Disordered ensembles of random matrices

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Heavy-tailed ensembles

"Levy matrices": Cizeau & Bouchaud
PRE 50, 1810 (1994)

uncorrelated non-invariant ensemble

Invariant ensembles

Bertuola, Bohigas and Pato, PRE 70, 065102(R) (2004)

F. Toscano, Vallejos and Tsallis, PRE 69, 066131 (2004

A. Y. Abul-Magd, PRE ~~71~~, 066207 (2005)
K. A. Muttalib and J. R. Klauder, PRE 71, 055101

Bohigas, de Carvalho and Pato,

PRE 77, 011122 (2008)

Cauchy variable
$(N = 1)$

$$\frac{1}{\pi}\frac{1}{1+x^2} = \frac{1}{\pi}\int_0^\infty d\xi \, e^{-\xi} e^{-\xi x^2} =$$

$$= \int_0^\infty d\xi \, \frac{e^{-\xi} \xi^{\frac{1}{2}-1}}{\sqrt{\pi}} \sqrt{\frac{\xi}{\pi}} \, e^{-\xi x^2} =$$

$$= \int_0^\infty d\xi \int_{-\infty}^{\infty} dh \, \frac{e^{-\xi} \xi^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2})} \frac{e^{-h^2}}{\sqrt{\pi}} \, \delta\left(x - \frac{h}{\sqrt{\xi}}\right)$$

$$x = \frac{h}{\sqrt{\xi}}$$

disordered Gaussian variable

combination of two independent random variables

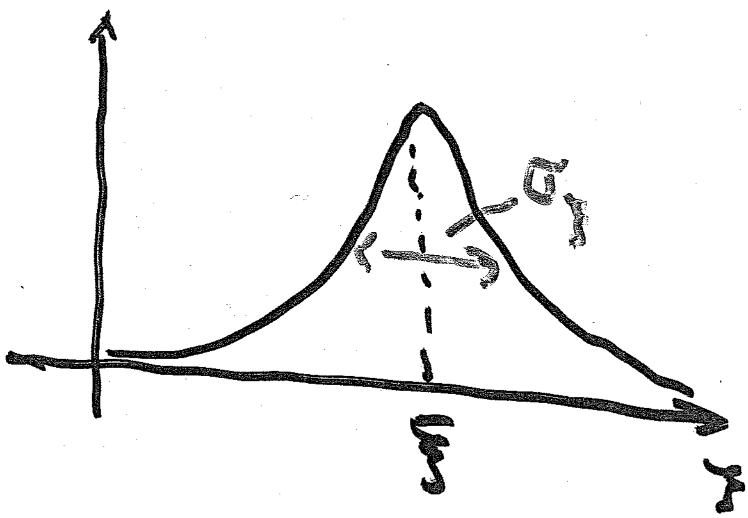
— $H_G(\alpha)$ is matrix with distribution

$$P_G(H;\alpha) = \left(\frac{\beta\alpha}{\pi}\right)^{f/2} e^{-\alpha\beta \,\mathrm{tr}\, H^2}$$

$$f = N + \beta \frac{N(N-1)}{2}, \quad \beta = 1, 2, 4, \ \alpha \text{ scale}$$

$$dH = \prod_{i=1}^{N} dH_{ii} \prod_{j>i} \prod_{k=1}^{\beta} \sqrt{2}\, dH_{ij}^{k}$$

— ξ is a positive random variable with distribution $w(\xi)$



Disordered ensemble

$$H(\alpha,\xi) = \frac{H_G(\alpha)}{\sqrt{\xi/\bar{\xi}}}$$

with $h_i = H_{ii}$ or $h_i = \sqrt{2}\, H_{ij}$

$$p(h_1, h_2 \dots h_n) = \left(\frac{\beta\alpha}{\pi\bar{\xi}}\right)^{\frac{n}{2}} \int d\xi\, w(\xi)\, \xi^{\frac{n}{2}}\, e^{-\frac{\beta\alpha\xi}{\bar{\xi}}\sum_1^n h_i^2}$$

for $n = f$

$$P(H) = \int d\xi\, w(\xi) \left(\frac{\beta\alpha\xi}{\pi\bar{\xi}}\right)^{f/2} e^{-\frac{\beta\alpha\xi}{\bar{\xi}}\,\mathrm{tr}\, H^2}$$

