Dynamics of Bose-Einstein Condensates in Trapped Atomic Gases at Finite Temperature
Lecture 2: The ZNG Equations

Eugene Zaremba
Queen’s University, Kingston, Ontario, Canada

Financial support from NSERC
Outline of These Lectures

Lecture 1: Basics

• quantum statistics, ideal gases and Bose-Einstein condensation
• single-particle density matrix, Wigner distribution
• interactions in dilute gases
• Gross-Pitaevskii equation, Thomas-Fermi approximation
• Bose broken symmetry, order parameter
• time-dependent GP equation, quantum hydrodynamic equations, Stringari equation, Kohn theorem

Lecture 2: The ZNG Equations

• condensate wavefunction, the generalized GP equation, quantum hydrodynamics
• thermal component, Boltzmann kinetic equation, collisions
• equilibrium solution, equilibrium collision times
• condensate formation, adiabatic and ergodic approximations
• recap of ZNG equations
Derivation of the ZNG Equations

- we are interested in the dynamical properties of a Bose-condensed system governed by the Hamiltonian

\[
\hat{H} = \int d^3r \hat{\psi}^\dagger(r) \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) \right] \hat{\psi}(r) + \frac{g}{2} \int d^3r \hat{\psi}^\dagger(r) \hat{\psi}^\dagger(r) \hat{\psi}(r) \hat{\psi}(r)
\]

- the Heisenberg equation of motion for the field operator is

\[
i\hbar \frac{\partial \hat{\psi}(r, t)}{\partial t} = [\hat{\psi}(r, t), \hat{H}] = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) \right) \hat{\psi}(r, t) + g \hat{\psi}^\dagger(r, t) \hat{\psi}(r, t) \hat{\psi}(r, t)
\]

- we now invoke the key idea in our approach: the nonequilibrium system of interest exhibits *Bose broken symmetry*. This means that the system can be described by a nonequilibrium ensemble in which the field operator has a *finite mean value*

\[
\Phi(r, t) = \langle \hat{\psi}(r, t) \rangle
\]

- we refer to this as the *condensate wavefunction*; it can be thought of as the most highly populated eigenfunction of the nonequilibrium single-particle density matrix
• taking the expectation value of the equation of motion for the field operator we obtain

\[ i\hbar \frac{\partial \Phi(r, t)}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) \right] \Phi(r, t) + g\langle \hat{\psi}^\dagger(r, t)\hat{\psi}(r, t)\hat{\psi}(r, t) \rangle \]

• this is essentially an exact equation of motion for the condensate wavefunction, however it is not closed. We now define the *noncondensate* field operator by (Beliaev, 1958)

\[ \hat{\psi}(r, t) = \Phi(r, t) + \tilde{\psi}(r, t), \quad \langle \tilde{\psi}(r, t) \rangle = 0 \]

• using this decomposition we find

\[ \langle \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \rangle = n_c \Phi + \tilde{m} \Phi^* + 2\tilde{n} \Phi + \langle \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \rangle \]

where

\[ n_c(r, t) \equiv |\Phi(r, t)|^2 = \text{local condensate density} \]
\[ \tilde{n}(r, t) \equiv \langle \tilde{\psi}^\dagger(r, t)\tilde{\psi}(r, t) \rangle = \text{non-condensate density} \]
\[ \tilde{m}(r, t) \equiv \langle \tilde{\psi}(r, t)\tilde{\psi}(r, t) \rangle = \text{off-diagonal (anomalous) density} \]

• in our work we make the simplification of neglecting the anomalous average: \( \tilde{m} \simeq 0 \)
finally, one must deal with the 3-field operator average; we find that it takes the form
\[ \langle \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \rangle \approx -iR(r, t) \Phi(r, t) \]

in this way we arrive at the generalized Gross-Pitaevskii equation
\[ i\hbar \frac{\partial \Phi}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) + gn_c(r, t) + 2g\tilde{n}(r, t) - iR(r, t) \right] \Phi \]

we see that the condensate wavefunction is coupled to the noncondensate which in the following we will also refer to as the thermal cloud. We will thus need an equation of motion for the latter. In addition, the GGP equation has a non-Hermitian term \(-iR\) which has the effect of changing the normalization of the wavefunction. The explicit form of this term will be given later, and as we shall see, it is determined by collisions between condensate and noncondensate atoms.

