POTSDAM INSTITUTE FOR CLIMATE IMPACT RESEARCH

Geometric approaches to detecting coupling directions and generalized synchronization – A complex network perspective

Reik V. Donner, Jan H. Feldhoff, Jonathan F. Donges
(with N. Marwan, J. Heitzig, J. Kurths)

Leibniz Gemeinschaft

CIDNET'14 --- Dresden, 19 June 2014

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Agenda

Disclaimer: This talk is not about information theory, causality and neuroscience.

1 Random geometric graphs and transitivity dimensions

2 Estimation of transitivity dimensions: recurrence networks

3 Multivariate generalization and generalized synchronization

4 Interacting network analysis and coupling direction

5 Summary

Agenda

Random geometric graphs

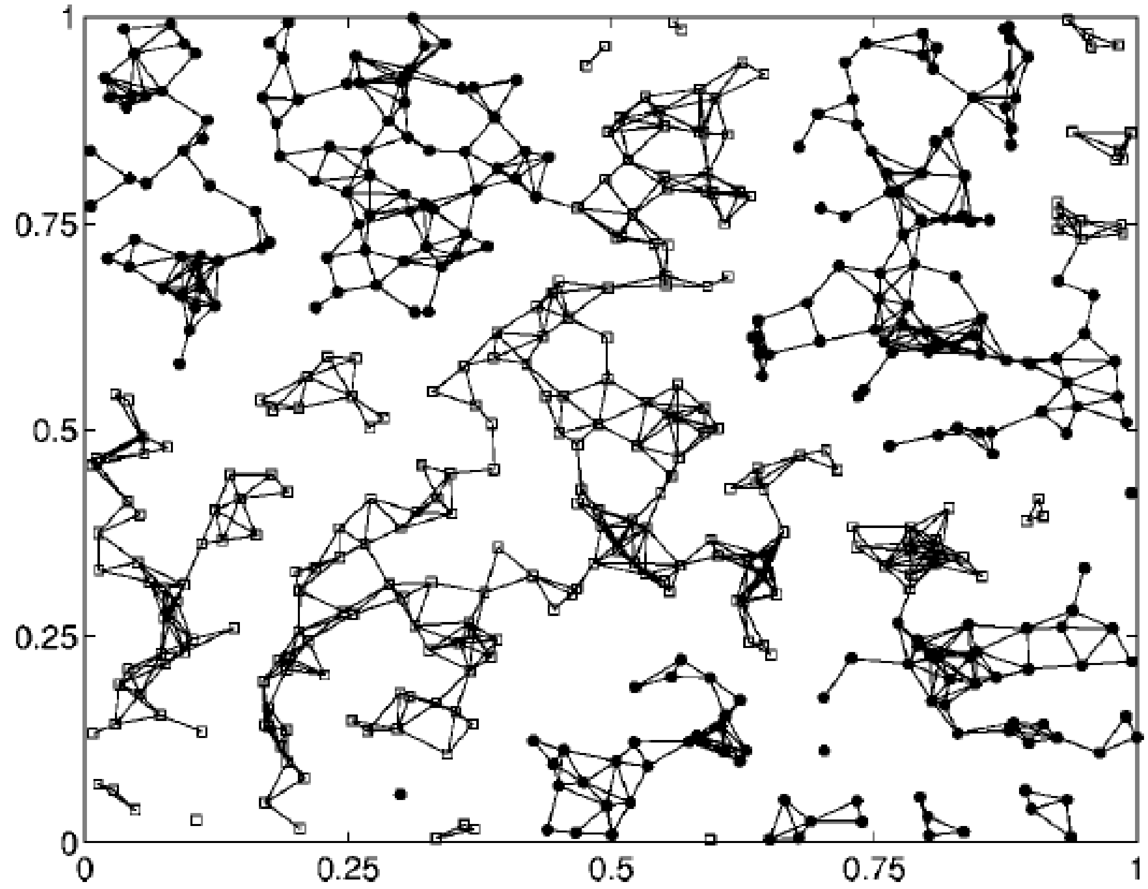
Suppose we have

1. **a bound measurable set S embedded in a d-dimensional metric space with a continuous PDF $p(x)$**
2. **a set V of N points drawn randomly according to $p(x)$**
3. **a function $f: S^2 \rightarrow [0,1]$ with $f(x,y)=f(|x-y|)$ being monotonically decreasing, describing with which probability two elements of V at positions x and y are mutually linked.**

An undirected graph $(V, V \times V)$ with the elements of the edge set being determined by f is called a random geometric graph.

In what follows: $f(x,y) = \Theta(\varepsilon - |x-y|)$

Random geometric graphs



(Dall and Christensen, PRE, 2002)

Random geometric graphs

<u>**Why is this relevant here?**</u>

Let *S* be the manifold describing a (chaotic) attractor of some (dissipative) dynamical system, *p(x)* being the associated invariant density.

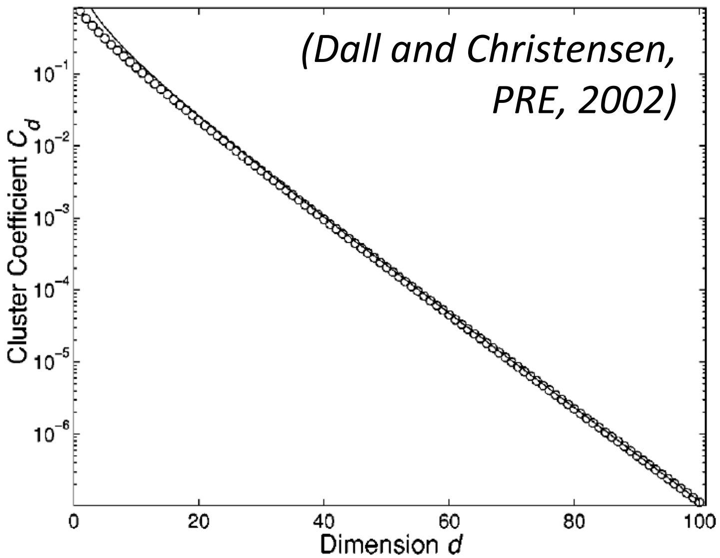
Then the geometric structure of the attractor can be approximated by a finite set of elements of *S* drawn at random according to *p(x)*.

The properties of the resulting random geometric graphs reveal geometric characteristics of the attractor, which are commonly related to dynamical properties.

Random geometric graphs: transitivity

Interesting property describing the attractor geometry: network transitivity (version of the global clustering coefficient)

$$\mathcal{T}(\varepsilon) = \frac{\sum_{i,j,k} A_{ij}(\varepsilon) A_{jk}(\varepsilon) A_{ki}(\varepsilon)}{\sum_{i,j,k} A_{ij}(\varepsilon) A_{ki}(\varepsilon)}$$



Classical result: for a random geometric graph in a metric space of integer dimension d, transitivity scales exponentially with d

Dall and Christensen, PRE, 2002: analytics for Euclidean norm

Donner et al., EPJB, 2011: analytics for maximum norm: $\mathcal{T}=(3/4)^d$

Random geometric graphs: transitivity

What if we construct a random geometric graph for a set *S* that does not fill the *d*-dimensional space, but has a lower (fractal) dimension?

Conjecture: $\mathcal{T}$ is larger than expected for *d* dimensions, since 3-loops occur more often than for an isotropic alignment of vertices.

Definition of a new notion* of generalized dimension: transitivity dimension

$$D_{\mathcal{T}}(\varepsilon) = \frac{\log \mathcal{T}(\varepsilon)}{\log(3/4)}$$

If *S* homogeneously fills a subset of dimension *d*: $D_{\mathcal{T}} = d$.

***This notion is different from the classical concept of fractal dimensions based on consideration of scaling characteristics with varying ε.**

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Random geometric graphs: transitivity

Comment P. Grassberger: Isn't this identical to the Renyi dimension of order 3?

