Exact Solution of a Model for Traffic Flow

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Outline

- Model
- Matrix Product Method
- Phase Diagram
- Conclusions

Nagel-Schreckenberg Model

Cars on a lattice. jth car has velocity $v_j(t)$ at time t.

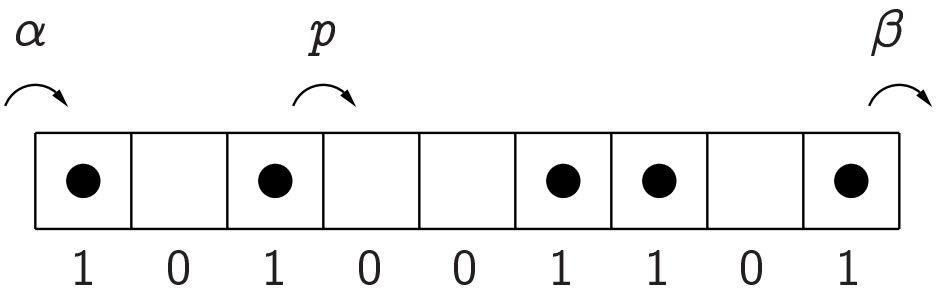
- Acceleration: $v_j(t+1/3) = \min(v_j(t), v_{\max})$.
- Braking: $v_j(t+2/3) = \min(d_j(t), v_j(t+1/3))$
- Randomization: With probability $1-p$, $v_j(t+1) = \max(v_j(t+2/3) - 1, 0)$
- Driving: car j moves $v_j(t+1)$ sites.

Open or periodic boundary conditions

Concentrate on $v_{\max} = 1$.

The ASEP

Asymmetric Simple Exclusion Process: particles hopping on a line with hardcore repulsion.



Different dynamical rules give rise to different correlation functions. The strongest correlations, e.g. $\langle \tau_i \tau_{i+1} \rangle$, arise with fully parallel updating.

Implementation of Fully Parallel Dynamics

Configurations labeled by

$$\{\tau\} = \{\tau_1, \ldots, \tau_L\}, \quad \tau_i = 0, 1.$$

With $\tau(t+1) = \tau'$ the rules are,

- Bulk:

$$\tau_i' = \hat{p}_{i-1}\tau_{i-1}(1-\tau_i)+(1-\hat{p}_i)\tau_i(1-\tau_{i+1})+\tau_i\tau_{i+1}.$$

- Boundary:

$$\begin{aligned} \tau_1' &= \hat{\alpha}(1-\tau_1)+(1-\hat{p}_1)\tau_1(1-\tau_2)+\tau_1\tau_2, \\ \tau_L' &= \hat{p}_{L-1}\tau_{L-1}(1-\tau_L)+(1-\hat{\beta})\tau_L. \end{aligned}$$

These rules are applied to all τ_i at the same time. The transition rates are obtained by time-averaging,

$$\langle \hat{p}_i \rangle = p, \quad \langle \hat{\alpha} \rangle = \alpha, \quad \langle \hat{\beta} \rangle = \beta.$$

Transfer matrix notation: Let $P_t(\tau)$ be the probability of finding configuration $\{\tau\}$ at time t, then

$$P_{t+1}(\tau) = \sum_{\sigma} T_{\tau\sigma} P_t(\sigma).$$

Free Flow: $p = 1,\ \beta = 1$

The bulk dynamics is simple:

$$\tau_i' = \tau_{i-1},$$

and there are no jams:

$$\tau_i \tau_{i+1} = 0.$$

The eigenvalue equation simplifies, e.g.:

$$P(\tau_1, \ldots, \tau_{L-1}, 1) = F_\alpha(\tau_1|\tau_2) P(\tau_2, \ldots, \tau_{L-1}, 1, 0),$$

Factorised solution:

$$P(\tau_1, \ldots, \tau_L) = R_{\tau_L} \prod_{i=1}^{L-1} F_\alpha(\tau_i|\tau_{i+1}).$$

Matrix Product Ansatz for Steady State

Ansatz: For the stationary state, each probability can be written as a product,

$$P(\tau_1, \ldots, \tau_L) = \langle L_{\tau_1} | M(\tau_1 \tau_2) \cdots M(\tau_{L-1} \tau_L) | R_{\tau_L} \rangle.$$

If $M(\tau_i \tau_{i+1})$ is a number, this would be a Mean Field Approximation. Here $M(\tau_i \tau_{i+1})$ is a matrix, generally infinite dimensional.