the GGP equation reduces to the usual GP equation in the absence of the thermal cloud. This is the zero temperature limit of the theory.
\[ i\hbar \frac{\partial \Phi}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) + gn_c(r, t) \right] \Phi \]
Quantum Hydrodynamics

- it is sometimes convenient to recast the GGP into the form of ‘hydrodynamic’ equations. These are not hydrodynamic equations in the sense of classical fluid mechanics or Landau’s two-fluid equations which rely on the concept of local thermodynamic equilibrium (more about this later). However, the equations look like hydrodynamic equations in the sense that they involve local fluid variables. To obtain these equations we make use of the amplitude-phase representation

\[ \Phi(r, t) = \sqrt{n_c(r, t)} e^{i\theta(r, t)} \]

- substituting this into the GGP and separating real and imaginary parts, we find

\[
\begin{align*}
\frac{\partial n_c}{\partial t} + \nabla \cdot n_c v_c &= -\Gamma_{12}[f, \Phi] \quad \text{ - continuity equation} \\
m \left( \frac{\partial v_c}{\partial t} + \frac{1}{2} \nabla v_c^2 \right) &= -\nabla \mu_c \quad \text{ - Euler equation}
\end{align*}
\]

_E: Derive these equations from the GGP._
Quantum Hydrodynamics, cont’d

- the condensate velocity is defined as
  \[ v_c(r,t) \equiv \frac{\hbar}{m} \nabla \theta(r,t) \]
  and the effective driving potential acting on the condensate is
  \[ \mu_c(r,t) = -\frac{\hbar^2 \nabla^2 \sqrt{n_c(r,t)}}{2m \sqrt{n_c(r,t)}} + V_{\text{trap}}(r) + gn_c(r,t) + 2g\tilde{n}(r,t), \]

- the first term is referred to as the \textit{quantum pressure term} and is associated with the kinetic energy of a \textit{nonuniform} condensate. The neglect of this term is referred to as the \textit{Thomas-Fermi approximation}. The remaining terms constitute the total \textit{mean-field potential} acting on the condensate:
  \[ U_c(r,t) = V_{\text{trap}}(r) + gn_c(r,t) + 2g\tilde{n}(r,t), \]

- in addition, the continuity equation has a source term given by
  \[ \Gamma_{12}[f,\Phi] \equiv \frac{2}{\hbar} n_c(r,t) R(r,t) \]
  which implies that the number of condensate atoms is \textit{not} necessarily conserved. As we shall see, this is due to the possibility of exchanging atoms with the thermal cloud.
Boltzmann Equation for the Thermal Cloud

- to complete the theory we need a dynamical equation for the thermal cloud. This equation takes the form of a Boltzmann kinetic equation. The derivation of this equation is lengthy and only an outline will be provided here. The starting point is the equation of motion for the noncondensate field operator:

\[ i\hbar \frac{\partial \tilde{\psi}}{\partial t} = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}} + 2gn \right) \tilde{\psi} - 2g\tilde{n}\tilde{\psi} + g\Phi^2 \tilde{\psi}^{\dagger} \]
\[ + g\Phi^* (\tilde{\psi}\tilde{\psi} - \tilde{n}) + 2g\Phi (\tilde{\psi}^{\dagger}\tilde{\psi} - \tilde{n}) + g (\tilde{\psi}^{\dagger}\tilde{\psi}\tilde{\psi} - \langle \tilde{\psi}^{\dagger}\tilde{\psi}\tilde{\psi} \rangle) \]