Answer: No! See continuum limit of transitivity:

$$\mathcal{T}(\varepsilon) = \left[\iiint_S d\mu(x)\, d\mu(y)\, d\mu(z) \Theta(\varepsilon - \|x - y\|) \times \Theta(\varepsilon - \|y - z\|) \Theta(\varepsilon - \|z - x\|) \right] \Big/ \left[\iiint_S d\mu(x)\, d\mu(y)\, d\mu(z) \Theta(\varepsilon - \|x - y\|) \times \Theta(\varepsilon - \|z - x\|) \right],$$

Random geometric graphs: transitivity

$$D_{\mathcal{T}}(\varepsilon) := \frac{\log \mathcal{T}(\varepsilon)}{\log 3/4} \overset{\mathcal{T}=I_1/I_2}{=} \frac{\log I_1(\varepsilon)}{\log 3/4} - \frac{\log I_2(\varepsilon)}{\log 3/4}$$

$$\begin{aligned}
I_1(\varepsilon) &= \int d\mu(x) \int d\mu(y) \int d\mu(z)\, \Theta(\varepsilon - \|x-y\|)\, \Theta(\varepsilon - \|x-z\|)\, \Theta(\varepsilon - \|y-z\|) \\
&= \int d\mu(x) \int d\mu(y)\, \Theta(\varepsilon - \|x-y\|) \int d\mu(z)\, \Theta(\varepsilon - \|x-z\|)\, \Theta(\varepsilon - \|y-z\|) \\
&= \int d\mu(x) \int d\mu(y)\, \Theta(\varepsilon - \|x-y\|)\, \mu(B_\varepsilon(x) \cap B_\varepsilon(y))
\end{aligned}$$

$$\begin{aligned}
I_2(\varepsilon) &= \int d\mu(x) \int d\mu(y) \int d\mu(z)\, \Theta(\varepsilon - \|x-y\|)\, \Theta(\varepsilon - \|x-z\|) \\
&= \int d\mu(x) \int d\mu(y)\, \Theta(\varepsilon - \|x-y\|) \int d\mu(z)\, \Theta(\varepsilon - \|x-z\|) \\
&= \int d\mu(x) \left[\mu(B_\varepsilon(x))\right]^2
\end{aligned}$$

$$D_3(\varepsilon) := \frac{1}{2} \frac{1}{\log \varepsilon} \int d\mu(x) \left[\mu(B_\varepsilon(x))\right]^2, \quad D_3 = \lim_{\varepsilon \to 0} D_3(\varepsilon)$$

$$\Rightarrow D_{\mathcal{T}}(\varepsilon) = \frac{\log I_1(\varepsilon)}{\log 3/4} - \frac{2 \log \varepsilon}{\log 3/4} D_3(\varepsilon) \neq D_3 \quad !!!$$

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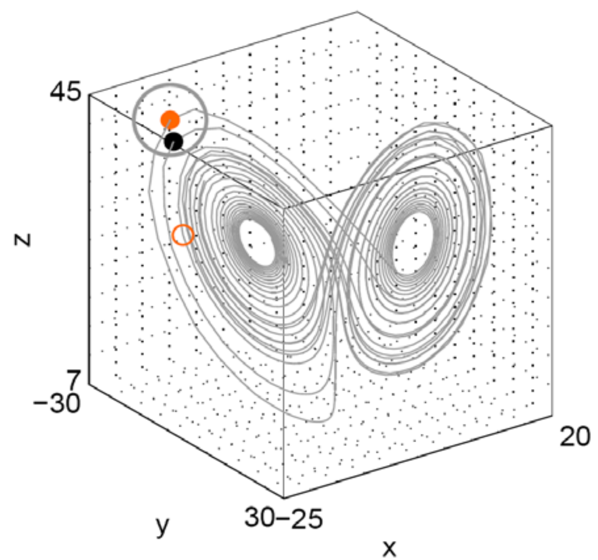
Recurrence: General idea

Recurrence of recent states is a typical feature of dynamical systems (Poincaré 1890)

Temporal pattern of recurrences encodes fundamental dynamical properties (Robinson & Thiel 2009)

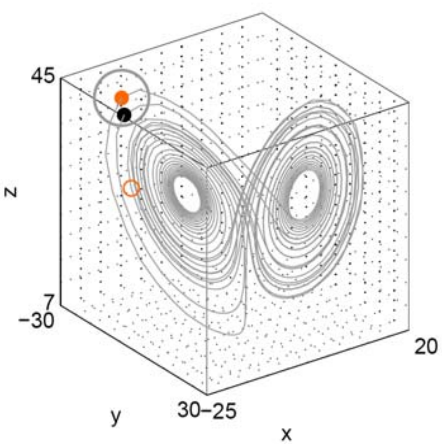


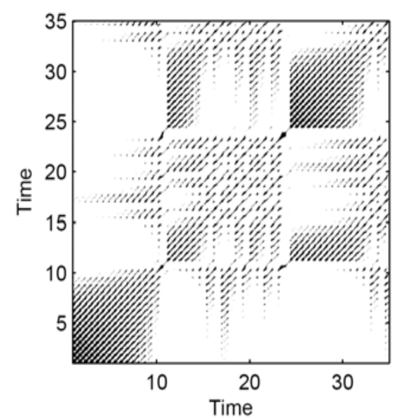
Recurrence plots

Eckmann *et al.* 1987: Visualization of recurrences in terms of "recurrence plots" based on the binary recurrence matrix

$$R_{i,j} = \Theta\big(\varepsilon - d(\vec{x}_i, \vec{x}_j)\big)$$

$\vec{x}_i$ **vector of simultaneously measured variables or reconstructed state vector using time-delay embedding** $\vec{x}_i = (u_i, u_{i+\tau}, \ldots, u_{i+\tau(m-1)})$



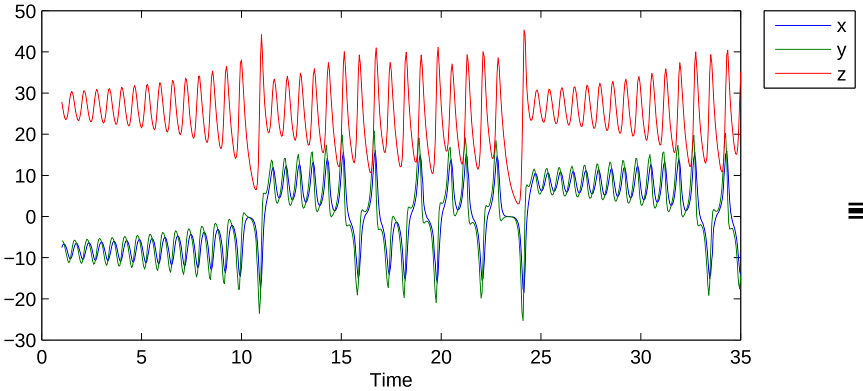


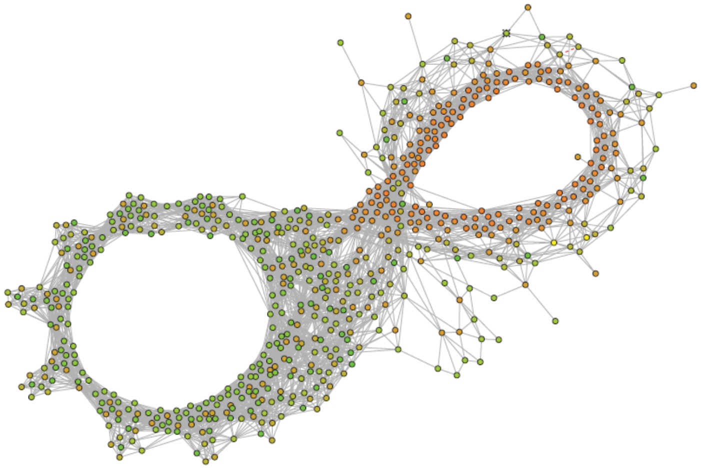
Recurrence networks

Recurrence matrix viewed as the adjacency matrix of a complex network encoding proximity relations between state vectors as edges

