial dimension. Here, we
demonstrate this procedure for Eqs. (2). Therein, a 1D wave source is studied as an
analogue of a spiral in 2D. We will show that the essential properties of spiral breakup
depend mostly on the asymptotic selected wavetrain and not on the detailed structure
of the source (e.g. the spiral core geometry or the center of a target pattern). Results
for 1D sources and 2D spirals should allow for a quantitative comparison if their selected
wavelengths are the same. A 1D source of periodic waves is given by the equations
\[
\frac{\partial u}{\partial t} = -\frac{1}{\epsilon} u(u-1)(u-\frac{b+v}{a}) + \frac{\partial^2 u}{\partial r^2},
\]
\[
\frac{\partial v}{\partial t} = h(u) - v .
\] (2)

Dirichlet \((u(0) = u_1)\) and zero-flux \((\partial u(r) / \partial r)|_{L = 0}\) boundary conditions are used at
respective ends of the 1D system of length \(L\). Eqs. (2) represent the radial part of Eqs.
(1) far from the core, where curvature effects can be neglected.

The Dirichlet boundary at \(r = 0\) constitutes a source of waves similar to the core of
the spiral in 2D. By changing \(u_1\) the wavelength of the emitted wavetrain can be varied.
Here, we picked \(u_1\) values that select the same wavelength as spirals in 2D just below
the breakup. In the oscillatory case, \(u_1 = u_0\) achieved that goal.

Space-time plots from the integration of equations (2) are presented in figure 2 both
for oscillatory and excitable conditions. In both cases, the wave train emitted from the
left boundary at \(r = 0\) exhibit an instability upon increase of \(\epsilon\) at a critical value \(\epsilon_C\).
For \(b = -0.045\) we find a scenario similar to the behaviour in 2D (cf. figure 2a, b). In
the excitable case with \(b = 0.07\), the instability appears always very close to the source
(see figure 2c,d). Breakup occurs first far away from the Dirichlet boundary, i.e. the
source of waves. Upon increase of \(\epsilon\) the breakup of the wavetrain moves towards the
source until it gets to a distance of about one wavelength.

The transition between the two scenarios happens around \(b = 0.04\). The values
of \(\epsilon_C\) are not very sensitive to changes in \(b\) and are slightly above 0.08. Thus, we find
the counterpart of breakup near and far away from the spiral core in the 1D model. In
particular, the appearance of different qualitative scenarios does not depend on specific
2D ingredients like meandering and curvature.

3. Numerical Stability Analysis of Spirals near Breakup

Let us consider the basic arguments in spiral stability analysis. The spiral selects for
a given set of parameters a particular wavelength \(\lambda_0\) and related temporal period \(\tau_0\).
At the same moment, there exists a one-parameter family of periodic wave solutions
with varying speed \(c\), temporal period \(\tau\) and wavelength \(\lambda = c\tau\). The one-parameter
family may be described by a dispersion curve \(c(\lambda)\). At small wavelength, the wavetrains
either exhibit an instability (Eckhaus in the CGLE, alternans in cardiac tissue models) or
cease to exist (saddle node or drift pitchfork bifurcations in FitzHugh-Nagumo systems).
These minimum stable wavelengths and periods shall be called \(\lambda_{\text{min}}\) respectively \(\tau_{\text{min}}\).
in the following. In a few cases (CGLE [15], Rinzel-Keller model [56]) it is possible to compute the instability or bifurcation analytically. In other cases, kinematic theories (alternans,[44, 45]) or approximations by singular perturbation theory (excitable media) may yield useful information on the nature of this instability. For the models treated here, none of these methods are sufficient to yield an understanding of the instability at $\lambda_{\text{min}}$. This calls for alternative methods, namely numerical stability analysis. Here, we will mostly present quantitative results found by numerical stability computations.