- notice that this equation preserves the condition \( \langle \tilde{\psi}(\mathbf{r}, t) \rangle = 0 \) as a function of time. Formally, the equation can be solved as

\[ \tilde{\psi}(\mathbf{r}, t) = \hat{S}^{\dagger}(t, t_0)\tilde{\psi}(\mathbf{r}, t_0)\hat{S}(t, t_0) \]

where the evolution operator is the solution of

\[ i\hbar \frac{d\hat{S}(t, t_0)}{dt} = \hat{H}_{\text{eff}}(t)\hat{S}(t, t_0) \]

with

\[ \hat{H}_{\text{eff}}(t) = \hat{H}_0(t) + \hat{H}'(t), \quad \hat{H}_0(t) = \int d\mathbf{r} \tilde{\psi}^{\dagger} \left( -\frac{\hbar^2 \nabla^2}{2m} + U(\mathbf{r}, t) \right) \tilde{\psi} \]
• the term $H_0(t)$ describes the dynamics of the thermal atoms moving in the noncondensate mean-field potential

$$U(r, t) = V_{\text{trap}}(r) + 2gn_c(r, t) + 2g\tilde{n}(rt),$$

• this level of approximation is known as the \textit{time-dependent Hartree-Fock approximation}. It implies that a thermal atom has the local energy

$$\tilde{\varepsilon}_p(r, t) = \frac{p^2}{2m} + V_{\text{trap}}(r) + 2g [n_c(r, t) + \tilde{n}(r, t)]$$

• the term $H'(t)$ is a perturbation containing several terms. We will not display it here but suffice it to say that it leads to collisions between the thermal excitations and between thermal excitations and the condensate.

• the expectation value of a noncondensate variable is given by

$$\langle \hat{O}(t) \rangle = \text{Tr} \hat{\rho}(t_0) \hat{O}(t) = \text{Tr} \hat{\rho}(t, t_0) \hat{O}(t_0)$$

with

$$\hat{\rho}(t, t_0) \equiv \hat{S}(t, t_0) \hat{\rho}(t_0) \hat{S}^\dagger(t, t_0)$$

• the evolution of the density matrix is now developed to lowest order in $H'(t)$
**Kinetic Equation**

- for the thermal cloud observable we choose the Wigner operator introduced earlier. Its equation of motion is given by

\[
\frac{\partial f(p, r, t)}{\partial t} = \frac{1}{i\hbar} \text{Tr} \hat{\rho}(t, t_0)[\hat{f}(p, r, t_0), \hat{H}_{\text{eff}}(t)]
\]

\[
= \frac{1}{i\hbar} \text{Tr} \hat{\rho}(t, t_0)[\hat{f}(p, r, t_0), \hat{H}_0(t)] + \frac{1}{i\hbar} \text{Tr} \hat{\rho}(t, t_0)[\hat{f}(p, r, t_0), \hat{H}'(t)]
\]

- the first term on the r.h.s. defines the free-streaming operator in the kinetic equation. Assuming that the mean-field potential \( U(r,t) \) is slowly varying, we find

\[
\frac{\partial f(p, r, t)}{\partial t} + \frac{p}{m} \cdot \nabla f(p, r, t) - \nabla U \cdot \nabla_p f(p, r, t) = C_{12}[f, \Phi] + C_{22}[f]
\]

- \( f(p, r, t) \) is the phase space distribution of thermal atoms. It gives the number of thermal atoms with momentum \( p \) at position \( r \) at time \( t \). Their dynamics on the one hand is governed by the mean-field potential \( U(r,t) \) which includes the interaction with the condensate. In addition, the r.h.s of the kinetic equation describes the collisions of the thermal atoms.
Collision Integrals

\[C_{12}[f, \Phi]\]

\[C_{12}[f] = \frac{2g^2n_c}{(2\pi)^2\hbar^4} \int dp_1 \int dp_2 \int dp_3 \delta(m\nu_c + p_1 - p_2 - p_3)\]
\[\times \delta(\varepsilon_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3})\]
\[\times [(1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3)]\]

\[C_{22}[f] = \frac{2g^2}{(2\pi)^5\hbar^7} \int dp_2 \int dp_3 \int dp_4 \delta(p + p_2 - p_3 - p_4)\]
\[\times \delta(\tilde{\varepsilon}_p + \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3} - \tilde{\varepsilon}_{p_4})\]
\[\times [(1 + f)(1 + f_2)f_3f_4 - ff_2(1 + f_3)(1 + f_4)]\]

\[\varepsilon_c(r, t) = \frac{1}{2}m\nu_c^2(r, t) + \mu_c(r, t), \quad \tilde{\varepsilon}_p(r, t) = \frac{p^2}{2m} + U(r, t)\]
Collisions Between Thermal Atoms

\[ C_{22}[f] = \frac{2g^2}{(2\pi)^5 \hbar^7} \int dp_2 \int dp_3 \int dp_4 \delta(p + p_2 - p_3 - p_4) \]
\[ \times \delta(\tilde{\varepsilon}_p + \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3} - \tilde{\varepsilon}_{p_4}) \]
\[ \times [(1 + f)(1 + f_2)f_3f_4 - ff_2(1 + f_3)(1 + f_4)] \]