$$A_{i,j}(\epsilon) = R_{i,j}(\epsilon) - \delta_{i,j}$$

$\Rightarrow$ **Random geometric graph estimated from a time series!**



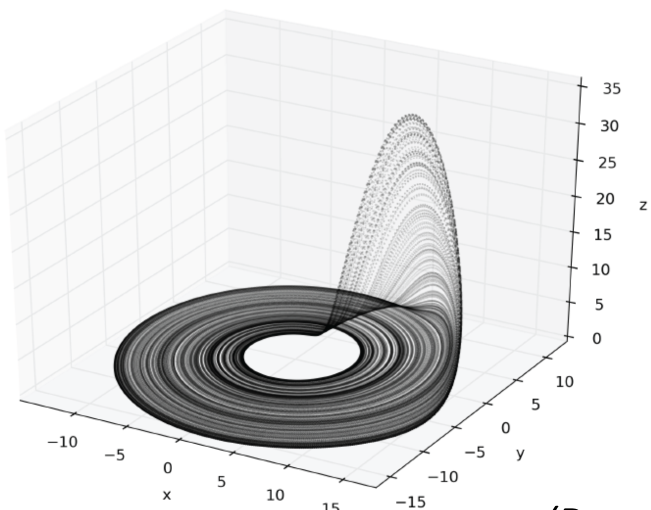


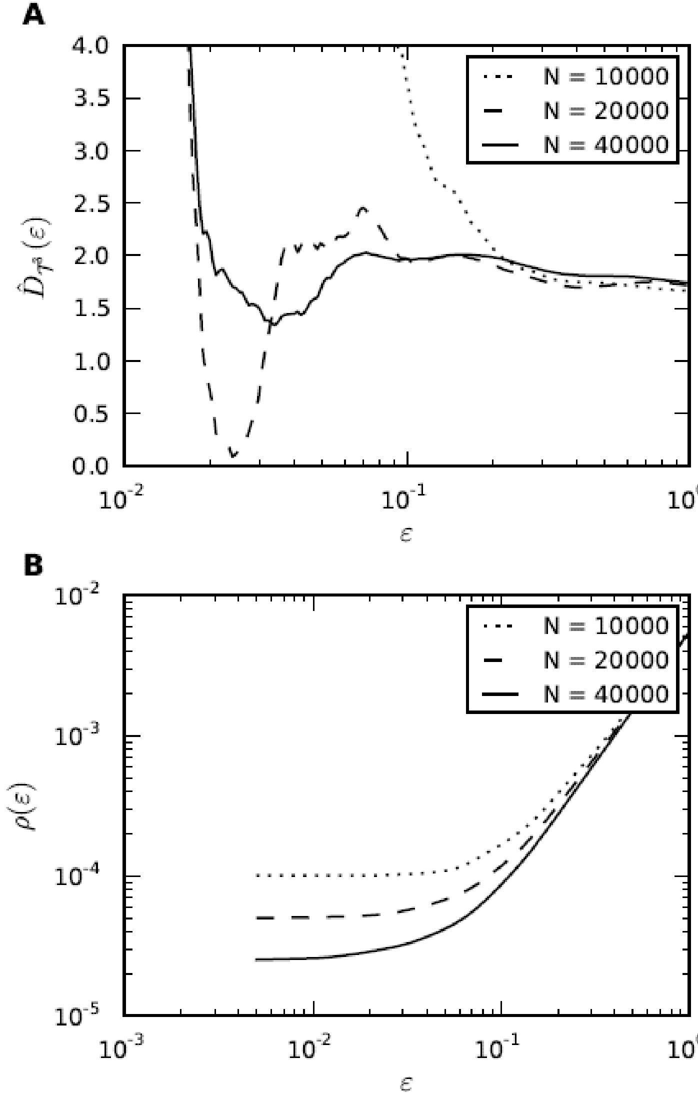
(Donner et al., IJBC, 2011)

Numerical examples

Rössler system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -y-z \\ x+ay \\ b+z(x-c) \end{pmatrix}$$





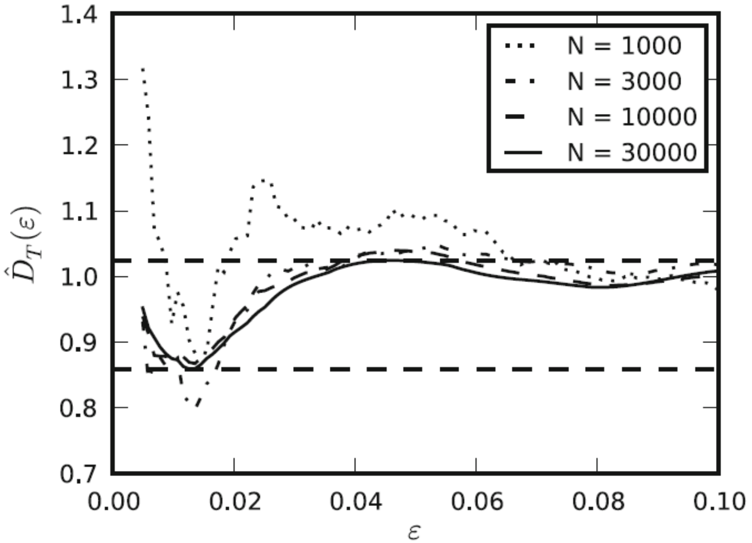
(Donner et al., in press)

Numerical examples

Henon map

$$x_{n+1} = y_n + 1 - a x_n^2$$
$$y_{n+1} = b x_n,$$

Result: for systems with fractal support of *p(x)*, *$\mathcal{T}(\varepsilon)$* exhibits two accumulation points – upper and lower transitivity dimension



(Donner et al., Eur. Phys. J. B, 2011)

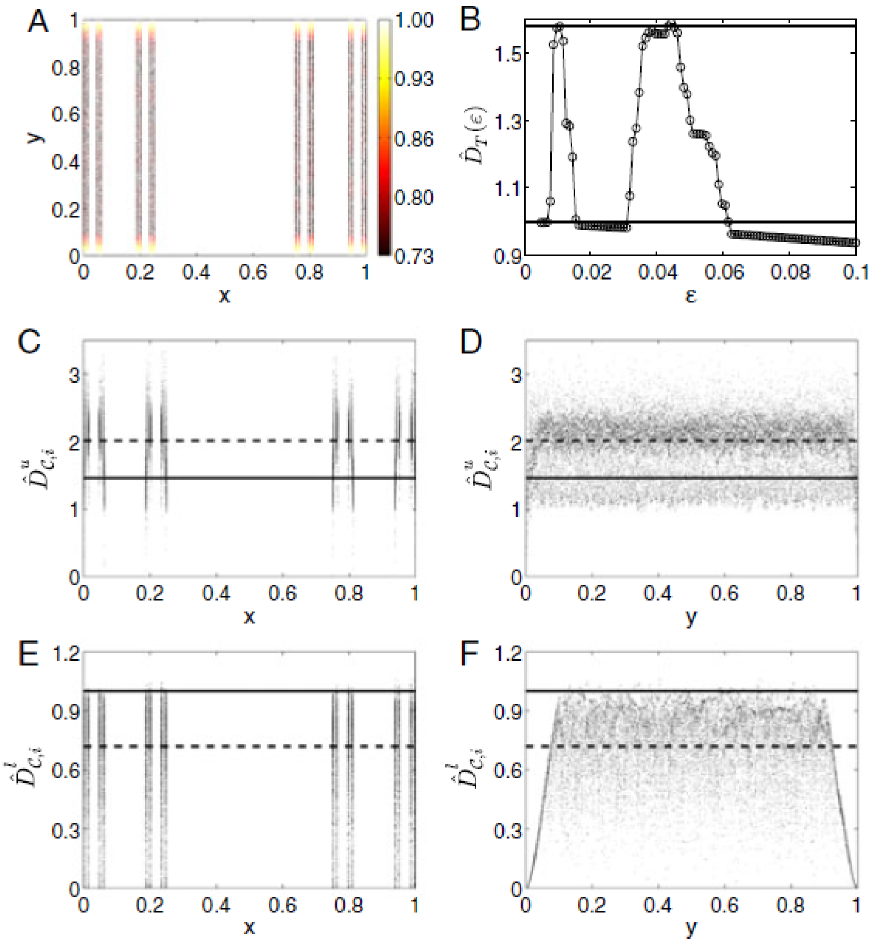
Numerical examples

Generalized baker's map

$$x_{n+1} = \begin{cases} \lambda_a x_n, & y_n < \alpha, \\ (1-\lambda_b) + \lambda_b x_n, & y_n > \alpha, \end{cases}$$

$$y_{n+1} = \begin{cases} y_n/\alpha, & y_n < \alpha, \\ (y_n - \alpha)/(1-\alpha), & y_n > \alpha \end{cases}$$

(Donner et al., Eur. Phys. J. B, 2011)

