EXTREME VALUE STATISTICS AND RANDOM MATRICES

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Chaos and Collectivity in Many-Body Systems,
MPIPKS, Dresden, 5–8 March, 2008

- Large deviations and random matrices.

Gaussian
Wishart-Laguerre } ensembles of RM

Maximum eigenvalue (edge)

Tracy-Widom distribution

Far from the edge (large deviations)

for { Gaussian
Wishart

Coulomb gas

D. S. Dean, S. N. Majumdar, Phys. Rev. Lett. 97 (2006) 160201

P. Vivo, S. N. Majumdar, O. Bohigas, J. Phys. A40 (2007) 4317

Acta Physica Polonica 38 (2007) 4139

- Entanglement of a random pure state.

Bipartite system. Reduced density matrix

Entanglement and minimum eigenvalue of reduced density matrix

Probability distribution of the minimum eigenvalue

for complex
real } random state

S. N. Majumdar, O. Bohigas, A. Lakshminarayan,
J. Stat. Phys. February 2008

- Exact maximum (minimum) intensity distribution of RMT eigenstates.

Convergence to Gumble (Weibull) distribution

Application to the standard map

(kicked rotor on the torus)

A. Lakshminarayan, Tomsovic, Bohigas, Majumdar,

Phys. Rev. Lett. 100 (2008) 044103

Extreme Value Theory (EVT)

X_i : independent random variables with a common distribution function F ← parent distribution

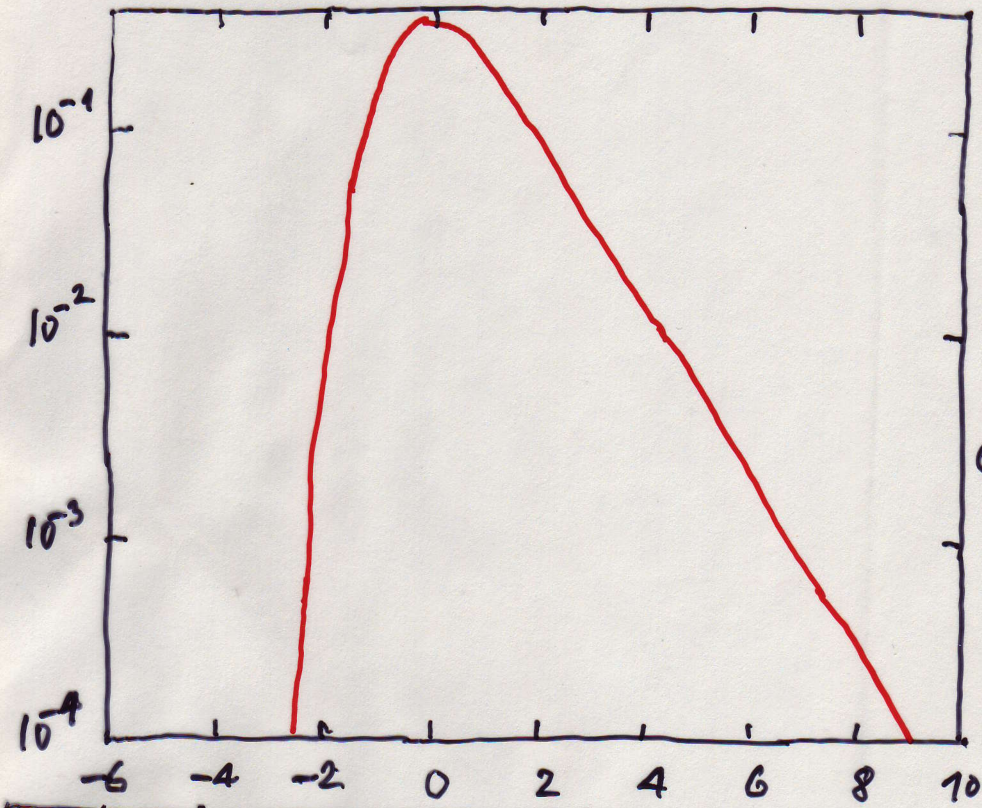
$$M_N = \max\{X_1, X_2, \ldots, X_N\}$$

$$\mathrm{Prob}\{M_N \le x\} = \mathrm{Prob}\{X_1 \le x, X_2 \le x, \ldots, X_N \le x\} = \{F(x)\}^N =$$

$$= \left[1 - \int_x^\infty p(x')\,dx'\right]^N \approx e^{-N\int_x^\infty p(x')\,dx'}$$

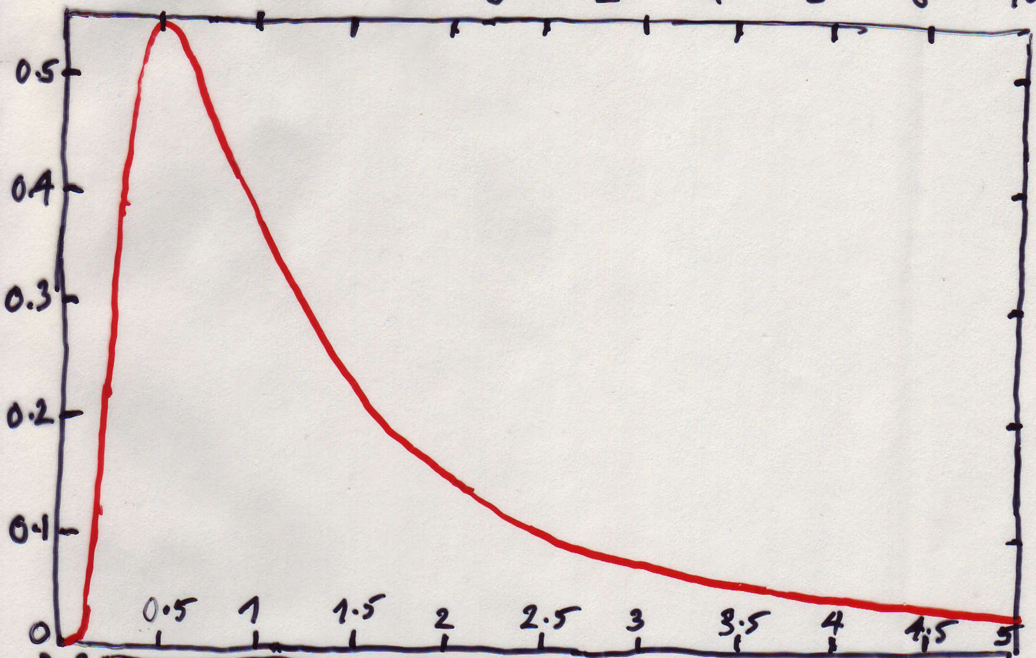
Limiting distribution of M_N has always one of the three forms, irrespective of the original distribution function

	Type I	Type II	Type III
parent distribution	$[\ ,\infty]$	$[\ ,\infty]$	bounded
	exponential, gaussian	heavy tailed (power law)	
	Gumbel	Fréchet Pareto	Weibull