- the collision integral accounts for Bose statistics. In scattering, we have the statistical factor \((1 + f_i)\) for the creation or \(f_i\) for the destruction of a particle in the state \(i\).
- one can show that the collision integral vanishes if \(f\) is an equilibrium Bose distribution. This means that collisions drive the distribution towards the equilibrium distribution, i.e. collisions equilibrate the gas.
- when \(f \ll 1\) the collision integral reduces to the usual collision integral for a classical gas – the square bracket reduces to \([f_3f_4 - ff_2]\).
- the collision integral is second order in the interaction strength \(g\), i.e. it is proportional to the binary collision cross-section.
Collisions Between Thermal Atoms and the Condensate

\[ C_{12}[f] = \frac{2g^2 n_c}{(2\pi)^2 \hbar^4} \int dp_1 \int dp_2 \int dp_3 \delta(mv_c + p_1 - p_2 - p_3) \]
\[ \times \delta(\varepsilon_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3}) \delta(p - p_1 - p_2 - p_3) \]
\[ \times [(1 + f_1)f_2 f_3 - f_1 (1 + f_2)(1 + f_3)] \]

- the wiggly lines represent condensate atoms; this collision integral arises *only* when there is a Bose condensate. These collisions play a crucial role in the theory.
- the \( C_{12} \) collision integral is directly related to the non-Hermitian term \( R \) in the GGP
  \[
  \Gamma_{12}[f, \Phi] \equiv \int \frac{dp}{(2\pi \hbar)^3} C_{12}[f(p, r, t), \Phi(r, t)] = \frac{2}{\hbar} n_c(r, t) R(r, t)
  \]
- \( C_{12} \) collisions establish equilibrium between the condensate and the thermal cloud
Total Number Conservation

- earlier we obtained the quantum hydrodynamic continuity equation
  \[ \frac{\partial n_c}{\partial t} + \nabla \cdot (n_c v_c) = -\Gamma_{12}[f, \Phi] \]

- we now integrate the kinetic equation over momenta:
  \[ \int \frac{d^3p}{(2\pi\hbar)^3} \left\{ \frac{\partial f(p, r, t)}{\partial t} + \frac{p}{m} \cdot \nabla f(p, r, t) - \nabla U \cdot \nabla_p f(p, r, t) \right\} = \int \frac{d^3p}{(2\pi\hbar)^3} \{ C_{12}[f, \Phi] + C_{22}[f] \} \]

- introducing the definitions
  \[ \tilde{n}(r, t) = \int \frac{d^3p}{(2\pi\hbar)^3} f(p, r, t), \quad \tilde{n}(r, t)v_n(r, t) = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p}{m} f(p, r, t) \]

  and using the result \[ \int d^3p C_{22}[f] = 0 \quad (E: \text{prove this}), \] we find
  \[ \frac{\partial \tilde{n}}{\partial t} + \nabla \cdot (\tilde{n}v_n) = \Gamma_{12}[f, \Phi] \]

- the source terms have opposite sign and so we find for the total density the continuity equation
  \[ \frac{\partial n}{\partial t} + \nabla \cdot j = 0 \quad \Rightarrow \quad \text{conservation of particle number!} \]

where
\[ n = n_c + \tilde{n}, \quad j = n_c v_c + \tilde{n}v_n \]
Equilibrium

• we now show what happens when the trapped Bose gas is in thermal equilibrium. We consider a Bose-condensed system containing $N$ atoms at temperature $T$. A stationary solution of the GGP has the form

$$
\Phi_0(r, t) = \Phi_0(r)e^{-i\mu_0 t/\hbar}
$$

where $\Phi_0(r)$ is the solution of the time-independent equation

$$\left[-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) + gn_c(r) + 2g\tilde{n}_0(r)\right] \Phi_0(r) = \mu_0 \Phi_0(r)
$$