For excitable media given by reaction-diffusion equations of the form presented in equations (1), a large body of theoretical work has been done in the limit of very small $\epsilon$. In particular, scaling laws originally predicted by Fife [57] have been explicitly derived for equations of the type used here by Karma [58] with the constraint $\epsilon^{1/3} \ll 1$. Applied to the model studied here, the argument according to the scaling argument goes as follows: for small enough $\epsilon$, the selected spiral wavelength scales like $\lambda_0 \propto \epsilon^{\alpha_1}$, while the smallest wavelength that is stable behaves like $\lambda_{\text{min}} \propto \epsilon^{\alpha_2}$. Usually, it is found that $\alpha_2 > \alpha_1$ and for $\epsilon = \epsilon_C$ the condition $\lambda_0 = \lambda_{\text{min}}$. For $\epsilon > \epsilon_C$, the selected wavelength correspond to an unstable wavetrain in the far-field and breakup should result. The analytically found scaling laws in the present case follow qualitatively the same trend as the numerically obtained data. They are, however, not very accurate quantitatively and cannot be used to estimate the value for $\epsilon_C$. Hence, one has to
carry out the stability analysis numerically as we will show below. Altogether, it is not too surprising that analytical scaling arguments fail quantitatively, because the breakup occurs at quite large values of $\epsilon$. Scaling arguments have been used with more success in the spiral breakup in cardiac tissue, where the instability appears upon decrease of $\epsilon$ and the scaling properties of the spiral waves are qualitatively different from the standard excitable media investigated here [22, 23].

In the following, we will numerically compute the one-dimensional wave solutions and their linear stability along the dispersion curve. Continuation software as AUTO has been used to obtain the spatial profiles of periodic waves as stationary solutions $u(z), v(z)$ in a comoving frame $z = r - ct$ [59]. Consequently, we compute periodic orbits that correspond to a single wavelength with periodic boundary conditions. The evolution of small perturbations of a spatially periodic, traveling wave $U_0(z)$ in the reaction diffusion equations is described by $W_{jn}(z)e^{\omega_{jn}t}$. Insertion of the perturbed wave solution into equation (2) allows to expand the equation in powers of the small perturbation which in linear order poses an eigenvalue problem for the eigenvectors $W_{jn}$ and eigenvalues $\omega_{jn}$. The indices $j, n$ will be specified below. There are infinitely many eigenvalues and eigenfunctions. For $L \to \infty$, $\omega_{jn}$ are located on continuous curves in the complex plane. The solution is stable when all eigenvalues $\omega_{jn}$ have negative real parts. The wavelength $\lambda$ is used as bifurcation parameter. The method yields accurate information on the minimum stable period $\lambda_{\text{min}}$ for the given medium, the eigenvalue spectra and the eigenfunctions describing the dynamics of perturbations to the wavetrains. These results are then compared to the selected wavelength of spirals $\lambda_0$ and 1D sources $\lambda_{\text{dir}}$ near the breakup threshold.

If we consider periodic traveling waves with constant shape $U_0(z)$ and constant speed $c$ in 1D, a few facts on the eigenvalues and the eigenfunctions are known a priori. Since our media are homogeneous and translation invariant, any translation of a given periodic solution is also a valid solution with identical stability properties. This translational symmetry of the wavetrains is reflected by an eigenvector $W_{00} = dU_0/dz$ with zero eigenvalue $\omega_{00} = 0$ (Goldstone mode). Symmetry arguments require the eigenfunctions of the periodic operator, obtained by linearization around a wavetrain with wavelength $\lambda$, to be Bloch functions $W_{jn}(z) = e^{i2\pi n \lambda / \lambda} \Phi_{jn}(z)$ with $\Phi_{jn}(z) = \Phi_{jn}(z + \lambda)$ and $n = 0, \ldots, N - 1$ [60]. The above Bloch form can be used in an Ansatz that reduces the stability problem for the infinite domain to a stability problem in a domain of length $\lambda$ with the wavenumber $k = 2\pi n / \lambda$ as an additional parameter.

### 3.1. Eckhaus Instability and Spiral Breakup Scenarios

We start by computing the spectra of periodic waves for Eq. (1), for a more detailed discussion see [43]. The eigenvectors corresponding to the eigenvalues with the largest real part are modulations of the Goldstone mode of the approximate form $W_{0n} \approx e^{i(2\pi n / \lambda)z}W_{00}(z)$. The amplitudes of the eigenfunctions are largest in the fronts and backs of the pulses in the wavetrain in contrast to the Fourier eigenfunctions in the
CGLE [15]. Thus, the fastest growing modes correspond to an alternating compression and expansion of subsequent pulses in line with the observations in the 2D simulations of figure 1 and experiments in the Belouzov-Zhabotinsky reaction [37]. The calculations for both models reveal an instability reminiscent of the Eckhaus instability in the CGLE. As the wavelength $\lambda$ is shortened, the spectra of the wavetrains shift towards larger real parts and cross the imaginary axis at $\lambda_{\text{min}}$. The leading part of the spectrum switches from a parabola opening to negative real parts and touching zero to a segment of a parabola opening to positive real parts. A symmetric curve results with two positive maxima of the real part of $\omega_{jn}$ at nonzero imaginary values. For an illustration, see the lower panels in figure 5.