Correlated invariant ensemble

Generating a matrix

$$p(h_1, h_2, \dots, h_f) = p(h_1)\,\frac{p(h_1 h_2)}{p(h_1)}\,\frac{p(h_1 h_2 h_3)}{p(h_1 h_2)} \cdots \frac{p(h_1 h_2 \dots h_f)}{p(h_1 h_2 \dots h_{f-1})}$$

where

$$\frac{p(h_1 h_2)}{p(h_1)} = p(h_2/h_1) \quad \text{conditional probability}$$

$$\frac{p(h_1 h_2 h_3)}{p(h_1 h_2)} = p(h_3/h_1 h_2) \quad \dots$$

we have $$h_n = \frac{h_G(\alpha)}{\sqrt{\xi_n/\bar{\xi}}}$$

ξ_n distribution

$$w_n(\xi) = w(\xi)\,\xi^{\frac{n-1}{2}}\, e^{-\frac{\beta d}{\bar{\xi}}\xi \sum_1^{n-1} h_i^2} \Big/ \text{norm}$$

matrices fluctuate $\longrightarrow$ non-ergodicity

Spectral properties

Invariant ensemble

$$P(E_1, E_2, \ldots, E_N) = \int_0 d\xi \, w(\xi) \left(\frac{\alpha \xi}{\bar{\xi}}\right)^{\frac{N}{2}} \times$$

$$\times P_G\left(\sqrt{\frac{\alpha\xi}{\bar{\xi}}}\, E_1, \ldots, \sqrt{\frac{\alpha\xi}{\bar{\xi}}}\, E_N\right)$$

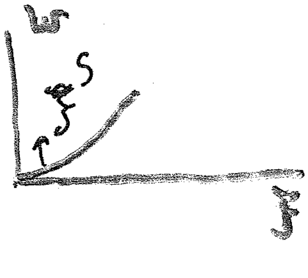
measures of P are weighted measures of P_G

Density:

$$\rho(E) = \sqrt{\frac{2\alpha}{\pi}} \int_0^{\xi_{max}} d\xi \, w(\xi) \left(\frac{\xi}{\bar{\xi}}\right)^{1/2} \sqrt{2N - \frac{2\alpha\xi}{\bar{\xi}} E^2}$$

Power-law:

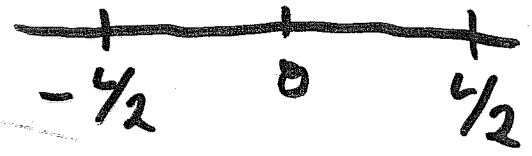
$$v = \frac{\alpha E^2}{N\bar{\xi}}\xi \;, \quad \rho(E) \propto \frac{1}{|E|^3} \int_0^1 dv \, w\left(\frac{N\bar{\xi} v}{\alpha E^2}\right) v^{\frac{1}{2}} \sqrt{1-v}$$



$$\rho(E) \xrightarrow{|E| \to \infty} \frac{1}{|E|^{2s+3}}$$

Nonergocity

spectral average $\neq$ ensemble average



$$N(L) = \int_{-\frac{L}{2}}^{\frac{L}{2}} dE\, \rho(E)$$

The variance of $\rho(E)$ in the interval $[-\frac{L}{2}, \frac{L}{2}]$ is given by

$$\mathrm{var}\,\rho = [\rho(0)]^2 \frac{\Sigma^2[N(L)]}{N^2(L)} \qquad \text{(Pandey, 1979)}$$