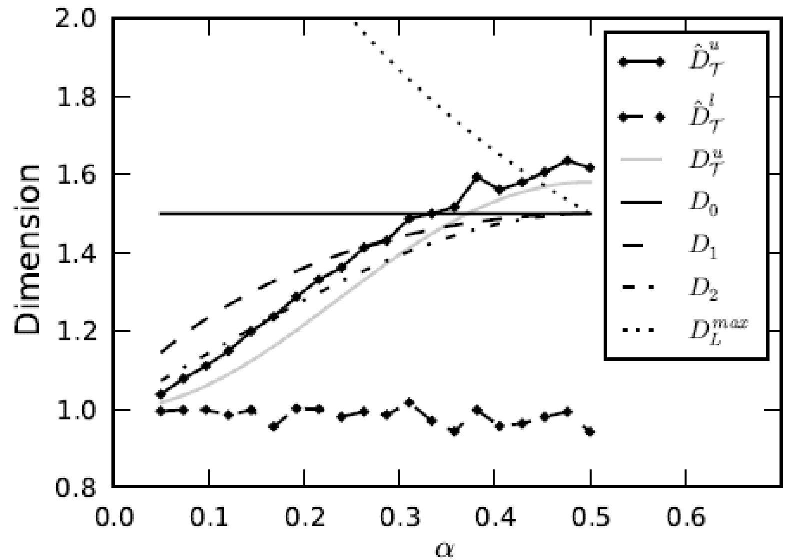


Numerical examples

Example for transitivity dimensions: Generalized baker's map

$$x_{n+1} = \begin{cases} \lambda_a x_n, & y_n < \alpha, \\ (1-\lambda_b) + \lambda_b x_n, & y_n > \alpha, \end{cases}$$

$$y_{n+1} = \begin{cases} y_n/\alpha, & y_n < \alpha, \\ (y_n - \alpha)/(1-\alpha), & y_n > \alpha \end{cases}$$



(Donner et al., Eur. Phys. J. B, 2011)

Result: (upper) transitivity dimension shows qualitatively similar behavior as other fractal dimensions

but: no scaling involved, therefore better estimation from short time series

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Synchronization

General meaning (in physical terms): <u>process</u> of mutual adjustment of <u>oscillations</u> of two or more <u>coupled</u> dynamical systems

Different types of sync:

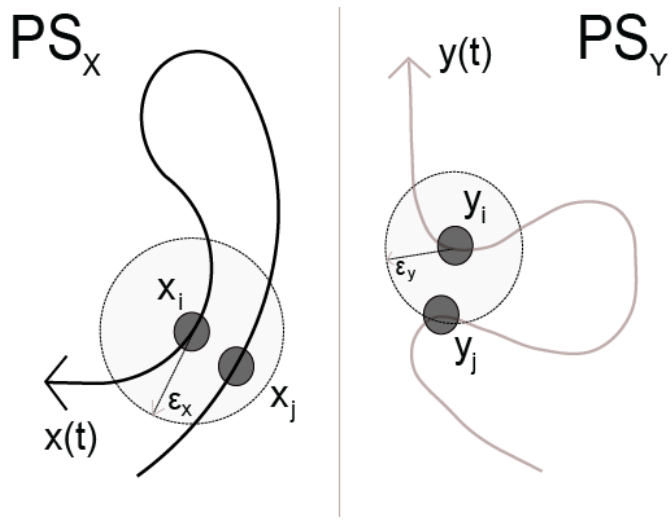
- **Complete sync: equal amplitude and phase dynamics**
- **Phase sync: equal phase dynamics**
- **Lag sync: equal dynamics with delay**
- **…**

Generalized synchronization: existence of an arbitrary deterministic functional relationship between the oscillations of two (or more…) systems
$\Rightarrow$ hard to be detected and quantified under general conditions
$\Rightarrow$ approach here: use phase space characteristics of both systems

Joint recurrences

Idea: study simultaneous recurrences of two or more systems (Romano et al., EPL, 2005; Feldhoff et al., EPL, 2013)

$$JR_{ij}(\varepsilon_x,\varepsilon_y) = \Theta(\varepsilon_x - \|x_i - x_j\|)\Theta(\varepsilon_y - \|y_i - y_j\|)$$
$$A_{ij}(\varepsilon_x,\varepsilon_y) = JR_{ij}(\varepsilon_x,\varepsilon_y) - \delta_{ij}$$



(Feldhoff et al., EPL, 2013)

Figure 1: Illustration of a joint recurrence of systems X and Y.

Joint recurrences

Romano et al., EPL, 2005: in presence of GS, recurrences of both systems occur at the same time – recurrence plots become the same, density of points in joint RP approaches that of single-system recurrence plots

$$RR^{\boldsymbol{x}} = \frac{1}{N^2} \sum_{i,j=1}^{N} \Theta(\varepsilon_{\boldsymbol{x}} - ||\boldsymbol{x}_i - \boldsymbol{x}_j||)$$

$$RR^{\boldsymbol{x},\boldsymbol{y}} = \frac{1}{N^2} \sum_{i,j=1}^{N} \Theta(\varepsilon_{\boldsymbol{x}} - ||\boldsymbol{x}_i - \boldsymbol{x}_j||)\Theta(\varepsilon_{\boldsymbol{y}} - ||\boldsymbol{y}_i - \boldsymbol{y}_j||)$$

$$S(\tau) = \frac{\frac{1}{N^2} \sum_{i,j}^{N} \Theta(\varepsilon_{\boldsymbol{x}}^i - ||\boldsymbol{x}_i - \boldsymbol{x}_j||)\Theta(\varepsilon_{\boldsymbol{y}}^i - ||\boldsymbol{y}_{i+\tau} - \boldsymbol{y}_{j+\tau}||)}{RR}$$

$$\boxed{JPR = \max_{\tau} \frac{S(\tau) - RR}{1 - RR}}$$

Joint recurrences

Problem: limit of *JPR=1* is hardly approached in complex scenarios (e.g., coupled Rössler systems in funnel regime)

Idea: use higher-order characteristics (three-point relations: transitivity)

absence of synchronization:

$$\mathcal{T}_X, \mathcal{T}_Y \gg \mathcal{T}_J$$

$$D_{\mathcal{T}_J} = \log \mathcal{T}_J / \log(3/4) \approx D_{\mathcal{T}_X} + D_{\mathcal{T}_Y}$$

generalized synchronization: locking of effective dynamical degrees of freedom

$$\mathcal{T}_J \to \mathcal{T}_X, \mathcal{T}_Y$$

Characteristic parameter: transitivity ratio

$$\boxed{Q_{\mathcal{T}} = \frac{\mathcal{T}_J}{(\mathcal{T}_X + \mathcal{T}_Y)/2}}$$

Example: Coupled Rössler systems

$$\dot{x}_1 = -(1+\nu)x_2 - x_3$$
$$\dot{x}_2 = (1+\nu)x_1 + ax_2 + \mu_{YX}(y_2 - x_2)$$
$$\dot{x}_3 = b + x_3(x_1 - c)$$
$$\dot{y}_1 = -(1-\nu)y_2 - y_3$$
$$\dot{y}_2 = (1-\nu)y_1 + ay_2 + \mu_{XY}(x_2 - y_2)$$
$$\dot{y}_3 = b + y_3(y_1 - c)$$