The dispersion curves describe the dependency between speed $c$, wavelength $\lambda$ and period $\tau$. It is sufficient to plot two of three quantities, since $\lambda = c\tau$. Here, we plot $c$ vs. $\lambda$ and display stable (unstable) wavetrain solutions along the dispersion curve with full (dashed) lines. The dispersion curve ends towards small wavelength and periods at a nonzero value of the speed $c$ and not at a saddle-node bifurcation or at zero velocity as known from standard excitable media [8, 56]. Closer inspection reveals that the wavetrains instead bifurcate off one of the additional unstable equilibria with zero amplitude. Such a scenario may be typical for excitable media with additional unstable fixed points and is of course prevented for models which just have the rest state of the medium as a fixed point. The Eckhaus-type instability, however, appears somewhere along the dispersion curve and cannot be deducted from the form of the dispersion curve alone (as could be a saddle-node bifurcation). The instability typically happens just before the speed has a local minimum along the dispersion curve.

The goal of the previous analysis has been to provide a comparison between the minimum stable wavelength $\lambda_{\text{min}}$ and the selected wavelength $\lambda_0$ and $\lambda_{\text{dir}}$ found in simulations near spiral breakup and its one-dimensional analogue.

Now, we can compare the results of the simulations with the Dirichlet source and the stability analysis of wavetrains. Figure 3 shows the comparison between $\lambda_{\text{min}}$ (red dashed line, triangles) and $\lambda_{\text{dir}}$, the selected period of the Dirichlet source (blue full line, circles). The upper panel of figure 3 shows the data for the oscillatory case ($b = -0.045$). There is a narrow interval where stable 1D sources and spirals exist with $\lambda_0$ resp. $\lambda_{\text{dir}} < \lambda_{\text{min}}$. This is presumably related to the convective nature of the Eckhaus instability. Perturbations are then simply advected out of the system boundaries. The radial dynamics of spirals in 2D are analog to the Dirichlet case; the amplitude of the waves in the spiral goes to zero in the center of rotation. This corresponds to a “self-imposed” Dirichlet boundary condition. We also note that the spiral breakup appears at lower values of $\varepsilon$ than the instability of the Dirichlet source despite the fact that in both cases practically the same period and wavelength are selected. This is most likely due to nonlinear effects not captured by the linear stability analysis.

For the excitable case (breakup close to the center, $b = 0.07$) shown in the lower panel of figure 3, one gets very close to $\lambda_{\text{dir}} = \lambda_{\text{min}}$ at the instability of the 1D source. While this is in line with naive expectations, the result is somewhat surprising at
second glance. It basically suggests that the instability does not strongly depend on the boundary conditions, which is typical for an absolute instability. The Eckhaus instability, however, is expected to be convective for generic reasons. To resolve the paradox, a plain linear stability analysis is not sufficient anymore - one needs to compute the propagation properties of unstable modes (see next subsection).

3.2. Core Breakup and Finite Wavenumber Instability

So far, we have established the existence of two different scenario of spiral breakup and showed that core breakup practically coincides with the convective instability of the planar wavetrains far from the spiral center, while far field breakup requires an absolute instability of the planar wavetrain similar to the scenario originally proposed for the CGLE [15]. Recent efforts have yielded a more substantial mathematical analysis of convective and absolute instabilities and their relation to boundary conditions [61]. Application of these results to the example described above indicate that the CGLE arguments holds indeed also for the case of far-field breakup in the modified Barkley model [43, 49]. In the same work, the results for core breakup turned out to be less clear. One difficulty is that new defects once created in the far field tend to be quite
robust and are usually not pushed towards the boundary, even if the instability is linearly convective. This leads typically to deviations between the predictions from linear stability analysis and two-dimensional simulations already noticed in extensive studies of the CGLE [16]. As we indicated above, a reduction to one-dimensional sources allows for a deeper understanding.