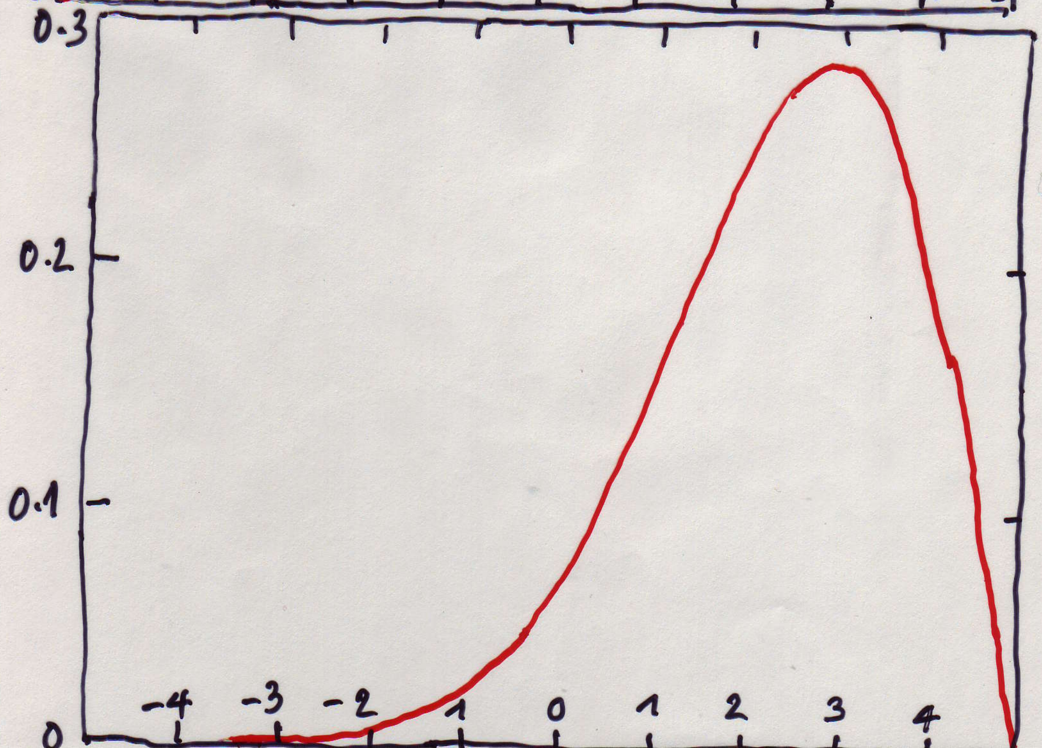
Gumbel

$$e^{-x}\cdot e^{-e^{-x}}$$



Fréchet

$$x^{-2}e^{-\frac{1}{x}}$$



Weibull

$$-xe^{-x^2}$$

Wigner-Dyson (Gaussian)

H : $N \times N$ hermitean matrix

H : real eigenvalues

$$P(H) \propto e^{-\frac{\beta}{2} \operatorname{Tr} H^2}$$

Wishart

X : $M \times N$ matrix $\quad M \geq N \quad \left(\frac{N}{M} = c \leq 1\right)$

$W = X^+ X$: hermitean $N \times N$ positive definite (Wishart) matrix

X^+ : conjugate transpose

$$P(X) \propto e^{-\frac{\beta}{2} \operatorname{Tr} X^+ X}$$

X : gaussian entries (i.i.d.)

Joint distribution of eigenvalues

Gaussian

$$\propto e^{-\frac{\beta}{2}\sum_1^N \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod_1^N d\lambda_i$$

Wishart

$$\propto \prod_1^N \lambda_i^{\frac{\beta}{2}(1+M-N)-1} e^{-\frac{\beta}{2}\lambda_i} \cdot \prod_{j<k} |\lambda_j - \lambda_k|^\beta \cdot \prod_1^N d\lambda_i$$

for $c=1, \beta=2$

$$\propto \prod_1^N e^{-\lambda_i} \cdot \prod_{j<k} |\lambda_j - \lambda_k|^2 \prod_1^N d\lambda_i$$

Gaussian

2d Coulomb gas submitted to external quadratic potential

Wishart

2d Coulomb gas confined to positive half line subject to an external linear + logarithmic potential

Eigenvalue density

Gaussian

$$\rho_N(\lambda) = \frac{1}{\sqrt{N}} f\left(\frac{\lambda}{\sqrt{N}}\right)$$

$$f(x) = \sqrt{\frac{1}{\pi}(2-x^2)} \qquad [-\sqrt{2}, \sqrt{2}]$$

Wigner semi-circle

Wishart

$$\rho_N(\lambda) = \frac{1}{N} f\left(\frac{\lambda}{N}\right)$$

Marčenko-Pastur

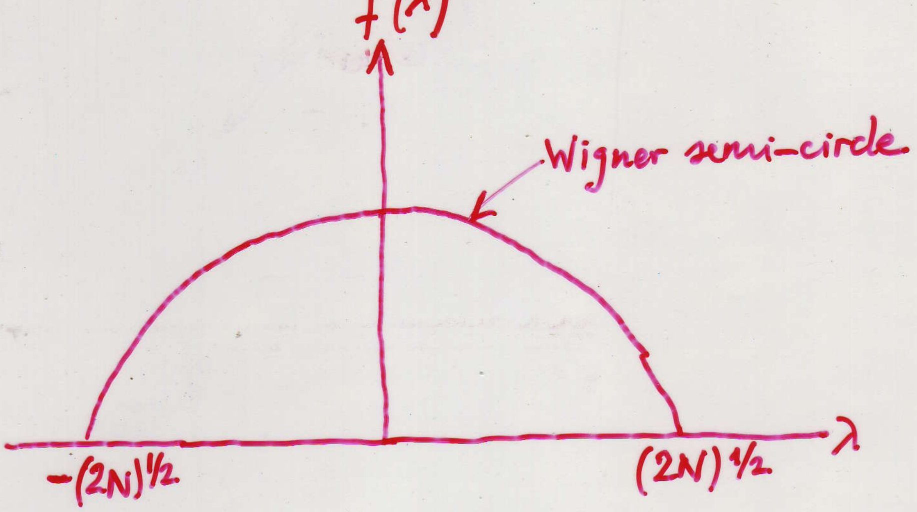
$$f(x) = \frac{1}{2\pi x}\sqrt{(x_+ - x)(x - x_-)}$$

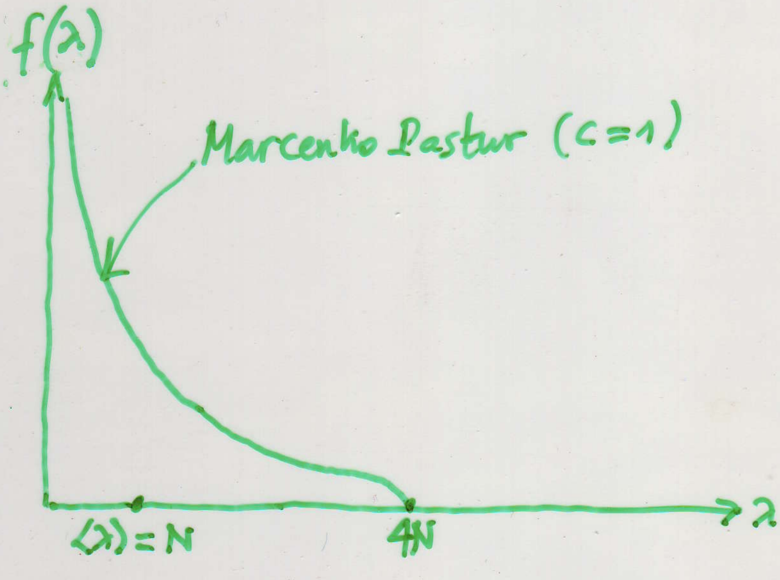
$$x_\pm = \left(\frac{1}{\sqrt{c}} \pm 1\right)^2 \qquad [x_-, x_+]$$