~~Ergodi~~ Ergodicity $\quad \mathrm{var}\,\rho \xrightarrow{L\to\infty} 0$

$$\Sigma^2(N) = \int d\xi\, w(\xi) \left[\Sigma_G^2(N_G) + N_G^2\right] - N^2$$

$$N_G(L) = \int_{-\frac{L}{2}}^{\frac{L}{2}} dE\, \rho_G(E;\xi)$$

Assuming that for large N the densities can be approximated by their values at the center of the spectrum

$$N(L) = \frac{\sqrt{4\alpha N}}{\pi} \left\langle \sqrt{\frac{\bar{\xi}}{\xi}} \right\rangle L$$

$$\langle N_G^2 \rangle_\xi = \frac{4\alpha N}{\pi^2} L^2$$

$$\langle N_G^2 \rangle - N^2 = \frac{4\alpha N}{\pi^2} \left(1 - \left\langle \sqrt{\frac{\bar{\xi}}{\xi}} \right\rangle^2 \right) L^2$$

But $$\left\langle \left(1 - \sqrt{\frac{\bar{\xi}}{\xi}} \right)^2 \right\rangle > 0$$

$$\Downarrow$$

$$\left\langle \sqrt{\frac{\bar{\xi}}{\xi}} \right\rangle < 1$$

and The ensemble is nonergodic independent of the choice of $w(\xi)$

$$w(\xi) = \frac{1}{\Gamma(\bar{\xi})} e^{-\xi} \xi^{\bar{\xi}-1}$$

Ensemble distribution:

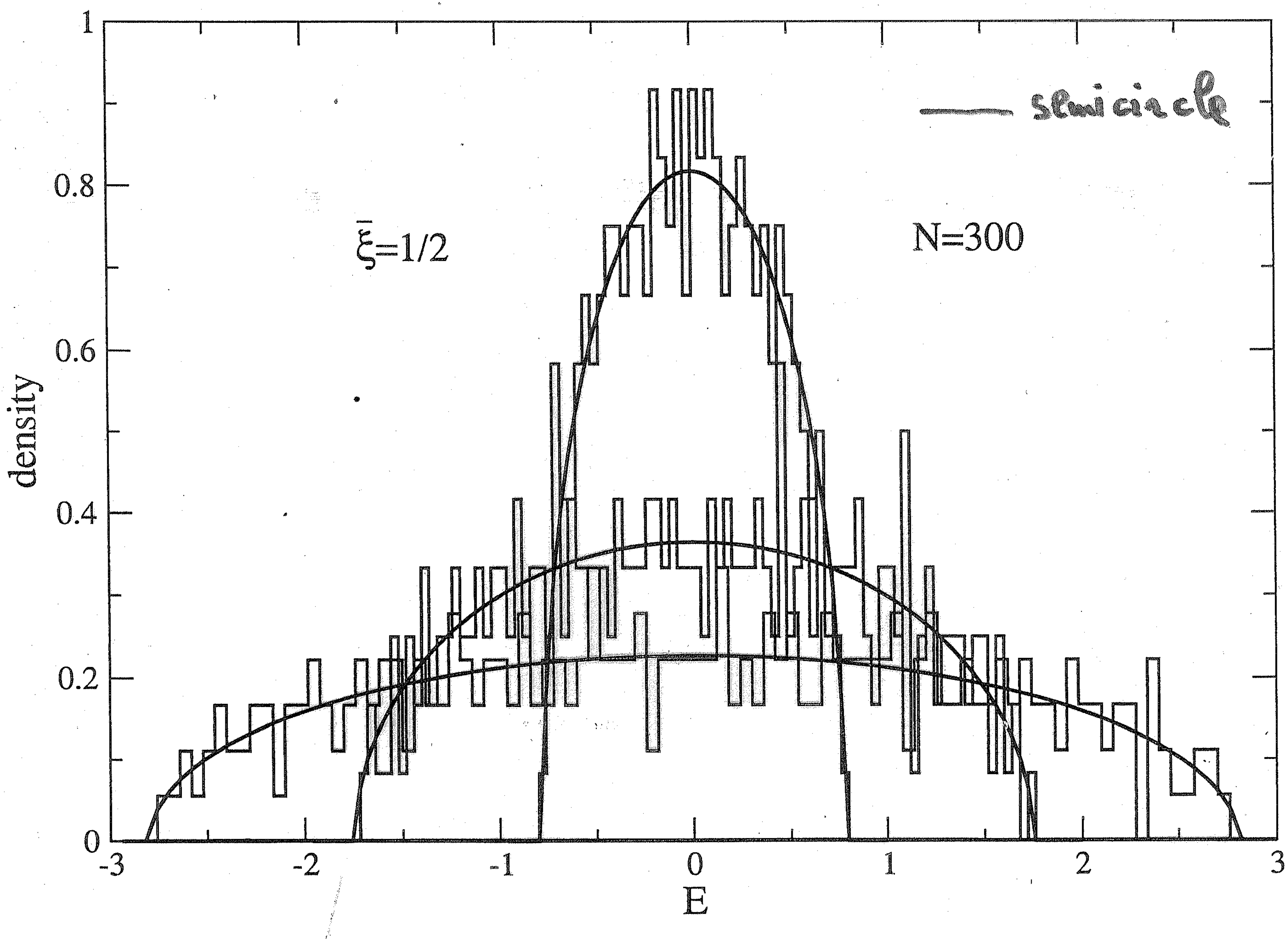
$$P(H) \propto \frac{1}{\left(1 + \beta \frac{\alpha}{\bar{\xi}} \operatorname{tr} H^2\right)^{\frac{1}{q-1}}}$$