Sandstede and Scheel argue that in the case of an Eckhaus instability perturbations are always traveling away from the core (convective instability) [49]. The window between the Eckhaus and corresponding absolute instability may however be very tiny and below the resolution of the numerical results presented above. Similar to the results in [49], the group velocity of the perturbation is also much smaller than in the case of the apparent „convective”, far-field breakup. Altogether such a convective Eckhaus instability leads to an apparent core breakup. Quantitative differences in the group velocity at the convective Eckhaus instability lead to a qualitative change in the breakup scenario. Below we present an interesting alternative: the Eckhaus instability is replaced by an instability to short wavelength modes. These modes propagate inward and therefore cause a core breakup.

Consider the following variant of the Barkley model, equation (3) with $h(u) = u^3$
\[ \frac{\partial u}{\partial t} = - \frac{1}{\ell} u(u-1)(u - \frac{b + v}{a}) + \frac{\partial^2 u}{\partial r^2}, \]
\[ \frac{\partial v}{\partial t} = h(u) - v. \] (3)
supplemented again with Dirichlet ($u(0) = u_1$) and zero-flux ($\partial u(r)/\partial r|_{L} = 0$) boundary conditions are used at respective ends of the 1D system of length $L$. 2D simulations of the model corresponding to Eq. (3) reveal a spiral core near the core influenced by a small amplitude meandering [62]. Here we consider only the one-dimensional source case. For fixed $a = 0.6, b = 0.06$ we compute wavetrain solutions $u(r - ct), v(r - ct)$ that depend on the choice of $\ell$ and their spatial period. The velocity $c$ is uniquely determined by a choice of the former two parameters. The comparison to the numerical stability analysis of wave trains will confirm the picture described above and in [43].

For small $\ell$ the wavetrains are linearly stable. As $\ell$ increases either the band of long wavelength perturbations (marked blue in the figures) or the band of finite wavelength perturbations (marked red) becomes unstable first, the second band follows upon further increase of $\ell$. We have determined the corresponding critical values of $\ell$ for a small range of wavelength of the wavetrains. Figure 4 shows the locations where the two instabilities occur (blue and red line) in parameter space. The leading part of the eigenvalue spectrum is shown in figure 5 for selected solutions indicated by colored dots in figure 4. The spectra in figure 5 have the same color as the corresponding dots and figure 5(a) belongs to wavetrains with period 7.7 and (b) to 8.0. For periods smaller than 7.9 the finite wavelength instability (red) occurs first. Similar spectra are obtained for the parameters that yield core breakup in Eq. (1); therein, however, the Eckhaus instability appears always slightly before the finite wavenumber band and the two instabilities do not cross as in figure 4. The period selected in numerical simulations with Dirichlet
boundary conditions are marked by squares whenever the solution appeared stable. For \( \epsilon > 0.0865 \) we observe breakup near the core and no squares are drawn in this area.

Near the onset of the instabilities we have computed the eigenvalue spectra of eigenmodes weighted by a factor \( \exp(-\alpha r) \). Sandstede and Scheel proved this method to reveal the direction of transport of small perturbations \[49\]. If eigenmodes transport toward positive \( r \) then a (sufficiently large) positive weight \( \alpha \) will damp their growth at positive \( r \) and stabilize them, i.e. shift their eigenvalues to smaller real parts. On the other hand negative \( \alpha \) will damp the growth of modes that transport to negative \( r \).

One example of these weighted spectra is shown in figure 6 for the wavetrain that was the last to be observed in numerical simulations (filled square in figure 4). The thick curve corresponds to the standard spectrum \((\alpha = 0)\) and thin curves are shifted due to weights of different magnitude. We find that long wavelength modes (marked blue) similar to the Goldstone mode transport in the same direction as the wavetrain travels but finite wavelength modes (marked red) transport in the opposite direction.

The corresponding numerical simulations of the one-dimensional system are shown in figure 7. The emitted waves are stable as long as the periodic wave is linearly stable (figure 7(a)). In figure 7(b) perturbations propagate inward corresponding to the band.
Core breakup from radial dynamics

Figure 5. Eigenvalue spectra of wave solutions corresponding to the dots in figure 4 with the same colour. The wavelength is chosen as (a) 7.7 and (b) 8.0, respectively. Upper and lower panels show the same spectrum on different scales to pronounce the finite wavelength instability (red, upper left) and the long wavelength instability (blue, lower right).

of finite wavenumber modes in the linear stability analysis.