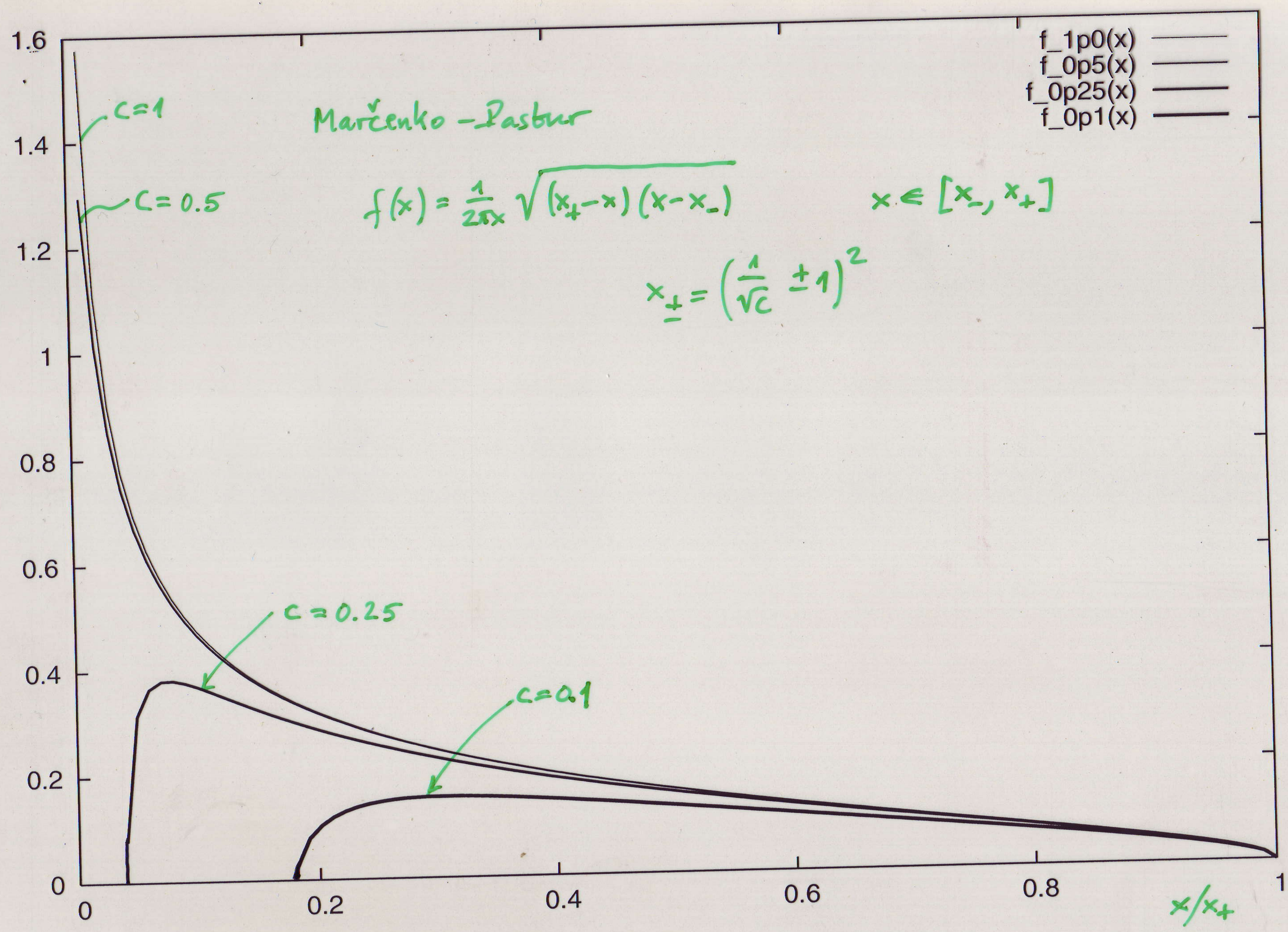
$$c = \frac{N}{M} \leq 1$$

For $c = 1$ $(M = N)$

$$f(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}} \qquad [0, 4]$$


f(λ)
Wigner semi-circle
λ
−(2N)^{1/2}
(2N)^{1/2}


f(λ)
Marcenko Pastur (c=1)
λ
⟨λ⟩ = N
4N



f_1p0(x)
f_0p5(x)
f_0p25(x)
f_0p1(x)
Marčenko – Pastur
$f(x) = \frac{1}{2\pi x}\sqrt{(x_+ - x)(x - x_-)}$
$x \in [x_-, x_+]$
$x_\pm = \left(\frac{1}{\sqrt{c}} \pm 1\right)^2$
c=1
c=0.5
c=0.25
c=0.1
x/x+
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6

$$\rho(\lambda) = \begin{cases} \frac{1}{\pi}\sqrt{\frac{2}{N}}\sqrt{1-\frac{\lambda^2}{2N}} & |\lambda| \le (2N)^{1/2} \\ 0 & \end{cases}$$

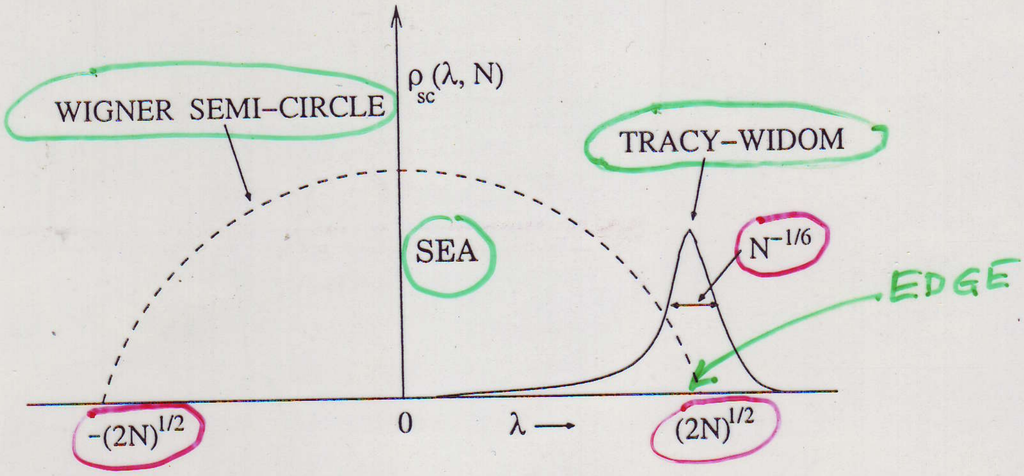


FIG. 1: The dashed line shows the semi-circular form of the average density of states. The largest eigenvalue is centered around its mean $\sqrt{2N}$ and fluctuates over a scale of width $N^{-1/6}$. The probability of fluctuations on this scale is described by the Tracy-Widom distribution (shown schematically).

scaling variable

$$\boxed{\xi = \sqrt{2}\, N^{1/6}\left[\lambda_{max} - \sqrt{2N}\right]}$$

$\text{Prob}\left[\xi \le x\right] = F_\beta(x) \leftarrow$ has a limit $N \to \infty$

(Tracy-Widom)

from Dean, Majumdar PRL 97 (2006) 160201

$$\rho(\lambda) = \frac{1}{N} f\left(\frac{\lambda}{N}\right)$$

$$f(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}} \qquad [0, 4]$$

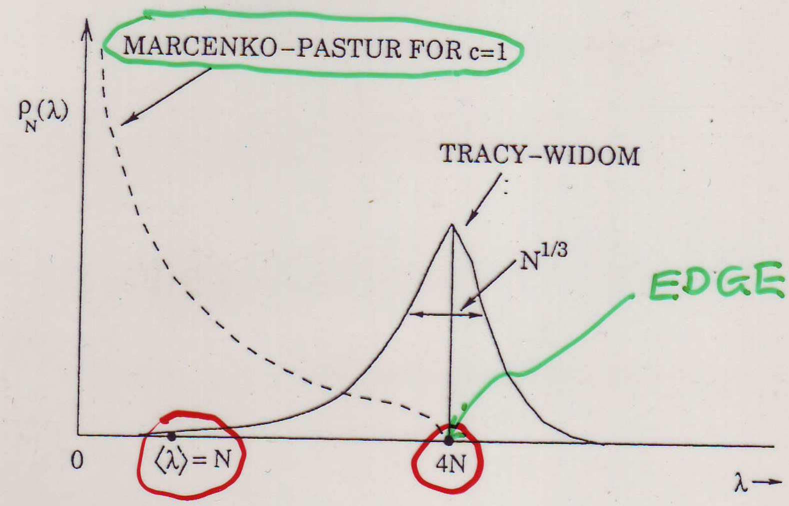


Figure 1. The dashed line schematically shows the Marčenko–Pastur form of the average density of states for $c = 1$. The average eigenvalue for $c = 1$ is $\langle\lambda\rangle = N$. For $c = 1$, the largest eigenvalue is centred around its mean $\langle\lambda_{\max}\rangle = 4N$ and fluctuates over a scale of width $N^{1/3}$. The probability of fluctuations on this scale is described by the Tracy–Widom distribution (shown schematically).

scaling variable

$$\boxed{\xi = c^{-1/6} x_+^{-2/3} N^{-1/3} (\lambda_{max} - x_+ N)}$$

$$x_+ = \left(\frac{1}{\sqrt{c}} + 1\right)^2, \quad c = \frac{N}{M}$$

$\mathrm{Prob}(\xi \le x) = F_\beta(x)$ has a large N limit (Tracy-Widom)