Example: Coupled Rössler systems

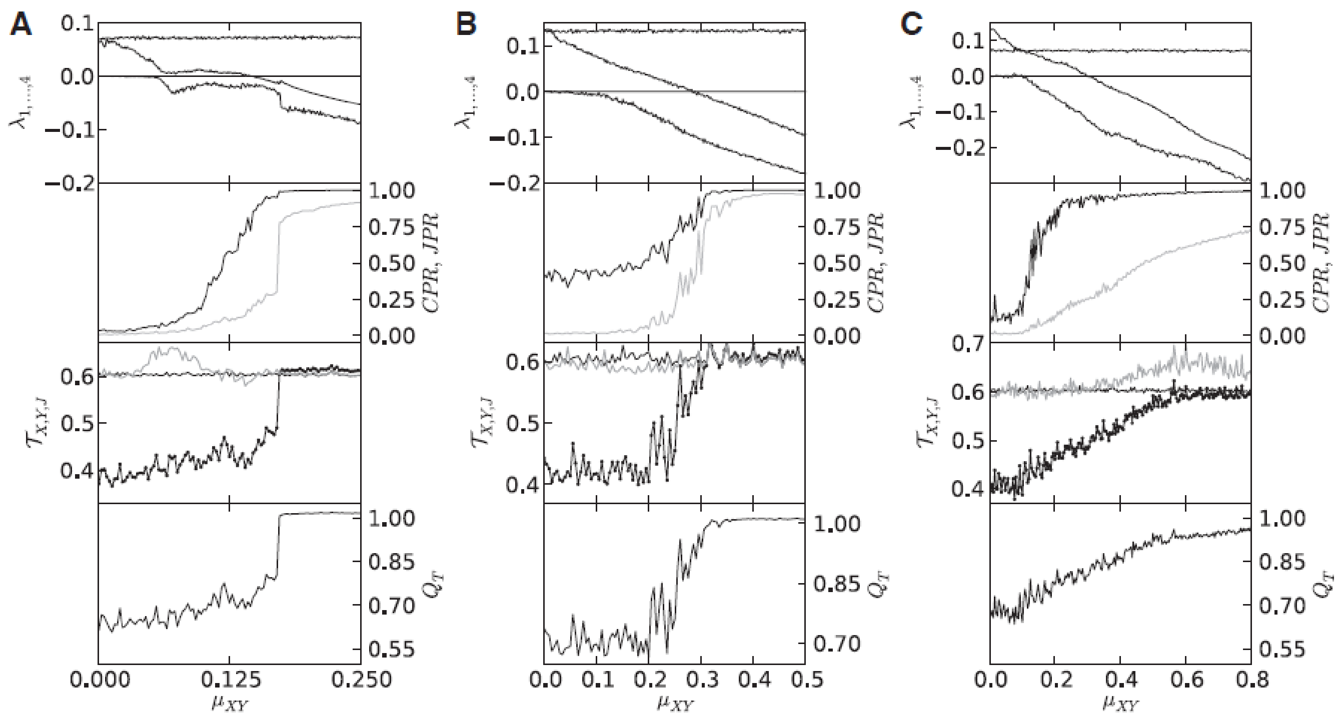


Figure 2: Results of synchronisation analysis for two unidirectionally ($X \rightarrow Y$) coupled Rössler systems being (A) both in the phase-coherent regime, (B) both in the funnel regime, and (C) in phase-coherent (X) and funnel regime (Y): the four largest Lyapunov exponents $\lambda_1, \ldots, \lambda_4$ estimated using the Wolf algorithm [46] (using $N = 9,000,000$ data points from simulations with step size $h = 0.001$ starting at $T = 1,000$); recurrence-based synchronisation indices CPR (black) and JPR (grey) [17]; transitivities of individual and joint recurrence networks $\mathcal{T}_X$ (dark grey), $\mathcal{T}_Y$ (light grey) and $\mathcal{T}_J$ (black); transitivity ratio $Q_\mathcal{T}$ (from top to bottom).

(Feldhoff et al., EPL, 2013)

Example: Coupled Rössler systems

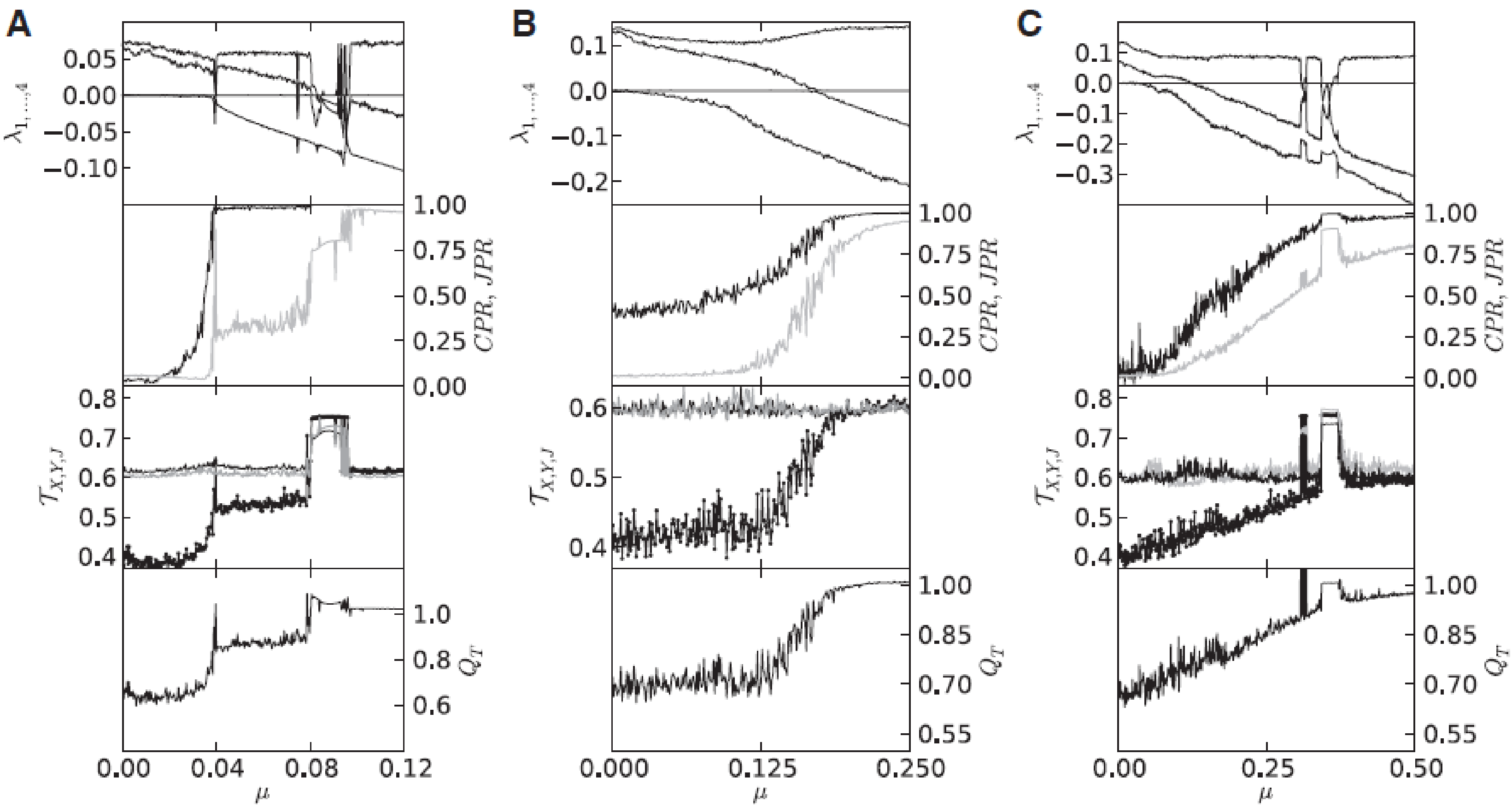


Figure 3: As in Fig. 2 for bidirectional coupling.

(Feldhoff et al., EPL, 2013)

Improvements

- Discriminatory skills of Q_T not yet satisfactory
  $\Rightarrow$ improvement: dimensional locking index defined in analogy to mutual information

$$\hat{\tilde{D}}_{\mathscr{T}^R} = \hat{D}_{\mathscr{T}^X} + \hat{D}_{\mathscr{T}^Y} - \hat{\tilde{D}}_{\mathscr{T}^J} \quad \Longrightarrow \quad \boxed{\widehat{\widetilde{DLI}} = \frac{\hat{\tilde{D}}_{\mathscr{T}^R}}{\hat{\tilde{D}}_{\mathscr{T}^J}}}$$

- Unlike for single systems, joint recurrences are not based on a consistent metric in the composed phase space of both systems
  $\Rightarrow$ No dimension interpretation of joint transitivity!
  Possible solution: replace by transitivity of composed system *X x Y*
  Problem: possibly different metric distances in both systems
- Overshooting of Q_T: values *>1* occur
  $\Rightarrow$ Possibly related to downward estimation bias of transitivity