• we note that the non-Hermitian term $R$ does not appear here (see below)

• in equilibrium, the thermal cloud distribution function is given by the static Bose distribution

$$
f^0(p, r) = \frac{1}{e^{\beta[p^2/2m+U_0(r)-\tilde{\mu}_0]} - 1}
$$

with the equilibrium thermal cloud mean-field potential

$$
U_0(r) = V_{\text{trap}}(r) + 2g[n_c(r) + \tilde{n}_0(r)]
$$

• the reason for the Bose distribution is that collisions drive the thermal cloud towards equilibrium for which $C_2[f_0, \Phi] = 0$
**Equilibrium, cont’d**

*E: Use energy and momentum conservation to show that*

\[
(1 + f^0)(1 + f_2^0) f_3^0 f_4^0 - f^0 f_2^0 (1 + f_3^0)(1 + f_4^0) = 0
\]

*and hence that* \(C_{22}[f_0, \Phi] = 0\).

- similarly, one finds that

\[
R[f^0, \Phi_0] = \frac{g^2}{(2\pi)^5 \hbar^5} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\
\times \delta(\mu c_0 + \tilde{\epsilon}_1 - \tilde{\epsilon}_2 - \tilde{\epsilon}_3)(1 + f_1^0)f_2^0 f_3^0 \left[ e^{\beta(\mu c_0 - \tilde{\mu}_0)} - 1 \right]
\]

- for this to vanish we require

\[\mu c_0 = \tilde{\mu}_0\]

i.e., in thermal equilibrium, the chemical potentials of the condensate and thermal cloud **must be the same**. We refer to this condition as *diffusive equilibrium*. This condition must therefore be **imposed** when obtaining a self-consistent solution of the stationary GGP equation and the equilibrium distribution of the thermal cloud.
\textbf{Summary of Equilibrium Equations}

\[
\left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) + g n_{c0}(r) + 2g \tilde{n}_0(r) \right] \Phi_0(r) = \mu_{c0} \Phi_0(r)
\]

\[
f^0(p, r) = \frac{1}{e^{\beta[p^2/2m + U_0(r) - \mu_{c0}]} - 1}
\]

\[
U_0(r) = V_{\text{trap}}(r) + 2g[n_{c0}(r) + \tilde{n}_0(r)]
\]

\[
n_{c0}(r) = |\Phi_0(r)|^2, \quad \tilde{n}_0(r) = \int \frac{d^3p}{(2\pi \hbar)^3} f^0(p, r) = \frac{1}{\lambda_T^3} g_{3/2}(z_0(r))
\]

with

\[
z_0(r) = e^{\beta(\mu_{c0} - U_0(r))}
\]

- this set of equations must be solved self-consistently for the condensate and thermal cloud densities
- note that the GGP equation is nonlinear and must therefore be solved using an iterative procedure
- also, the iterative procedure must ensure that the solution provides the desired total number of particles

\[N_{c0} + \tilde{N}_0 = N\]
Example of an Isotropic Trap

• in this example we have 5000 $^{87}\text{Rb}$ atoms in an isotropic trap with harmonic frequency $\nu_0 = 200$ Hz

• the dashed line is the thermal cloud density and the solid line is the total density
• the thermal atoms form a low density halo around the condensate
• repulsion suppresses the thermal atom density in the vicinity of the condensate

• the solid line is the effective condensate potential and the dashed line is the effective potential for the thermal atoms
• the zero of energy is the chemical potential. Note that it lies above the minimum of the condensate potential
Comparison with Experiment

Fraction of condensate atoms vs. the temperature (normalized to the transition temperature of a noninteracting trapped Bose gas). The results of calculations are shown for an ideal Bose gas (broken line) and the Hartree–Fock TF approximations in which the thermal cloud mean field is omitted (dotted line) and retained (solid line) (from Gerbier et al., 2004)
Collision Times