Johansson
Johnston

From P. Vivo, S. Majumdar, O. Bohigas
cond-mat/0701371
J. Phys. A 40 (2007) 4371

Large deviations and random matrices

Gaussian
Wishart $\Big\}$ distribution of λ_{max}

given by Tracy-Widom for typical fluctuations

scale of typical fluctuations

$O(N^{-1/6})$ $\qquad\qquad \sqrt{2N}$ $\qquad$ Gaussian

around mean

$O(N^{1/3})$ $\qquad\qquad x_+(c)\,N$ $\qquad$ Wishart

Study of atypical and large fluctuations of λ_{max} around its mean, over a range

$O(N^{1/2})$ $\qquad$ for $\qquad$ Gaussian

$O(N)$ $\qquad\qquad\qquad$ Wishart

Question

$$\text{Prob}\{\lambda_{max} < t\} = \Pr\{\lambda_1 < t, \lambda_2 < t, \dots; \lambda_N < t\} = Q_N(t)$$

Find the equilibrium charge density of the Coulomb gas with a barrier at t

Restricted partition function

$$Z_N(t) = \int_{I(t)} \prod_{i=1}^{N} d\lambda_i \exp\left[-\frac{\beta}{2}\left(\sum_{i=1}^{N} V(\lambda_i) - \sum_{i<j} \ln(|\lambda_i - \lambda_j|)\right)\right]$$

$I(t)$: allowed range for eigenvalues

$(-\infty, t)$ Gaussian

$[0, t]$ Wishart

$$Q_N(t) = \frac{Z_N(t)}{Z_N(\infty)}$$

$V(x) = x^2$ Gaussian

$V(x) = x - [(1+M-N) - \frac{2}{\beta}] \log x$ Wishart

Rescaled variables

$$\mu = \lambda N^{-\alpha} \qquad \alpha = \tfrac{1}{2} \quad \text{Gaussian WD}$$

location of the barrier $\longrightarrow$ $z = t N^{-\alpha}$ $\qquad \alpha = 1 \quad$ Wishart

$$Z_N(z) \propto \int \mathcal{D}[\hat{f}] \exp\{ \beta N^2 S[\hat{f}(\mu; z] + O(N) \}$$

$$S[\hat{f}(\mu, z)] = -\frac{1}{2} \int_{I(z)} d\mu\, \hat{f}(\mu)\, V(\mu)$$

$$+ \frac{1}{2} \int_{I(z)} \int_{I(z)} d\mu\, d\mu'\, \hat{f}(\mu) \hat{f}(\mu') \ln|\mu - \mu'|$$

$I(z)$: allowed range of eigenvalus

Leading contribution to action, from

$$\frac{\delta S}{\delta f} = 0$$

with solution $\hat{f}$

Leads to equation of the form

$$\mathcal{P}\int_0^z \frac{f(x')}{x-x'}\,dx' = g(x)$$

Its general solution (Tricomi)

$$f(x) = \frac{1}{\pi^2\sqrt{x(z-x)}}\left[\mathcal{P}\int_0^z \sqrt{w(z-w)}\;\frac{g(w)}{w-x}\,dw + B\right]$$

WD Gaussian $\quad g(w) = w$

Wishart $\quad g(w) = \frac{1}{2}$

For Wishart ($c=1$)

$$\hat{f}(\mu) = \frac{1}{2\pi\sqrt{\mu(z-\mu)}}\left[\frac{z}{2} - 2 - \mu\right]$$

$$0 \leq \mu \leq z$$

From $\hat{f}(\mu)$ one can calculate the saddle point action

$$S[\hat{f}(\mu); z] \quad ,$$

and the restricted partition function

$$Z_N(z) \approx \exp\{\beta N^2 S(z)\}$$

The deformed semi-circle

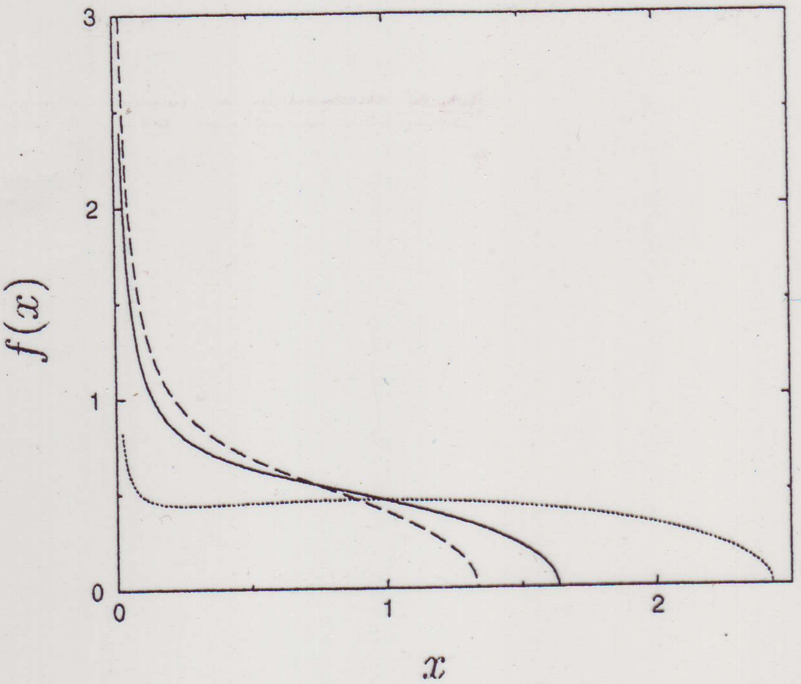


FIG. 2. The average density of states $f(x)$ plotted as a function of the shifted variable x for $z = -1$ (dotted line), $z = 0$ (solid line), and $z = 0.5$ (dashed line).

D. S. Dean, S. N. Majumdar, Phys. Rev. Lett. 97 (2006) 160201

C=1

The deformed Marčenko-Pastur

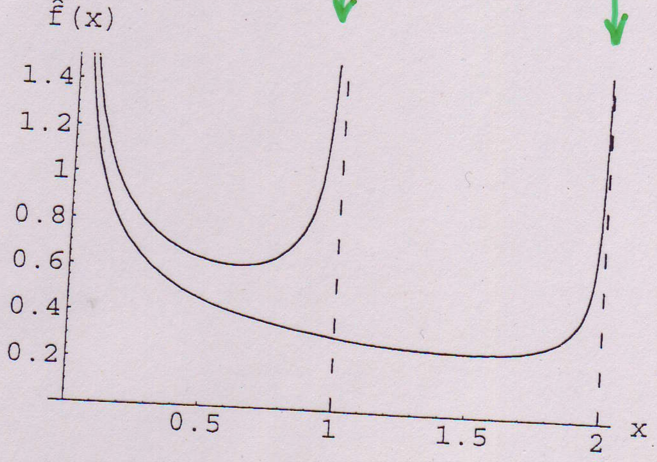


Figure 2. Constrained spectral density $\hat{f}(x)$ for the barrier at $\zeta = 1$ and $\zeta = 2$.