Numerical results

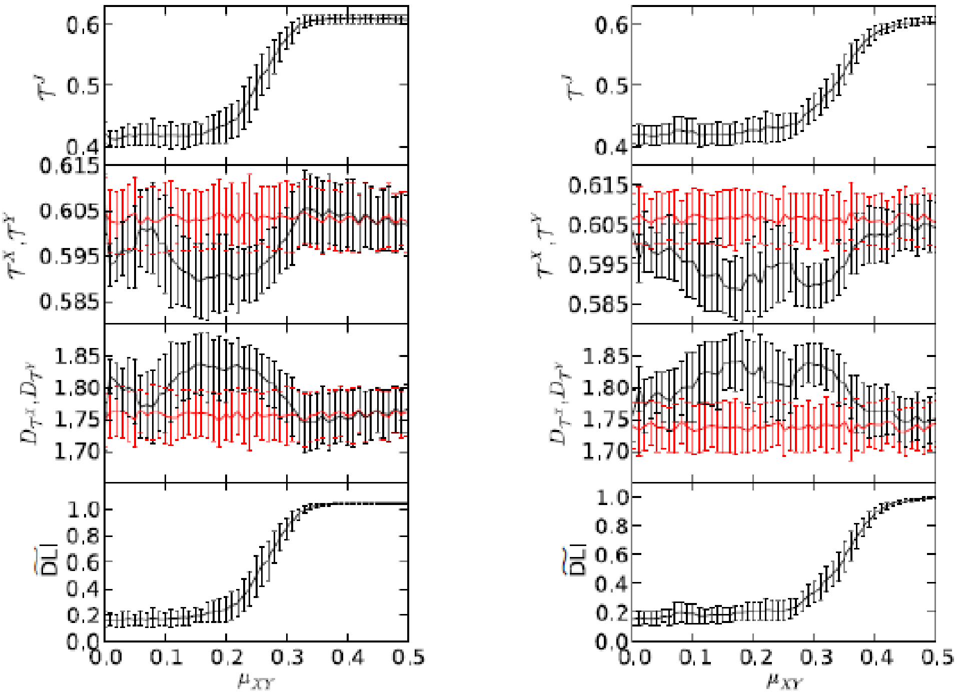
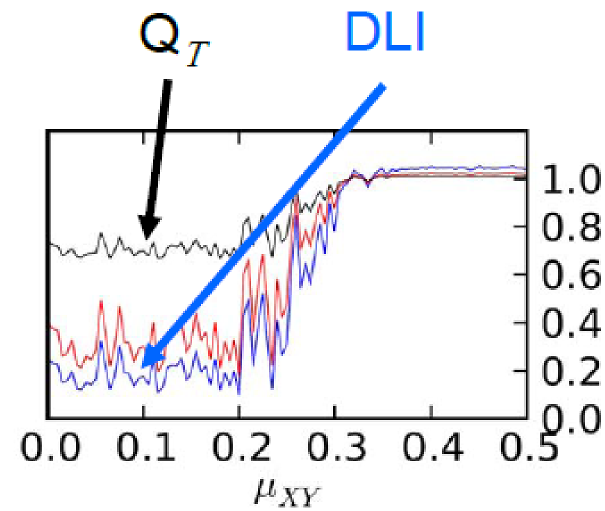


Fig. 16 Joint transitivity $\hat{\mathcal{T}}_J$, single-system RN transitivities $\hat{\mathcal{T}}_{X,Y}$, corresponding transitivity dimensions $\hat{D}_{\mathcal{T}_X}$, $\hat{D}_{\mathcal{T}_Y}$ and derived dimensional locking index $\widehat{DLI}$ (from top to bottom) for two unidirectionally coupled Rössler systems ($X \rightarrow Y$) in the funnel regime with $\nu = 0.02$ (driver oscillates faster than driven system, left panels) and $\nu = -0.02$ (right panels). The error bars indicate mean values and standard deviations estimated from 100 independent network realizations for each value of the coupling strength μ_{XY}. For transitivities and transitivity dimensions, red (black) lines correspond to the values for system X (Y).



(Donner et al., in press)

Agenda

Inter-system recurrence networks

Idea: Combing recurrence and "cross-recurrence" matrices with different densities of intra- and inter-system edges

$$\mathbf{IR}(\boldsymbol{\varepsilon}) = \begin{pmatrix} \mathbf{R}^X(\varepsilon^X) & \mathbf{CR}^{XY}(\varepsilon^{XY}) \\ [\mathbf{CR}^{XY}(\varepsilon^{XY})]^T & \mathbf{R}^Y(\varepsilon^Y) \end{pmatrix}$$

$$\text{with } \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon^X & \varepsilon^{XY} \\ \varepsilon^{XY} & \varepsilon^Y \end{pmatrix}$$

$$\begin{aligned} CR_{ij}(\varepsilon^{XY}) &= CR^{XY}(x_i, y_j \mid \varepsilon^{XY}) \\ &= \Theta(\varepsilon^{XY} - d^{XY}(x_i, y_j)) \end{aligned}$$

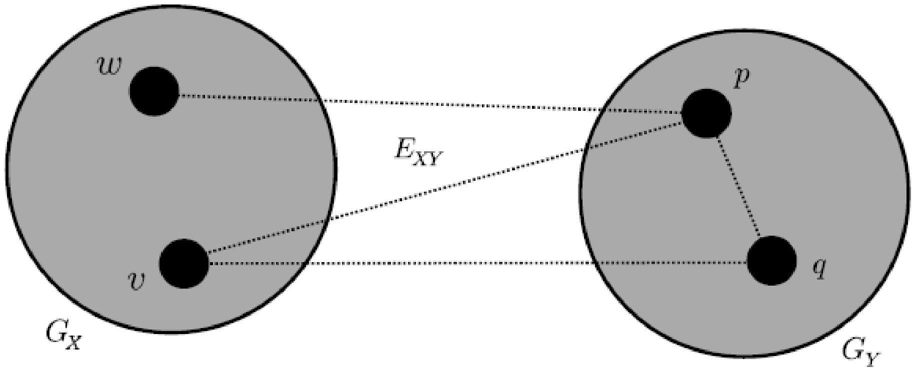


Fig. 2. Two coupled subnetworks. The graph has global cross-clustering coefficients of $\mathcal{C}^{XY} = 0.5 \neq \mathcal{C}^{YX} = 0$ and cross-transitivities $\mathcal{T}^{XY} = 1 \neq \mathcal{T}^{YX} = 0$.

Inter-system recurrence networks

Asymmetries between interacting network measures: coupling direction

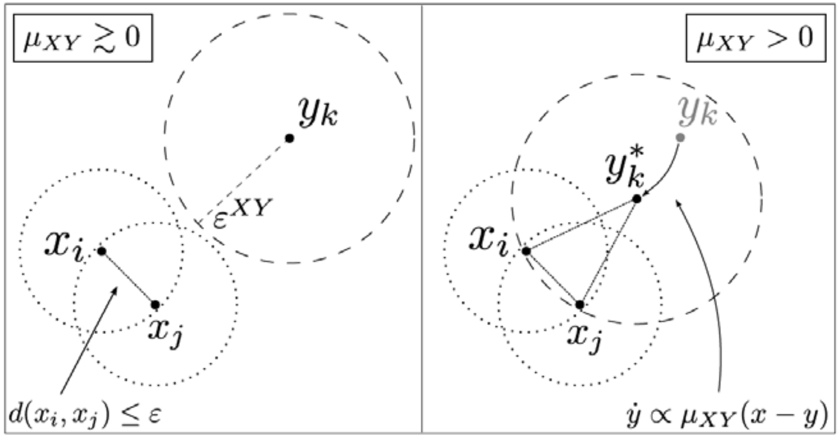
Expected qualitative behaviour of IRN measures in different coupling situations for systems with comparable properties in the absence of (generalised) synchronisation.

Coupling direction	Expected relation in network measures
no coupling	$\mathcal{T}^{XY} \approx \mathcal{T}^{YX}$, $\mathcal{C}^{XY} \approx \mathcal{C}^{YX}$
$X \rightarrow Y$	$\mathcal{T}^{XY} < \mathcal{T}^{YX}$, $\mathcal{C}^{XY} < \mathcal{C}^{YX}$
$Y \rightarrow X$	$\mathcal{T}^{XY} > \mathcal{T}^{YX}$, $\mathcal{C}^{XY} > \mathcal{C}^{YX}$
$X \leftrightarrow Y$	$\mathcal{T}^{XY} \approx \mathcal{T}^{YX}$, $\mathcal{C}^{XY} \approx \mathcal{C}^{YX}$

so far only heuristic arguments!