$\frac{1}{q-1} = \bar{\xi} + \frac{f}{2}$, q Tsallis entropic parameter

Marginal distribution:

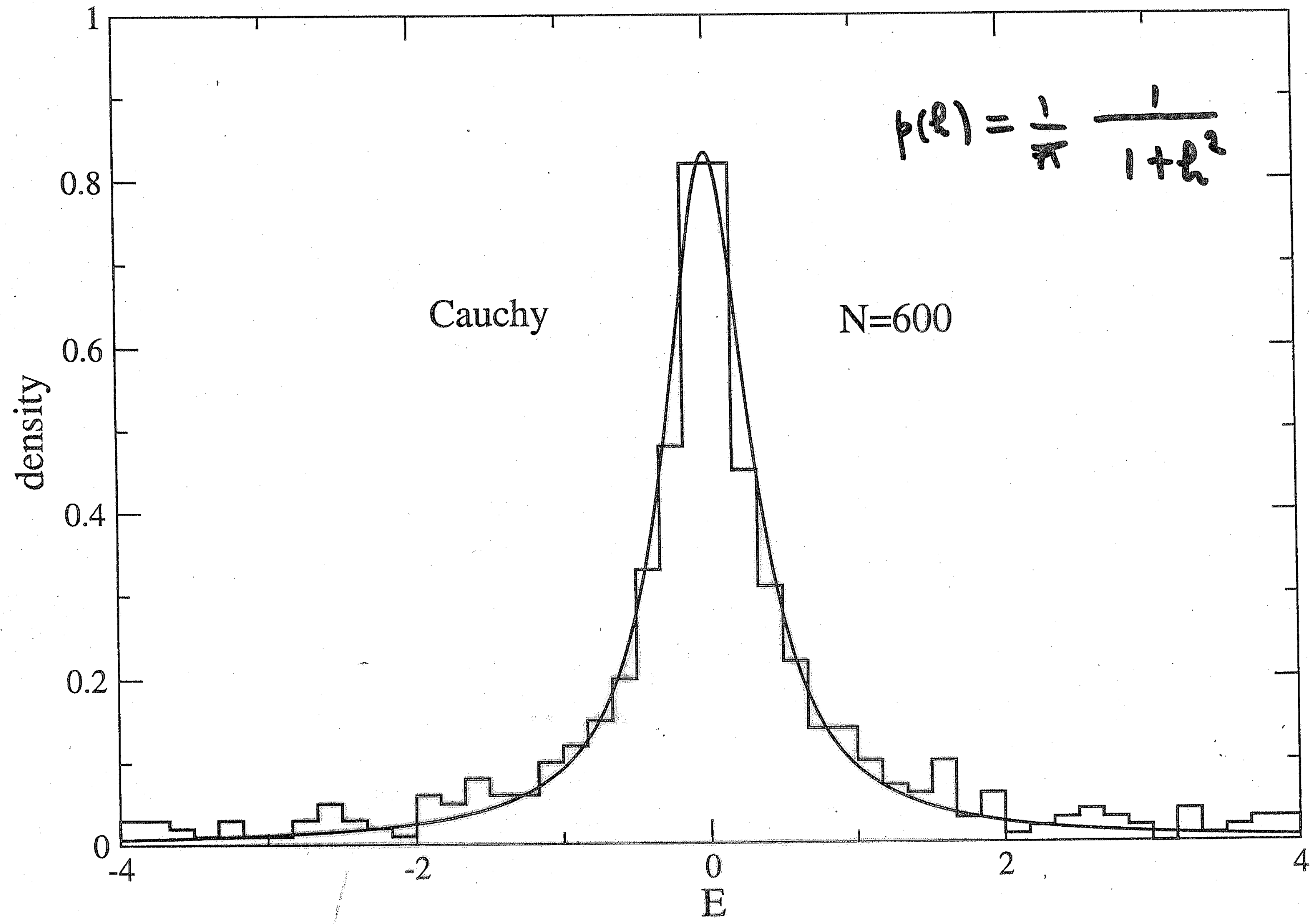
$$p(h) \propto \frac{1}{\left(1 + \frac{\alpha\beta}{\bar{\xi}} h^2\right)^{\bar{\xi}+\frac{1}{2}}} \longrightarrow \frac{1}{|h|^{2\bar{\xi}+1}}$$