C=1

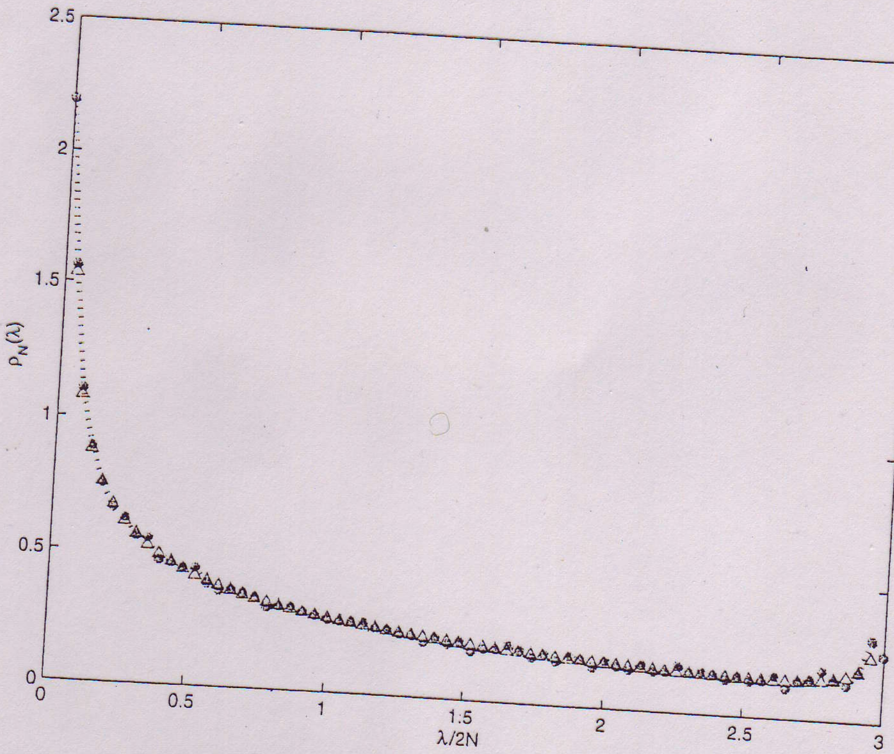


Figure 6. Constrained spectral density $\varrho_N(\lambda)$ for $N = M = 30$. The barrier is at $\zeta = 3$. In dotted green is the histogram of rescaled eigenvalues over an initial sample of 3×10^5 matrices ($\beta = 2$). In triangled red is the theoretical distribution.

Vivo, Majumdar, OB

cond-mat/0701371

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c=0.1

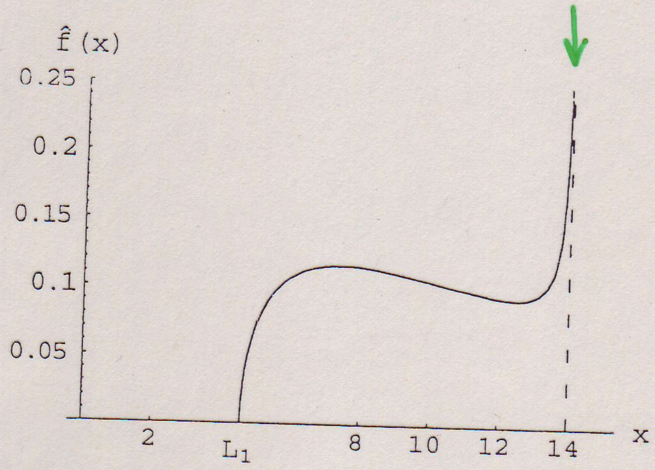
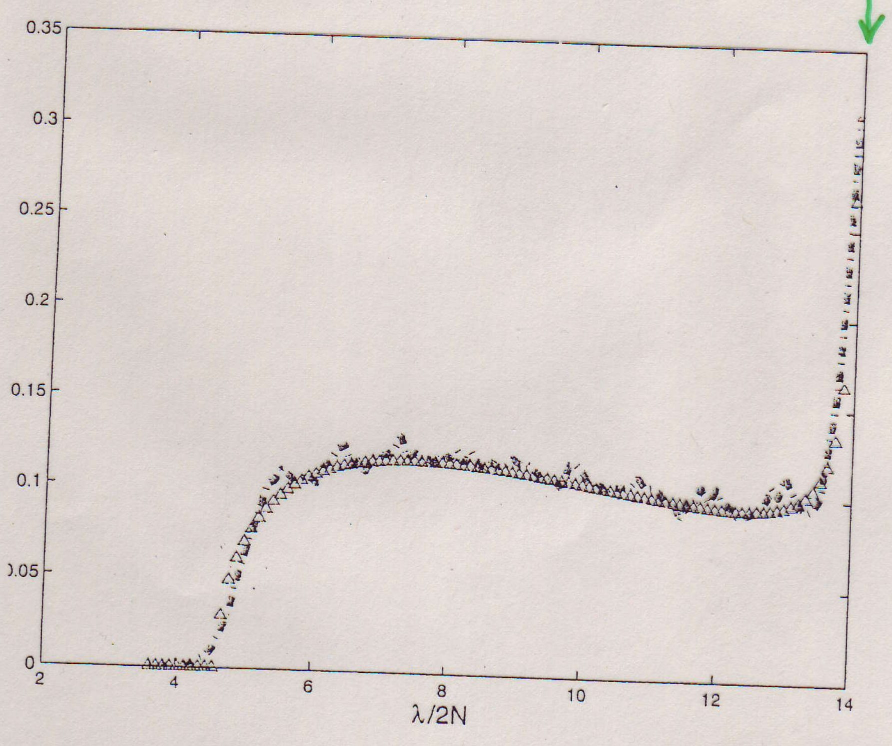


Figure 4. Constrained spectral density $\hat{f}(x)$ for $c = 0.1$ and $\zeta = 14$.



c=0.1

re 8. Constrained spectral density $\hat{\varrho}_N(\lambda)$ for $N = 10$, $M = 100$ ($c = 0.1$). The barrier is at $\zeta=14$. In dash-dotted green is the histogram of rescaled eigenvalues over an initial sample of $5\cdot10^5$ matrices ($\beta = 2$). In triangled red is the theoretical distribution.

Vivo, Majumdar, O B

Cond-mat/0701371

J. Phys. A40 (2007) 4317

Saddle point action

Gaussian WD

$$S'(z) = -\frac{1}{216}\Big\{72z^2 - 2z^4 + (30z + 2z^3)\sqrt{6+z^2}$$
$$+ 27\big[3 + \ln 1296 - 4\ln\big(-z + \sqrt{6+z^2}\big)\big]\Big\}$$

Wishart

$$S'(z) = 2\log 2 - \log z + \frac{z}{2} - \frac{z^2}{32}$$

z : location of the barrier

$$Q_N(t) = \frac{Z_N(t)}{Z_N(+\infty)} = \text{Prob}\{\lambda_{max} < t\}$$

$$Q_N(t = zN^{\alpha}) \qquad \begin{pmatrix} \alpha = 1/2 & \text{Gaussian} \\ = 1 & \text{Wishart} \end{pmatrix}$$

$$\approx \exp\left[-\beta N^2 \Phi(z)\right]$$

$$\Phi(z) = \begin{cases} S(z) - S\sqrt{2} & \text{Gaussian WD} \\ S(z) - S(x_+) & \text{Wishart} \end{cases}$$

$Q_N(t)$ matches exctly the $\overset{\text{left}}{\text{tail}}$ of the Tracy-Widom distribution

$$\text{Prob}(\lambda_{max} \leq 0, N) = \exp\left(-\beta \tfrac{1}{4} \ln 3\right) N^2$$

$\langle\lambda\rangle$

$$= \exp\left(-0.2746\, \beta N^2\right)$$

Wigner Dyson

$$\text{Probability}\{\lambda_{max} < N; N\} = \exp\left(-\beta \left(\log 2 - \tfrac{33}{64}\right) \cdot N^2\right.$$

$\langle\lambda\rangle$

$$= \exp\left(-\beta \left(\log 2 - \tfrac{33}{64}\right) \cdot N^2\right)$$

Wishart

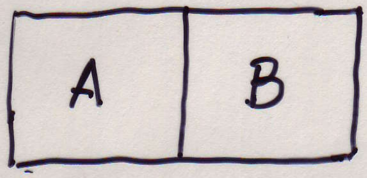
$$= \exp\left(-0.1775 \,.\, \beta N^2\right)$$

Entanglement of a random pure state

In quantum information and quantum computation

entanglement of states

Here, bipartite entanglement



dimension: N $\quad M$ $\qquad N \leq M$

basis $|i^A\rangle$ $\quad |\alpha^B\rangle$

$\mathcal{H}_A^{(N)}$ $\quad \mathcal{H}_B^{(M)}$

Hilbert space of composite system

$$\mathcal{H}^{(NM)} = \mathcal{H}_A^{(N)} \otimes \mathcal{H}_B^{(M)}$$

Any quantum state of the composite system

$$|\Psi\rangle = \sum_{i=1}^{N} \sum_{\alpha=1}^{M} x_{i,\alpha} \, |i^A\rangle \otimes |\alpha^B\rangle$$