How does this observation relate to an absolute or convective nature of the instability? If the eigenvalues of the stationary mode(s) with zero group velocity have positive real part then the instability is called absolute. If only a single band of eigenmodes is unstable that contains only modes transporting into the same direction, the corresponding instability is called convective. In the present example the eigenvalues of stationary modes have negative real parts hence it is not an absolute instability. Instead, we observe two distinct bands that transport into opposite directions. Sandstede and Scheel have predicted this complementary type of instability and named it remnant [61]. We found this type for positive $b$ at fixed $a = 0.6$. In general, the finite wavelength instability with modes travelling towards the core occurs first for small wavelength $\lambda$.

These results strongly suggest a generalization of the criterion derived for the CGLE by Kramer and Aranson that requires an absolute instability of the emitted wavetrains. Namely, spiral breakup occurs if destabilizing modes propagate towards the core. For the convective Eckhaus instability, the critical modes travel away from the core. Consequently, the first unstable modes that travel towards the core appear, if the instability becomes absolute and the original argument does not contradict our new criterion. The outcome is different in the case of a convective instability with modes
that propagate inward, then no absolute instability is required and core breakup can be expected right at the onset of the (short wavelength) instability.

4. Summary

In this paper, we have studied the phenomenon of spiral breakup by comparing numerical simulations to a stability analysis of periodic wave trains. We used a modification of the Barkley model, which is very similar to the FitzHugh-Nagumo model. Therein, one observes two distinct phenomenologies of spiral breakup: core breakup and far-field breakup (see figure 1). The core breakup is accompanied by the meander-instability, which makes the analysis more difficult, because it introduces a Doppler effect into the waves emitted from the spiral core. It is possible to suppress the meandering in simulations of a 1D source with a fixed “core” boundary condition of Dirichlet type. Such a source is in some cases a good approximation of the radial dynamics of a non-
meandering spiral. For the systems studied here, we found that the phenomena of core and far field breakup persist in the 1D source and thus should not depend crucially on the presence or absence of meandering. Altogether, one can distinguish four distinct phenomenologies of spiral breakup:

- **Far field breakup without meandering** as seen in figure 1(b), the CGLE and the Belousov-Zhabotinsky reaction
- **Core breakup without meandering** as shown in the modified Barkley model by Sandstede and Scheel [49]
- **Far field breakup with meandering** as seen in experiments by Zhou and Ouyang [64, 55], in simulations of calcium waves by Falcke et al. [47]. For a theoretical analysis see the work of Brusch et al. [65]
- **Core breakup with meandering** as seen in figure 1a, in many cardiac models [22, 26, 33] and in experiments with the Belousov-Zhabotinsky reaction [55]. This route is often termed Doppler induced instability.

The second main topic treated in this paper is the numerical stability analysis. We suggest the following strategy:

- Identify spiral breakup in two-dimensional simulations of a reaction-diffusion model
• Try to simulate the radial dynamics for a one-dimensional source and suppress potential meandering influences on breakup
• Find all periodic wavetrain solutions from continuation and compute their spectra numerically
• In case of instability, determine the direction of propagation of unstable modes and decide if the instability is convective or absolute
• Is the selected wavetrain in the 2D spiral resp. 1D source linearly stable?
• Try to establish the relation between the stability results and the simulations

In the above examples, we found that the instability of the wavetrain is always of convective nature and coincides with the breakup of waves near a source if the perturbations at onset propagate towards the core (see Section 3.2). In the opposite case of perturbations traveling away from the core region, an absolute instability is required (see Section 3). This observation is related to the fact, that the absolute and essential spectra of periodic waves are not identical in the latter case, while our results imply that the maximum real part coincides in the first case.