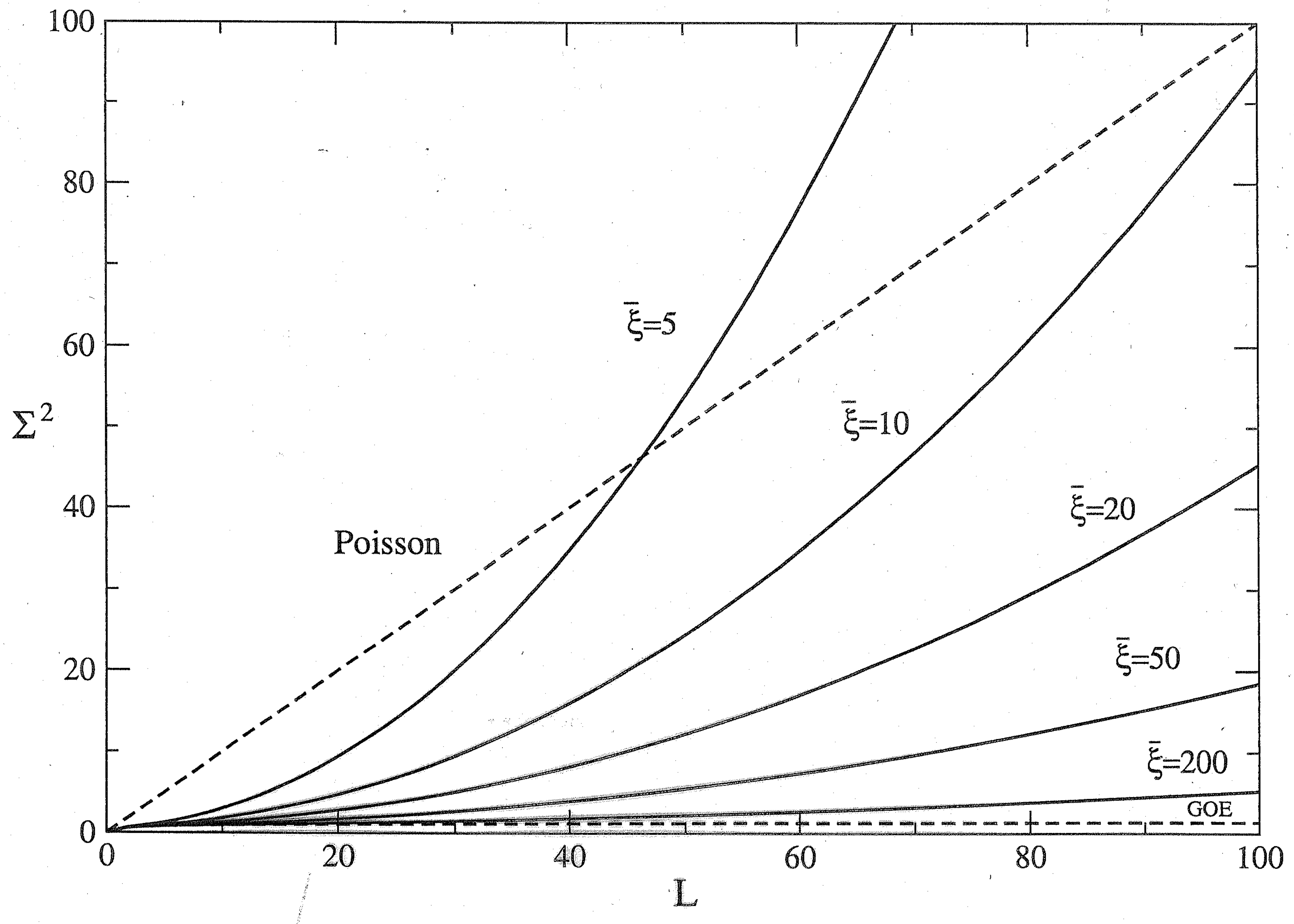
$\bar{\xi} = \frac{1}{2}$: elements are Cauchy distributed