$|i^A\rangle$: complete basis of $\mathcal{H}_A^{(N)}$

$|\alpha^B\rangle$: complete basis of $\mathcal{H}_B^{(M)}$

entries of a rectangular $(N \times M)$ matrix X

Properties (mutually non exclusive) of $|\Psi\rangle$

- entangled

fully unentangled when $x_{i,\alpha} = a_i b_\alpha$ for all i and α

$$\therefore \quad |\Phi^A\rangle = \sum_{i=1}^{N} a_i \, |i^A\rangle$$

$$\Rightarrow |\Psi\rangle = |\Phi^A\rangle \otimes |\Phi^B\rangle$$

$$|\Phi^B\rangle = \sum_{\alpha=1}^{M} b_\alpha \, |\alpha^B\rangle$$

- random $\quad x_{i,\alpha}$: random variables

- pure

if density matrix of composite system

$$\rho = |\Psi\rangle\langle\Psi|$$

$\mathrm{Tr}\,\rho = 1$, from $\langle\Psi|\Psi\rangle = 1$

((for composite system

$$\rho = \sum_k p_k \, |\Psi_k\rangle\langle\Psi_k|$$

$0 \leq p_k \leq 1$

$|\Psi_k\rangle$ are pure states of the composite system

$0 \leq p_k \leq 1 \qquad \sum_k p_k = 1$))

density matrix of pure state $|\Psi\rangle$

$$\rho = \sum_{i,\alpha} \sum_{j,\beta} x_{i,\alpha} x^*_{j,\beta} \, |i^A\rangle\langle j^A| \otimes |\alpha^B\rangle\langle\beta^B|$$

reduced density matrix of subsystem A

$$\rho_A = \mathrm{Tr}_B \, \rho = \sum_{i,j=1}^{N} \sum_{\alpha=1}^{M} x_{i,\alpha} x^*_{j,\alpha} \, |i^A\rangle\langle j^A|$$

$$= \sum_{i,j=1}^{N} W_{i,j} \, |i^A\rangle\langle j^A|$$

W_{ij} : entries of $N \times N$ square matrix W

$$W = X X^+$$

W : eigenvalues $\lambda_1, \dots, \lambda_N$; $\lambda_i \geq 0$; $\sum_1^N \lambda_i = 1$

eigenvectors $|\lambda_i^A\rangle$

In this diagonal representation

$$\rho_A = \sum_{i=1}^{N} \lambda_i \, |\lambda_i^A\rangle\langle\lambda_i^A|$$

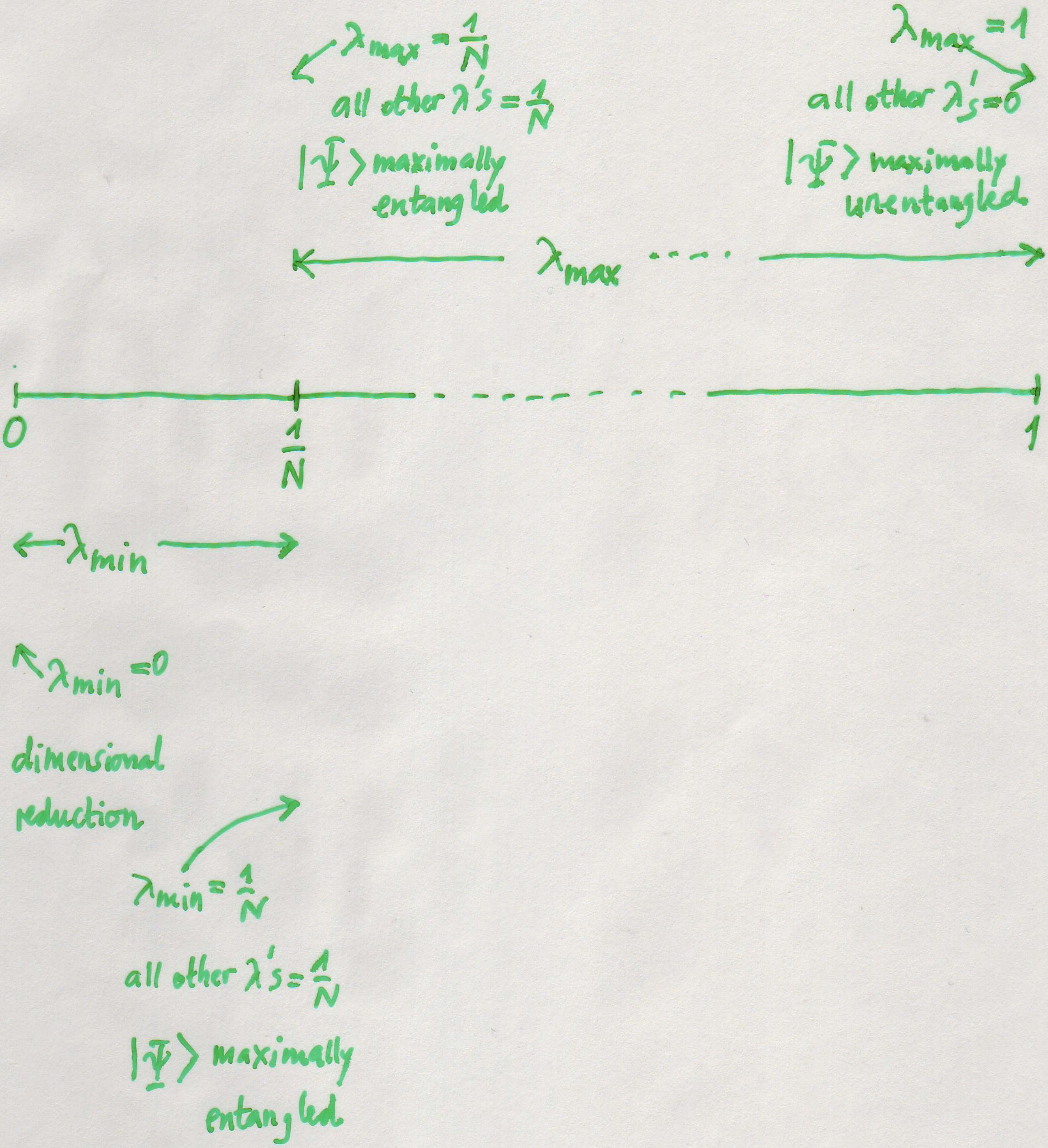
$$|\Psi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i} \, |\lambda_i^A\rangle \otimes |\lambda_i^B\rangle$$

$\nwarrow X^+ X$

Useful measure of entanglement

von Neumann entropy

$$S = -\sum_{i=1}^{N} \lambda_i \log \lambda_i$$



Proximity of λ_{min} to borders provides information on entanglement and dimensional reduction

Random, Pure State $|\Psi\rangle$:

$x_{i,\alpha}$: independent and identically distributed Gaussian variables, real or complex

$W = XX^+$: is then (almost) a random Wishart matrix $(\sum \lambda_i = 1)$

Joint distribution of the N (non-negative) eigenvalues of W

$$P^W(\lambda_1, \dots, \lambda_N) = B_{M,N}\, \delta\Big(\sum_{i=1}^N \lambda_i - 1\Big) \cdot \prod_{i=1}^N \lambda_i^{\frac{\beta}{2}(M-N+1)-1} \cdot \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

$$\text{Prob}\{\lambda_{min} \geq x\} = \text{Prob}\{\lambda_1 \geq x, \lambda_2 \geq x, \dots, \lambda_N \geq x\} =$$

$$\left.\begin{array}{l} M=N \\ \beta=2 \end{array}\right\} = Q_N(x) = B_{N,N} \int_x^\infty \cdots \int_x^\infty \delta\Big(\sum \lambda_i - 1\Big) \prod_{j<k} (\lambda_j - \lambda_k)^2 \cdot \prod_{i=1}^N d\lambda_i$$

distribution function of λ_{min}

its derivative : probability density $P_N(x)$

Page, '95 Bandyopadhyay & Lakshminarayan 2003

$1 \ll N \leq M$ $\qquad \langle S \rangle \approx \ln N - \frac{N}{2M}$

Complex case $\beta = 2$

probability density
$$P_N(x) = N(N^2-1)\,(1-Nx)^{N^2-2}\;\theta(1-Nx)$$

exact, finite N

Real case $\beta = 1$

Complicated expression in terms of hypergeometric function

(exact, finite N)

$$\langle \lambda_{min} \rangle = \frac{1}{N^3}$$

(Zdinaric conjectured, 2007)