rigorous mathematical formulation: ongoing MSc thesis…

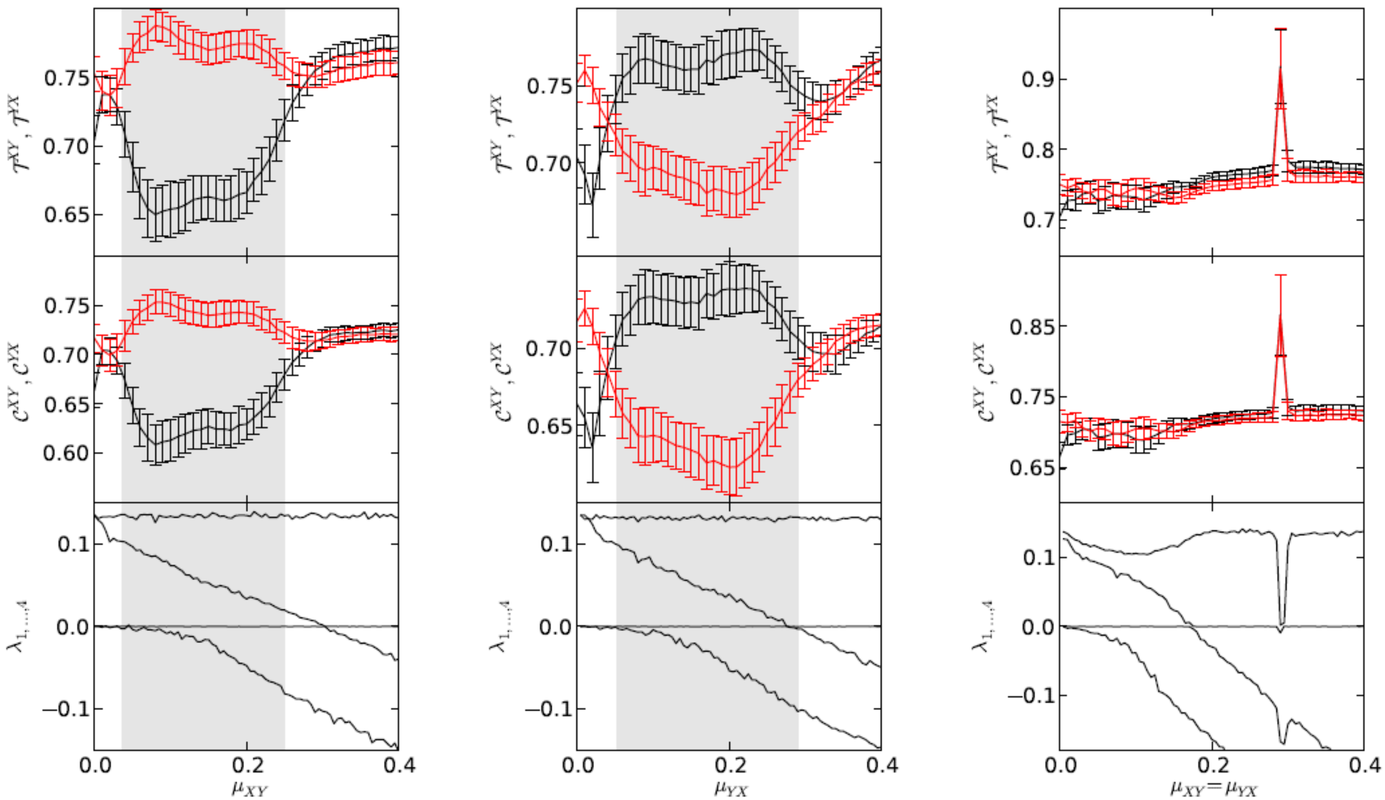
(Feldhoff et al., Phys. Lett. A, 2012)



Inter-system recurrence networks

Example: Two diffusively coupled Rössler oscillators

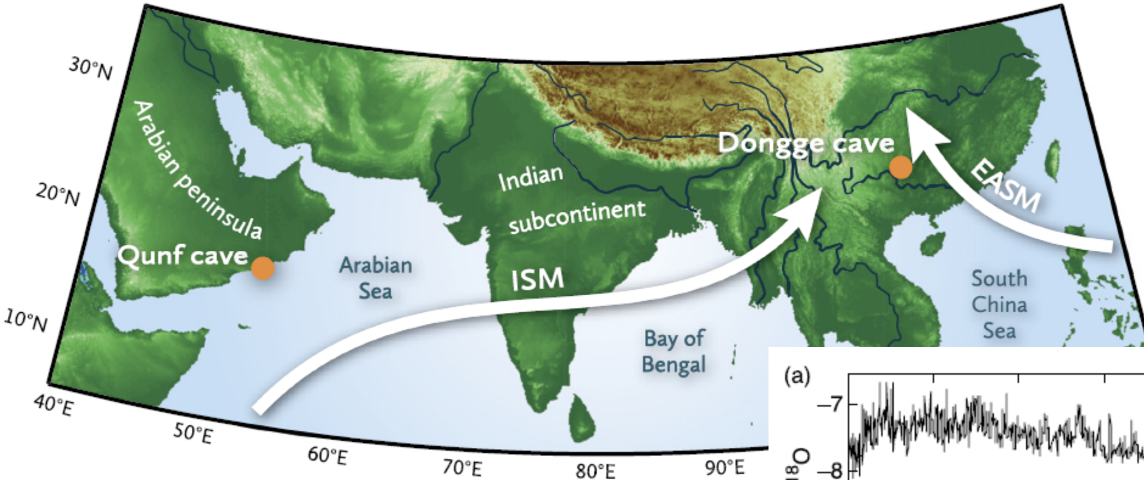
(Feldhoff et al., Phys. Lett. A, 2012)



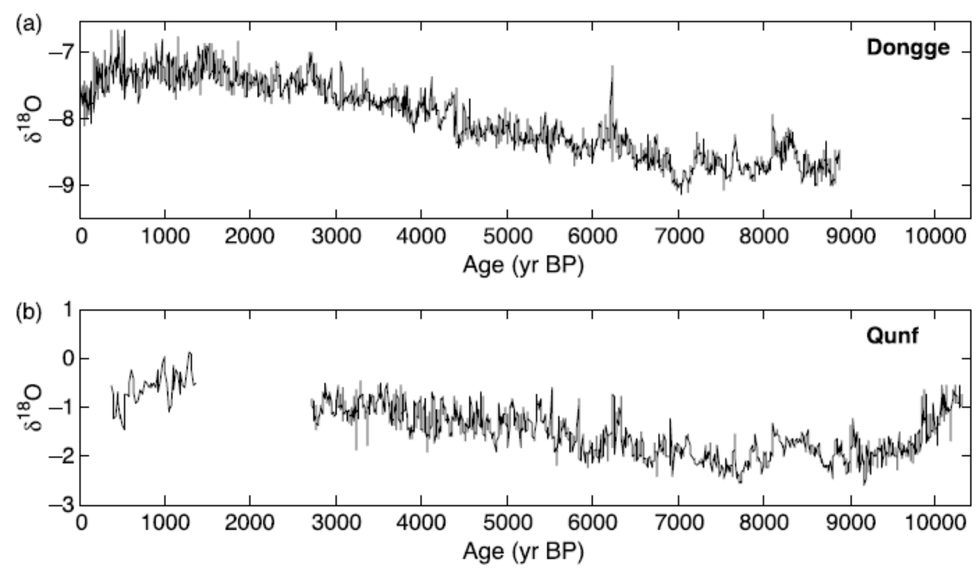
(a) Coupling: $X \to Y$ (b) Coupling: $Y \to X$ (c) Coupling: $X \leftrightarrow Y$

Inter-system recurrence networks

Example: Coupling between Indian and East Asian Summer Monsoon



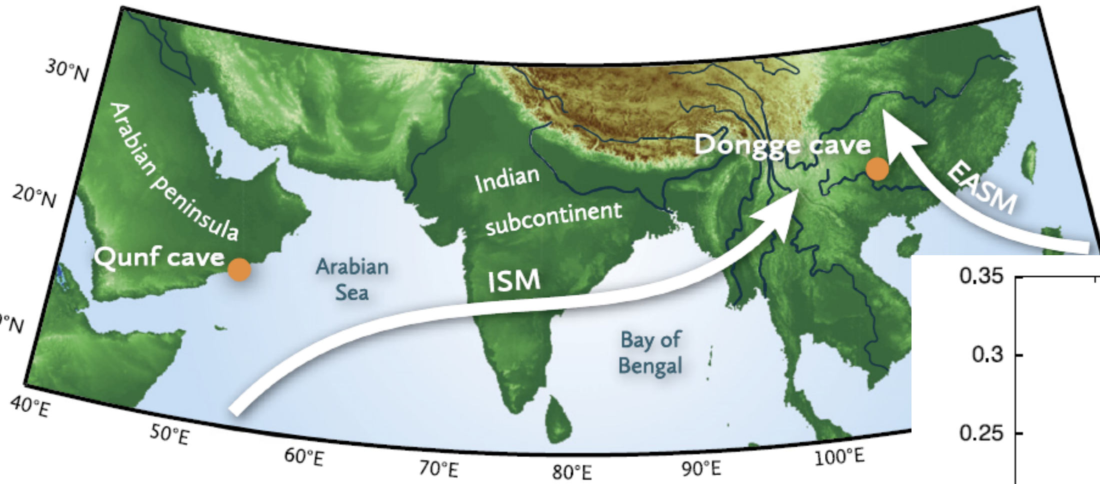
(Feldhoff et al., Phys. Lett. A, 2012)