density
E
ξ̄=1/2
N=300
semicircle

Levy (Cauchy) matrix

$$p(\ell) = \frac{1}{\pi}\frac{1}{1+\ell^2}$$



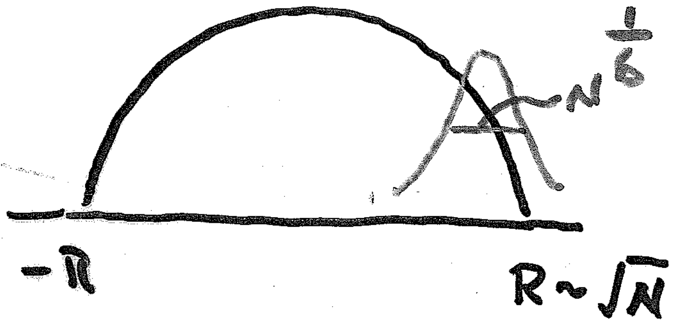


Σ^2
L
Poisson
$\bar{\xi}$=5
$\bar{\xi}$=10
$\bar{\xi}$=20
$\bar{\xi}$=50
$\bar{\xi}$=200
GOE
0
20
40
60
80
100

Largest eigenvalue distribution

Gaussian case

semi circle



$$P(\lambda_{max} < t)$$

$$t = 2\sigma\sqrt{N} + \frac{\sigma}{N^{1/6}} s \;, \quad \sigma^2 = \langle |H_{ij}|^2 \rangle$$

$$P(\lambda_{max} < t) \xrightarrow{N\to\infty} E_\beta(s) \;, \quad \beta = 1, 2, 4$$

$E_\beta(s)$ are the Tracy-Widom dist.

Occurrences:

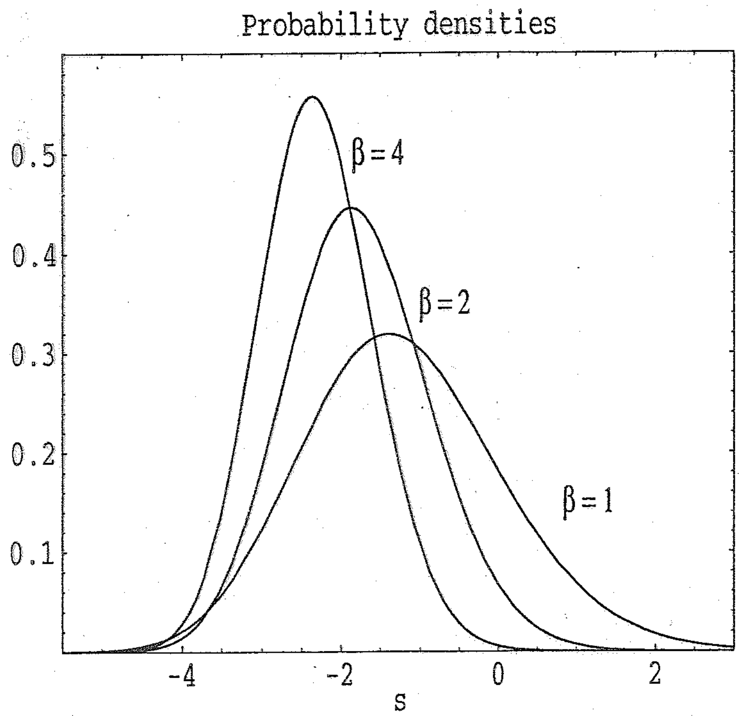
Combinatorics
Growth processes
Random tilings
Queuing theory
Superconductors
⋮ ⋮ ⋮