For large N
$$\langle \lambda_{min} \rangle \approx \frac{c}{N^3}$$

$$c = 2\left[1 - \sqrt{\frac{\pi e}{2}}\,\mathrm{erfc}\left(\frac{1}{\sqrt{2}}\right)\right] = 0.688641\ldots$$

$x \to 0$ $\quad N(N^2-1)\exp[-N(N^2-1)x]$ $\quad C_N \cdot x^{-1/2}$

$x \to \frac{1}{N}$ $\quad N(N^2-1)(1-Nx)^{N^2-2}$ $\quad A_N N^{-N/2}(1-Nx)^{\frac{N^2+N-4}{2}}$

$\uparrow$ $\beta = 2$ $\qquad\qquad$ $\uparrow$ $\beta = 1$

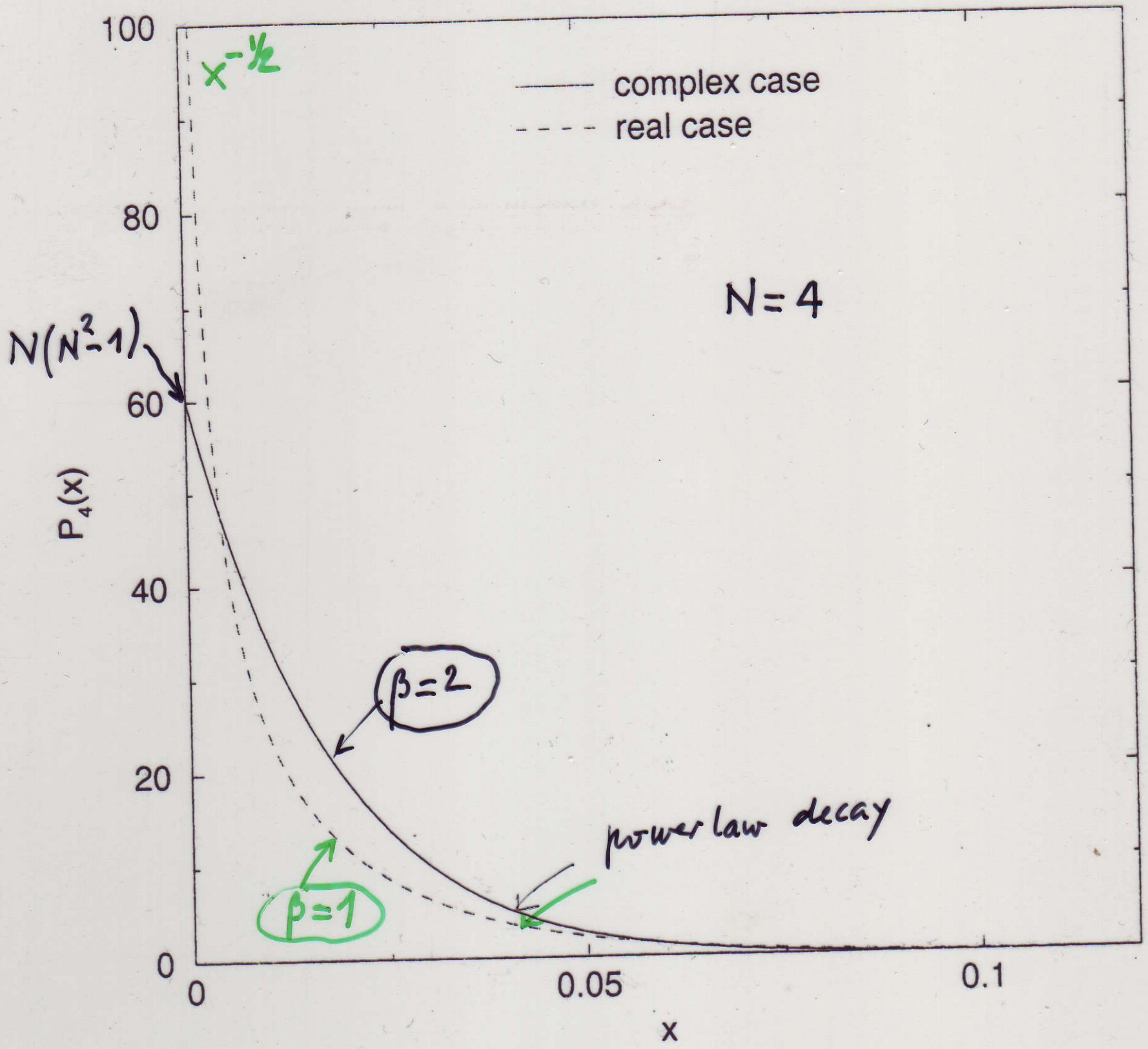


FIG. 1: The p.d.f $P_N(x)$ of the minimum eigenvalue $\lambda_{\min}$ vs. x for $N = 4$, for the complex and the real cases (Eqs. 28, 52) respecively). In the complex case, the density approaches a constant as $x \to 0$, whereas for the real case, it d diverges as $x^{-1/2}$ as $x \to 0$.

Lakshminarayan, Majumdar, OB
arXiv 0711.0677
J. Stat. Phys.

maximum (minimum) intensity distribution of RMT eigenstates

In an N-dimensional Hilbert space a general state is represented in a fixed orthonormal basis as

$$|\Psi\rangle = \sum_{i=1}^{N} z_i |i\rangle$$

z_i: complex components of a random state

constraint: normalization

$$\sum_{i=1}^{N} |z_i|^2 = 1$$

$|z_i|^2$: distributed uniformly on an $(N-1)$ simplex

Maximum intensity distribution $F(t,N)$

$$t = \max\{ |z_1^2|, |z_2|^2, \dots, |z_N|^2 \}$$

$$F(t \in I_k, N) = \sum_{m=0}^{k} \binom{N}{m} (-1)^m (1 - mt)^{N-1}$$

$$k = 1, 2, \dots, N-1$$

$$I_k = \left[\frac{1}{k+1}, \frac{1}{k} \right]$$

piecewise smooth

$$\langle t \rangle = \frac{\gamma + \ln N}{N} + O\left(\frac{1}{N^2}\right)$$

$N \to \infty$

$$F(t, N\to\infty) \approx \exp\left\{ -e^{-N[t - \ln N/N]} \right\}$$

Gumbel

Minimum intensity distribution

$$s = \min\{ |z_1|^2, |z_2|^2, \dots, |z_N|^2 \}$$

$$F(s,N) = \begin{cases} 1 - (1 - Ns)^{N-1} & 0 \le s \le 1/N \\ 1 & 1/N \le s \le 1 \end{cases}$$