• let’s take a closer look at collisions; the quantity

\[
[drdp/(2\pi\hbar)^3]C_{22}^{\text{out}}
\]

represents the number of atoms *leaving* the phase-space volume \( drdp \) per unit time as a result of thermal atom collisions

• the mean number of atoms at \( r \) suffering a collision per unit time per unit volume is therefore

\[
\Gamma_{22}^{\text{out}} = \int \frac{dp}{(2\pi\hbar)^3} C_{22}^{\text{out}}
\]

• by definition the probability of an atom at \( r \) suffering a collision in time \( dt \) is \( dt/\tau_{22} \) where \( \tau_{22} \) is called the *mean collision time*. We thus have

\[
\Gamma_{22}^{\text{out}} = \frac{\tilde{n}}{\tau_{22}}
\]

• in equilibrium

\[
\frac{\tilde{n}_0(r)}{\tau_{22}^0(r)} = \int \frac{dp_1}{(2\pi\hbar)^3} f_1^0 \int \frac{dp_2}{(2\pi\hbar)^3} f_2^0 \int \frac{d\Omega}{4\pi} \sigma |v_1 - v_2|(1 + f_3^0)(1 + f_4^0)
\]
Collision Times, cont’d

• in the classical (Maxwell-Boltzmann) limit,

\[ \frac{1}{\tau_0^{22}} = \sqrt{2} \sigma v_{th} \tilde{n}_0 \]

with

\[ \sigma = 8\pi a^2, \quad v_{th} = (8kT/\pi m)^{1/2} \]

• consider an isotropic trap with \( \nu_0 = 200 \text{ Hz} \) containing \( 10^5 \) \(^{87}\text{Rb} \) atoms

• 1–5 correspond to temperatures 100–500 nK in steps of 100 nK

• the mean collision time is smallest at the edge of the condensate where the thermal atom density is largest (Bose enhancement)

• the inequality \( \omega_0 \tau_{22}^{-0} \ll 1 \) implies one is locally in the collisional hydrodynamic regime
Collision Times, cont’d

- for $C_{12}$ collisions we similarly find

the mean collision time falls off rapidly beyond the edge of the condensate

- at elevated temperatures, the inequality $\omega_0 \tau_{12}^0 << 1$ is satisfied throughout the region of the condensate
Condensate Formation in Traps

- we have already emphasized that collisions drive the system towards equilibrium. This is nowhere more evident than when a Bose gas above the transition temperature is suddenly cooled by means of a quench. If the quench is sufficiently deep, the gas can equilibrate to a Bose-condensed state.

- images of a cloud following an evaporative quench (MIT experiment, Science 279, 1005 (1998))
Condensate Formation in Traps


Evaporative Cooling

- in the experiments, an rf field is used to rapidly remove particles above a certain energy $E_{cut}$
- the quench removes both particles and energy from the system; the system is now ‘colder’
- this defines the initial nonequilibrium state of the gas
- the gas begins to equilibrate via interparticle collisions
- if the cut is sufficiently deep, the gas will equilibrate to a temperature $T < T_C$ and a condensate will begin to grow
- this is clearly a highly nonequilibrium process
Condensate Growth within the ZNG Theory

Bijlsma, Zaremba and Stoof, PRA (2000)

- the calculations are based on two main approximations:
  a) assumption of **adiabatic growth**
  b) assumption of **ergodicity**

a) Adiabatic Growth

- at each instant of time, the condensate is in its ground state

\[ n_c(r, t) = \frac{1}{g} \left[ \mu_c(t) - V_{\text{trap}}(r) \right] \]

\[ \mu_c(t) = \frac{1}{2} \hbar \omega \left[ 15 N_c(t) \frac{a}{l} \right]^{2/5} \]

- this ignores any internal dynamics of the condensate; everything is determined by the condensate number \( N_C(t) \)
b) Ergodic Approximation

- all points in phase space having the same energy are *equally probable*
  \[ f(p, r, t) \rightarrow g(E(p, r, t), t) \]
  \[ E(p, r, t) = \frac{p^2}{2m} + V_{\text{trap}}(r) + 2gn_c(r, t) \]