The linear stability analysis serves to answer two main questions: (i) Is spiral breakup caused by a linear instability of a simpler pattern? (ii) Can the phenomenology of breakup be deduced from a particular type of instability and its properties like wavenumber of the critical modes and their direction of propagation? The evidence presented in this paper and in the recent literature allows to answer the first question in a positive way. This is not at all trivial, because there are alternative possibilities. Other potential scenarios include the non-normal dynamics responsible for turbulence in simple flow geometries [66], extremely long-lived chaotic transients towards a final non-chaotic or even periodic attractor [67, 68] or a coexistence of simple periodic solutions like spirals or rotating spirals and complex spiral-defect-chaos patterns. The latter possibility has been investigated in detail, for chaotic convection patterns in Rayleigh-Benard convection experiments and simulations [69]. These cases have been ruled out for the reaction-diffusion systems studied here. We found that spiral breakup in simulations is clearly correlated with linear instabilities. The second question is more difficult to answer. Far-field breakup of simple spirals can in some cases be linked with the convective nature of the Eckhaus instability, while we showed in Section 3.2 that core breakup stems from a finite wavenumber instability with critical modes that propagate towards the core. Earlier work however points to the fact, that the core breakup reported in figure 1b is also related to an Eckhaus instability [43]. Sandstede and Scheel noted first that the critical modes can also propagate outward in the case of core breakup, though their group velocity is in this case much smaller than for far-field breakup [49]. In Section 3.2, we applied the same method of spatially weighted perturbations as developed by Sandstede and Scheel [49]. For our slightly different model we found a new instability with finite wavenumber instability that could precede the also possible Eckhaus instability. Application of the same method to the case shown in figure 3, showed that there core breakup results also from an Eckhaus instability with small
group velocity critical mode. For small group velocity, the parameter window between the convective Eckhaus and the absolute instability becomes extremely narrow and for coarser numerical resolution the difference between the two points may be overlooked. It is important to note, that (i) core breakup consequently can result from different types of instability and that (ii) the same instability scenario (here: absolute version of Eckhaus instability with dominating modes propagating away from the core) may lead to phenomenologically different appearances like the different scenarios in figures 1 and 3. This statement is further illustrated by the many cardiac models, in which core breakup is presumably caused by a period doubling known as alternans [23, 26, 33]. For far-field breakup, one may also find explanations different from the absolute Eckhaus instability unveiled in the CGLE and the modified Barkley model studied above.

A further complication in the analysis of spiral breakup is spiral meandering. A Dirichlet boundary condition suppresses the meandering instability which could therefore be neglected in the above analysis. Simulations in 2D reveal this meandering instability and yield breakup near the core at \( \epsilon \) slightly smaller than the critical values for the instability of the emitted wavetrain. Meandering leads to a Doppler effect in the waves emitted from the spiral core. Due to the modulation of the spiral tip curvature a motion of the source of waves results in a modulation of the wavelength and frequency of the emitted wave trains. Originally, this effect has been analyzed in simulations with core breakup [27]. Meandering spirals can, however, also display far field breakup as has been demonstrated also with simulations of the Atri model for intracellular calcium waves [47]. The influence of meandering and the resulting Doppler effect in core breakup has recently been verified in experiments with the Belousov-Zhabotinsky reaction [46]. Far field breakup of (presumably) meandering spirals with a non-decaying modulation of the wavetrain in the far field ("superspirals" [63]) has been studied in the same reaction [64, 55]. Here, an interpretation of the breakup in terms of the nonlinear consequences of the Eckhaus instability is possible already within the CGLE [65]. Another variant of far-field breakup has recently been identified in resonantly forced oscillatory media and is caused by a merging instability of subsequent wavefronts [70].

From the four main phenomenologies, we have discussed in detail examples for the two scenarios without meandering. One case of far-field breakup with meandering is analyzed in the CGLE and related to the properties of modulated amplitude waves [65]. The fourth scenario, core breakup with meandering has not been subject to a comparable analysis. If the meandering amplitude gets too large, waves may break already close to the center and new spirals can appear. A stability analysis would however require a consideration of time-periodic waves, which is technically more difficult than the stability analysis of traveling waves with constant profiles. The technique has however proved successful in other circumstances [71, 72]. In retrospect, it is not surprising that linear stability analysis of spirals leads to a much richer variety of breakup scenarios than the simple phenomenology, because the only requirement is a instability of the planar wave train far from the spiral core. This leaves a large number of possibilities, while the phenomenological distinction of far-field and core breakup is only a crude scheme.
The good news is however that the spiral’s stability analysis may be often reduced to the analysis of radial dynamics, which can be done with standard numerical procedures for most reaction-diffusion models. As the above examples demonstrate, a comparison of stability analysis and simulations leads to a much advanced understanding of the mechanism of spiral breakup and allows for a mathematically precise classification of such a transition way beyond the simple phenomenology of Doppler, core and far-field breakup.

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