$$\langle s \rangle = \frac{1}{N^2}$$

$$F(s, N\to\infty) = 1 - \exp[-N^2 s]$$

Weibull

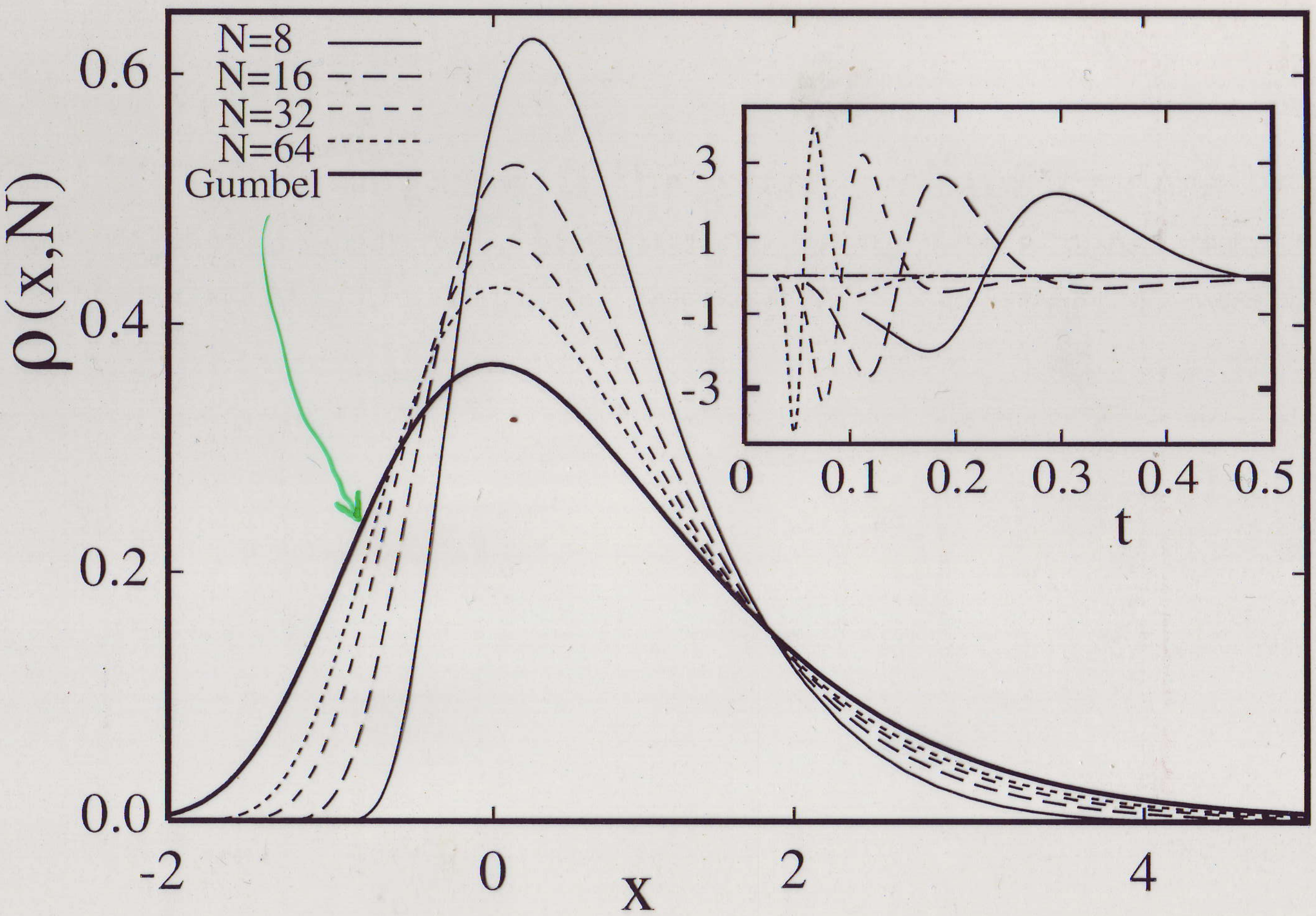


FIG. 1: The convergence of the exact probability density to the asymptotic Gumbel distribution using the scaled variable $x = N(t - \ln(N)/N)$ with increasing N. The inset shows the difference between the exact and the Gumbel distribution for the same values of N, but in the unscaled variable.

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Kicked rotor.

$$H(q,p) = \frac{p^2}{2} - \frac{K}{4\pi^2}\cos(2\pi q)\sum_{n=-\infty}^{+\infty}\delta(t-n)$$

Classical mapping

$$\begin{cases} p_{i+1} = p_i - \dfrac{K}{2\pi}\sin 2\pi q_i \\ q_{i+1} = q_i + p_{i+1} \end{cases}$$

Quantum evolution operator

$$U_{nn'} = \frac{1}{N}\exp\left\{\frac{iNK}{2\pi}\cos\frac{2\pi(n'+\alpha)}{N}\right\}\times$$

$$\times\sum_{m=0}^{N-1}\exp\left\{-\pi i\frac{(m+\beta)^2}{N} + \frac{2\pi i(m+\beta)(n-n')}{N}\right\}$$

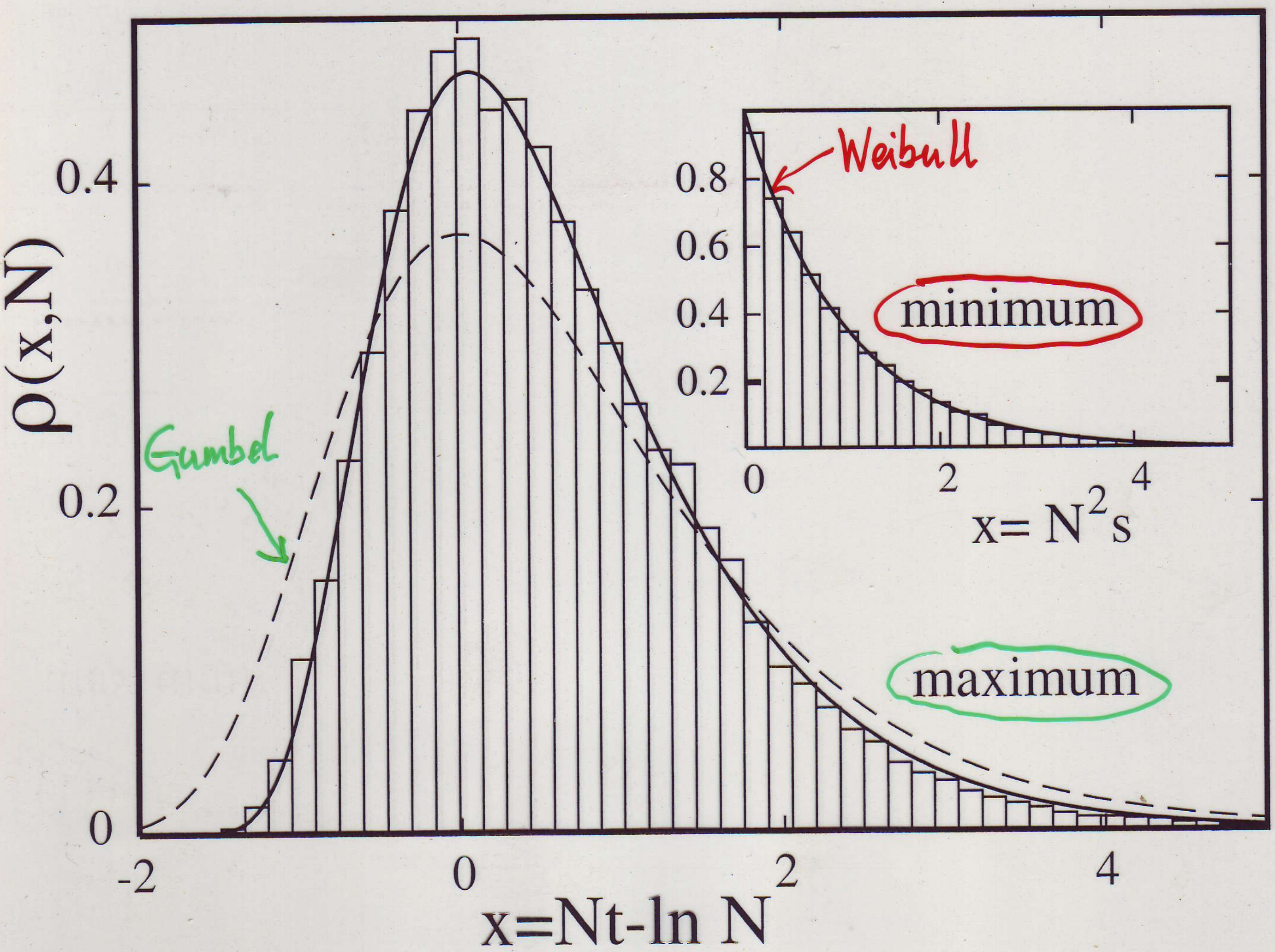

FIG. 2: The probability densities (histograms) of the scaled maximum and minimum (inset) intensity of eigenfunctions in the position basis of the quantum kicked rotor for $N = 32$ in the parameter range $13.8 < K < 14.8$. Shown as a continuous line is the exact density for random states while the dotted ones are the respective Gumbel and Weibull densities.

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