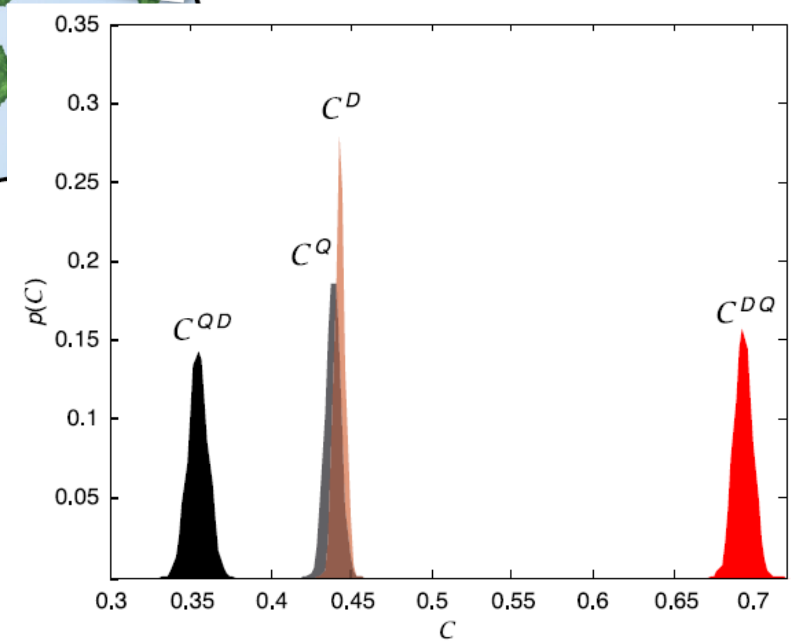


Inter-system recurrence networks

Example: Coupling between Indian and East Asian Summer Monsoon

(Feldhoff et al., Phys. Lett. A, 2012)

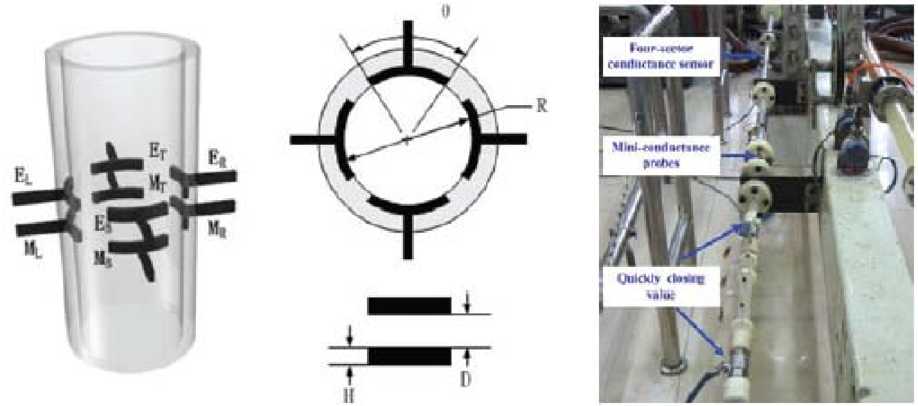




Result: indication for a coupling ISM -> EASM

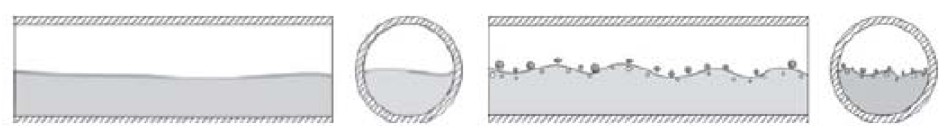
Application

Pattern formation in stratified horizontal two-phase flows



(a) (b)

Fig. 1: (Color online) (a) The four-sector conductance sensor: E_T, E_R, E_B, E_L are the exciting electrodes, and M_T, M_R, M_B, M_L are the measuring electrodes, respectively, θ is the exciting and measuring electrode angle, H is the electrode height, R is the inner pipe radius, and D the distance between the exciting and the measuring electrode; (b) experimental flow loop facility.



(a) (b)

Fig. 2: Horizontal oil-water stratified flow patterns. (a) ST flow; (b) ST&MI flow.

(Gao et al., EPL, 2013)

Application

Pattern formation in stratified horizontal two-phase flows

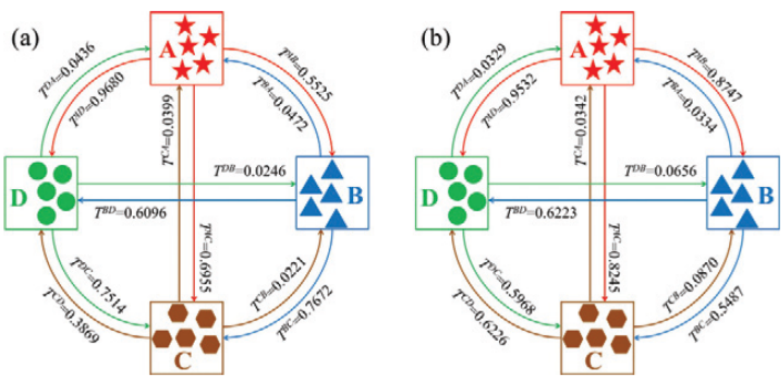
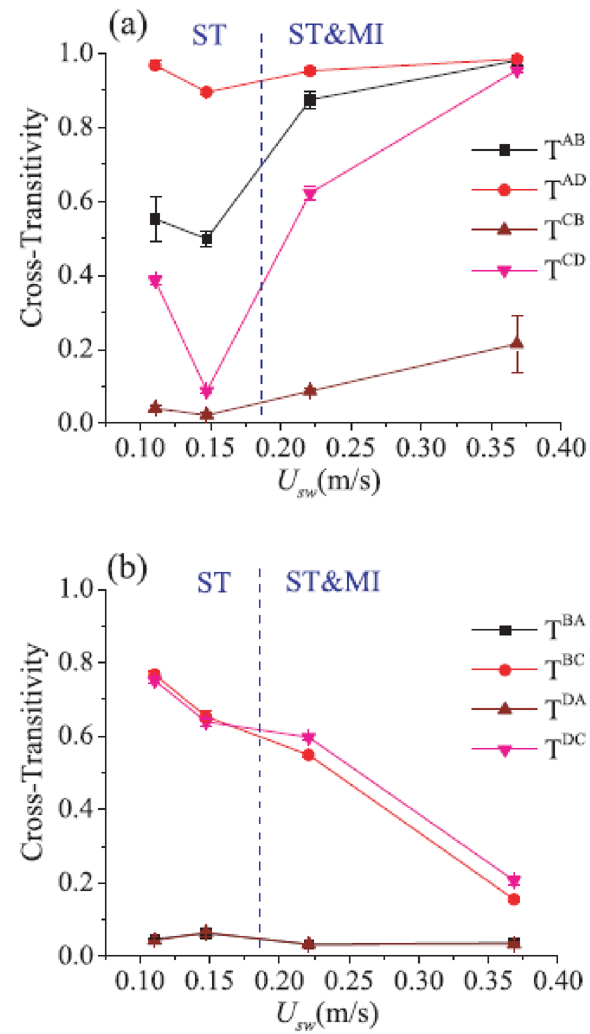


Fig. 6: (Color online) Cross-transitivity of all pairs of sub-networks for a multivariate recurrence network generated from (a) ST flow pattern ($U_{so} = 0.1945\,\mathrm{m/s}$, $U_{sw} = 0.1105\,\mathrm{m/s}$) and (b) ST&MI flow pattern ($U_{so} = 0.1945\,\mathrm{m/s}$, $U_{sw} = 0.2216\,\mathrm{m/s}$), where A, B, C, D represent different subnetworks.



(Gao et al., EPL, 2013)

Fig. 7: (Color online) Cross-transitivity for the transitions from ST flow to ST&MI flow, where some error bars are smaller than the size of the symbols.

Agenda

Take home messages

- **Chaotic attractors can be approximated by random geometric graphs estimated from recurrence plots**
- **New concept of transitivity dimension that is easily computable**
- **Meaningful generalizations of recurrence network concept to bi- and multivariate problems**
- **Geometric signatures of effects due to directional coupling of different systems**
- **Improved detection of onset of generalized synchronization**

Open problems

- **Formal link between transitivity dimension and classical fractal dimension concepts, generalized transitivity dimensions (similar to Renyi dimensions, taking higher-order transitivities into account)**

- **Theoretical justification of behavior of cross-transitivities in the presence of unidirectional coupling – geometric theory of coupled dynamical systems (MSc thesis M. Khotyakov)**

- **Formal approach to generalized synchronization by means of projections from combined phase space of coupled systems**

Questions? Comments?