- define the *projector*
  \[ \rho(\epsilon, t)g(\epsilon, t) \equiv \int_{2\pi\hbar} d^3p d^3r \delta(\epsilon - E(p, r, t))f(p, r, t) \]

  where the *density of states* is
  \[ \rho(\epsilon, t) = \int_{2\pi\hbar} d^3p d^3r \delta(\epsilon - E(p, r, t)) \]

- applying the projector to the kinetic equation, we obtain
  \[ \frac{\partial}{\partial t}(\rho g) + \frac{\partial}{\partial \epsilon}(\rho_w g) = I_{12} + I_{22}, \quad \frac{\partial N_c}{\partial t} = \int d\epsilon I_{12}(\epsilon, t) \]
**Projected Collision Integrals**

\[ I_{22}(\epsilon_1, t) = \frac{m^3 g^2}{2\pi^3 \hbar^7} \int d\epsilon_2 \int d\epsilon_3 \int d\epsilon_4 \rho(\epsilon_{\text{min}}) \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \]
\[ \times [(1 + g_1)(1 + g_2)g_3 g_4 - g_1 g_2 (1 + g_3)(1 + g_4)] \]

\[ I_{12}(\epsilon_1, t) = \frac{m^3 g^2}{2\pi^3 \hbar^7} \int d\epsilon_2 \int d\epsilon_3 \int d\epsilon_4 \left[ \delta(\epsilon_1 - \epsilon_2) - \delta(\epsilon_1 - \epsilon_3) - \delta(\epsilon_1 - \epsilon_4) \right] \]
\[ \times [(1 + g_2)g_3 g_4 - g_2 (1 + g_3)(1 + g_4)] \]
\[ \times \int_{U \leq \epsilon_{\text{min}}} d\mathbf{r} \, n_c(\mathbf{r}, t) S(p_2, p_3, p_4) \delta(E_c(\mathbf{r}, t) + \epsilon_2 - \epsilon_3 - \epsilon_4) \]

**Time Evolution**

\[ \rho g(t_{i+1}) = \rho g(t_i) + \Delta t \left[ -\frac{\partial (\rho_w g)}{\partial \epsilon} + I_{12} + I_{22} \right]_{t_i} \]

\[ N_c(t_{i+1}) = N_c(t_i) - \Delta t \int d\epsilon \, I_{12}(\epsilon, t_i) \]

\[ N_c \rightarrow \mu_c \rightarrow U \rightarrow \rho_w \rightarrow g(\epsilon) \]
Growth Curves vs. Depth of Cut

- the cloud is initially in equilibrium right at the transition temperature: \( T_i = T_c \)
- particles with energies \( \varepsilon > kT_{cut} \) are removed at \( t = 0 \)
- with decreasing \( T_{cut} \), the total number and total energy of the gas decreases
- the saturation number increases with decreasing \( T_{cut} \) for the solid curves and decreases with \( T_{cut} \) for the dashed curves

\[
\tilde{N}(0) = 4 \times 10^7 \, ^{23}\text{Na atoms}
\]

\[
\nu_z = 18.0 \, \text{Hz}, \quad \nu_{\perp} = 82.3 \, \text{Hz}
\]
Growth vs. Initial Temperature

- the initial temperature $T$ is varied for a fixed depth of cut, $T_{cut}/T_c = 2.5$
- the onset time increases with $T$; it takes longer for the cloud to reach a supercritical state at low energies when the gas is heated
- also, the gas relaxes to equilibrium more slowly with increasing $T$

$T/T_c = 1 \rightarrow 1.3$

$\tilde{N}(0) = 4 \times 10^7 \ 23Na \ atoms$
**Energy Distribution vs. Time**

- distribution function at equal time intervals; the curves are displaced vertically for clarity
- the peak at low energies corresponds to a supercritical build up of the population: \( g(\epsilon) \sim \epsilon^{-1.63} \)
- distribution function at fixed energies as a function of time
- the increase at early times corresponds to the build up of a supercritical population
Comparison with the MIT Experiment

$T_{\text{cut}}/T_c = 1.9$ (a), 0.6 (b), 5.7 (c). (b) and (c) reproduce the final number of condensate atoms, but not the total number. (a) reproduces the total final number. The dashed curve shows the effect of switching off the mean field interactions.