Random environment:

TW — Normal

Table 1: The mean (μ_β), standard deviation (σ_β), skewness (S_β) and kurtosis (K_β) of F_β.

β	μ_β	σ_β	S_β	K_β
1	-1.20653	1.2680	0.293	0.165
2	-1.77109	0.9018	0.224	0.093
4	-2.30688	0.7195	0.166	0.050



The Airy kernel is an example of an *integrable integral operator* [19] and a general theory is developed in [49]. A vertex operator approach to these distributions (and many other closely related distribution functions in random matrix theory) was initiated by Adler, Shiota and van Moerbeke [1]. (See the review article [51] for further developments of this latter approach.)

Historically, the discovery of the connection between Painlevé functions (P_{III} in this case) and Toeplitz/Fredholm determinants appears in work of Wu et al. [53] on the spin-spin correlation functions of the two-dimensional Ising model. Painlevé functions first appear in random matrix theory in Jimbo et al. [20] where they prove the Fredholm determinant of the sine kernel is expressible in terms of P_V.

Largest eigenvalue with disorder

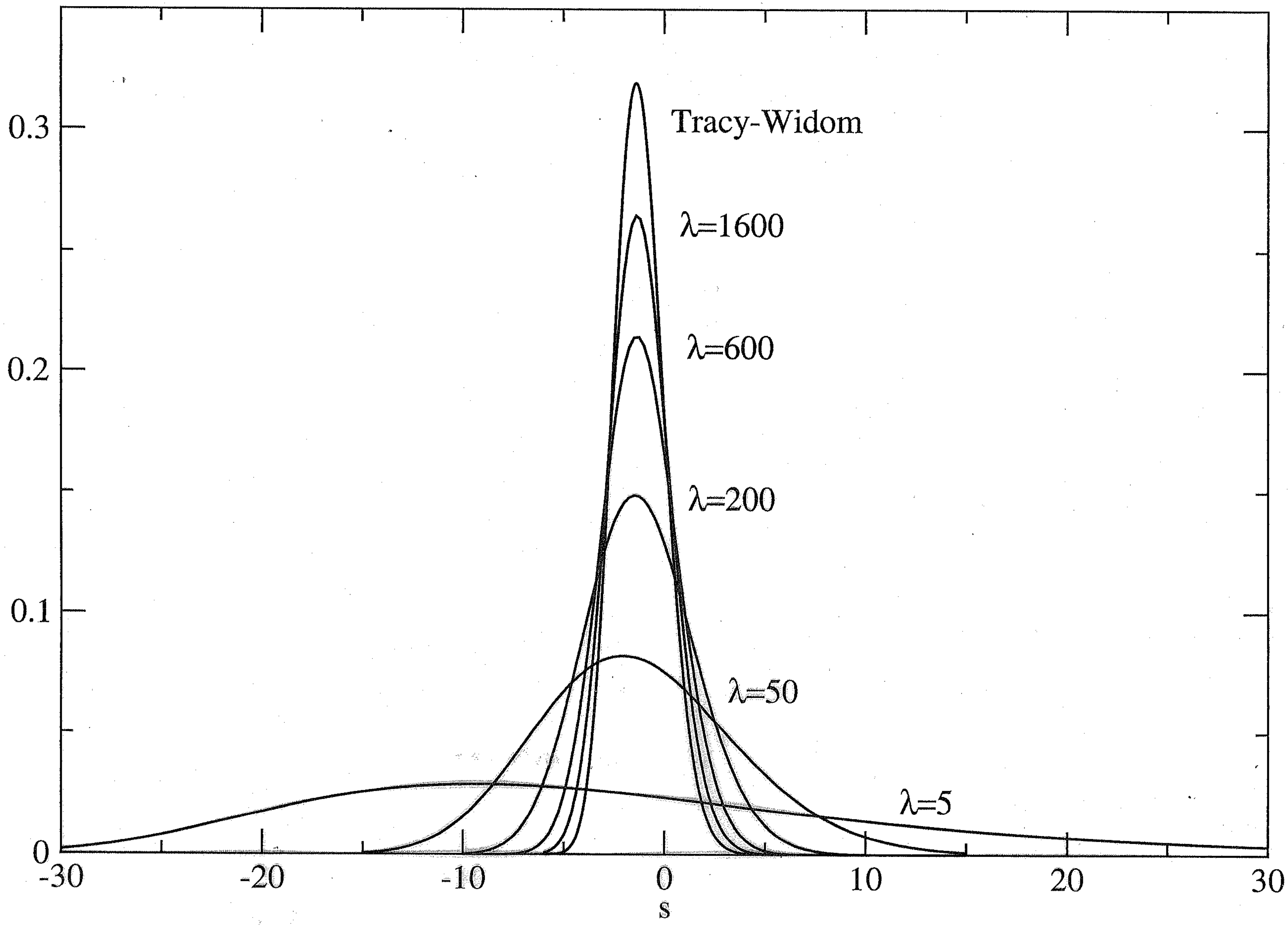
$$E_\beta(\lambda_{max} < t) = \int d\xi\, w(\xi)\, E_{G,\beta}[S(\xi,t)]$$

$$S(\xi,t) = N^{1/6}\left[\frac{t}{\sigma(\xi)} - 2\sqrt{N}\right]$$

$$\sigma(\xi) = \sqrt{\frac{\xi}{4d\xi}}$$, variance of H_{ij}

Density:

$$\frac{dE_\beta}{dt}(t) = N^{1/6}\int d\xi\, \underbrace{\frac{w(\xi)}{\sigma(\xi)}}_{\text{localized}}\, \underbrace{E'_{G,\beta}(S)}_{\text{localized}}$$


Tracy-Widom
λ=1600
λ=600
λ=200
λ=50
λ=5
0.3
0.2
0.1
0
-30
-20
-10
0
10
20
30
s

$N \to \infty$ asymptotics

$$\frac{\sigma_\xi}{\bar{\xi}} = N^{-z} \quad , \quad z > 0$$

assume that with

$$\xi = \bar{\xi} + v\,\sigma_\xi \ , \quad w(\xi) \to \frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} = N(0,1)$$

$$E_\beta(\lambda_{max} < t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dv\, e^{-\frac{v^2}{2}}\, E_{G,\beta}\left[S(v;s)\right]$$

$$\alpha = N \ , \quad s = 2N^{2/3}(t-1)$$

$$S(v,s) = s + v\, N^{2/3 - z}$$

$z > \frac{2}{3} \qquad E_\beta \longrightarrow TW$

$z = \frac{2}{3} \qquad E_\beta \longrightarrow$ convolution

$z < \frac{2}{3} \ , \quad s = 2N^{z}(t-1) \ , \quad E_\beta \longrightarrow N(0,1)$

$$E_\beta(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dv\, e^{-\frac{v^2}{2}}\, E_{G,\beta}\left[\frac{N^{2/3}}{N^{z}}(s-v)\right]$$

$$z = 0$$

$w(\xi)$ is independent of N

when $N \to \infty$ TW become step function

$$E_\beta(\lambda_{max} < t) = \int_{\bar{\xi}/t^2}^{\infty} d\xi \, w(\xi)$$

$$E'_\beta(t) = \frac{2\bar{\xi}\, w(\bar{\xi}/t^2)}{t^3} \xrightarrow{t\to\infty} \frac{1}{t^{2s+3}}$$

For large t E_β behaves like Fréchet distribution of i.i.d. with a power-law density

$$w(\xi) = \frac{e^{-\xi}\, \xi^{\bar{\xi}-1}}{\Gamma(\xi)}$$, E_β is Student dist. that generalizes Fréchet for correlated points