In this case, the experimental values of $N$ and $T_{\text{cut}}$ are not available and were chosen to reproduce the final observed number of condensate atoms. It should be regarded simply as a fit to the experimental data.
Recap of the ZNG Formalism

- the condensate is described by the generalized Gross-Pitaevskii equation
  \[ i\hbar \frac{\partial \Phi}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}} + gn_c + 2g\tilde{n} - iR \right] \Phi + g\tilde{n}\Phi^* \]
  
  **Popov approximation:** \( \tilde{m} = 0 \)

- the thermal cloud is described by the kinetic equation
  \[ \frac{\partial f}{\partial t} + \frac{p}{m} \cdot \nabla f - \nabla U \cdot \nabla_p f = C_{12}[f, \Phi] + C_{22}[f] \]
  \[ n_c(r, t) = |\Phi(r, t)|^2, \quad \tilde{n}(r, t) = \int \frac{d^3p}{(2\pi\hbar)^3} f(p, r, t) \]
  \[ U(r, t) = V_{\text{trap}}(r) + 2gn_c(r, t) + 2g\tilde{n}(r) \]
Collisions

- collisions occur between thermal atoms and between thermal atoms and the condensate; they are essential for establishing equilibrium within the system

\[
\frac{\partial f}{\partial t} + \frac{p_i}{m} \cdot \nabla f - \nabla U \cdot \nabla_p f = C_{12}[f, \Phi] + C_{22}[f]
\]

\[
C_{12}[f] = \frac{2g^2 n_c}{(2\pi)^2 \hbar^4} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \delta(m\mathbf{v}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)
\times \delta(\varepsilon_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3}) [\delta(\mathbf{p} - \mathbf{p}_1) - \delta(\mathbf{p} - \mathbf{p}_2) - \delta(\mathbf{p} - \mathbf{p}_3)]
\times [(1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3)]
\]

\[
C_{22}[f] = \frac{2g^2}{(2\pi)^5 \hbar^7} \int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4 \delta(\mathbf{p} + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)
\times \delta(\tilde{\varepsilon}_p + \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3} - \tilde{\varepsilon}_{p_4})
\times [(1 + f)(1 + f_2)f_3f_4 - f_2(1 + f_3)(1 + f_4)]
\]
Mean-field Coupling

\[ i\hbar \frac{\partial \Phi}{\partial t} = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}} + g[n_c + 2\tilde{n}] - iR \right) \Phi \]

\[ \frac{\partial f}{\partial t} + \frac{p}{m} \cdot \nabla f - \nabla U \cdot \nabla_p f = C_{22}[f] + C_{12}[f] \]

\[ U(r, t) = V_{\text{trap}}(r) + 2gn_c(r, t) + 2g\tilde{n}(r, t) \]
Collisional Coupling

\[ i\hbar \frac{\partial \Phi}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}} + gn_c + 2g\tilde{n} - iR \right] \Phi \]

\[ R = \frac{\hbar}{2n_c} \int \frac{dp}{(2\pi \hbar)^3} C_{12} \quad \text{exchange of atoms} \]

\[ \frac{\partial f}{\partial t} + \frac{p}{m} \cdot \nabla f - \nabla U \cdot \nabla_p f = C_{22}[f] + C_{12}[f] \]
**Equilibrium**

- the thermal cloud is represented by the equilibrium distribution

\[ f^0(p, r) = \frac{1}{e^{\beta[p^2/2m+U_0(r)-\tilde{\mu}_0]} - 1} \]

\[ U_0(r) = V_{\text{trap}}(r) + 2g[n_{c0}(r) + \tilde{n}_0(r)] \]

\[ \tilde{n}_0(r) = \int \frac{d^3p}{(2\pi \hbar)^3} f^0(p, r) = \frac{1}{\lambda^3_T} g^{3/2}(z_0(r)) \]

- \( \tilde{n}_0 \) must be determined selfconsistently with \( n_{c0} = |\Phi_0|^2 \)

\[ \left(-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}} + gn_{c0} + 2g\tilde{n}_0 \right) \Phi_0 = \mu_{c0} \Phi_0 \]