If matrices are known the calculation of correlation functions become easy. The normalisation is

$$\begin{aligned} Z &= \sum_{\{\tau\}} P(\{\tau\}) = \langle L | S^{L-1} | R \rangle = 1, \\ S &= M(00) + M(01) + M(10) + M(11). \end{aligned}$$

The current is,

$$J = p \langle \tau_i (1 - \tau_{i+1}) \rangle = p \langle L | S^{i-1} M(10) S^{L-i-1} | R \rangle.$$

Algebra for Matrices

The matrices may be found by deducing algebraic rules from explicit solutions for small system sizes.

Examples

$$P_L(\ldots 1101\ldots) = (1-p)P_{L-1}(\ldots 101\ldots) + P_{L-1}(\ldots 111\ldots) + pP_{L-2}(\ldots 11\ldots),$$

which leads to

$$M(11)M(10)M(01) = (1-p)M(10)M(01) + \ldots$$

Similarly, there are are boundary relations, e.g.,

$$P_L(00\ldots) = p\frac{(1-\alpha)}{\alpha}P_{L-1}(0\ldots),$$

leading to

$$\langle L_0|M(00) = p\frac{(1-\alpha)}{\alpha}\langle L_0|.$$

Find representations of this algebra

Representation for Algebra

An example of a representation:

$$M(00) = M(10) = q \begin{pmatrix} q & 0 & 0 & 0 & \cdots \\ 1 & q & 0 & 0 & \\ 0 & 1 & q & 0 & \\ 0 & 0 & 1 & q & \\ \vdots & & & & \ddots \end{pmatrix}$$

$$M(11) = (1-p)M(01) = q \begin{pmatrix} q & 1 & 0 & 0 & \cdots \\ 0 & q & 1 & 0 & \\ 0 & 0 & q & 1 & \\ 0 & 0 & 0 & q & \\ \vdots & & & & \ddots \end{pmatrix}$$

$$|R\rangle = (1, b, b^2, b^3, \ldots)^t,$$
$$\langle L| = (1, a, a^2, a^3, \ldots),$$

where

$$q = \sqrt{1-p}, \quad a = \frac{p-\alpha}{q\alpha}, \quad b = \frac{p-\beta}{q\beta}.$$

Phase boundaries

Many correlations are expressed in

$$R_L(c) \sim \oint_{|z|=1} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \frac{\Lambda(z)^L}{(z-c)(z^{-1}-c)}, \quad c = a, b > 1.$$

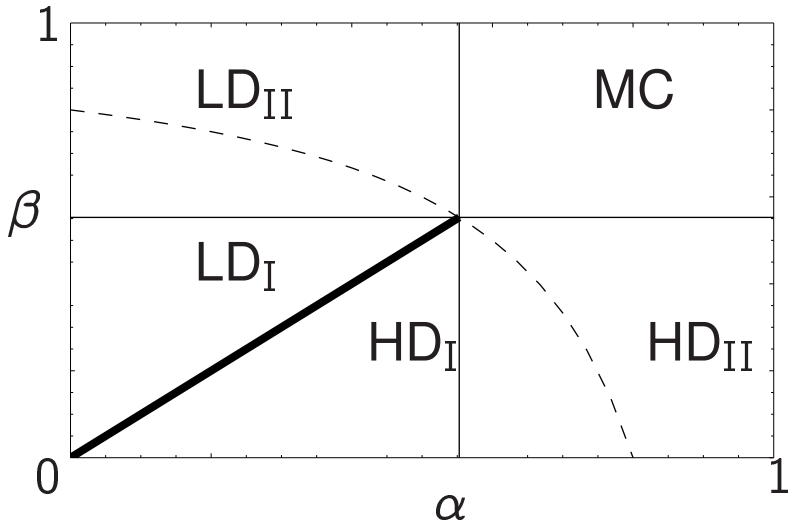
Implies phase boundaries at $a = 1$ and $b = 1$ and $a = b$, or

$$\alpha = 1 - \sqrt{1-p},\ \beta = 1 - \sqrt{1-p},\ \alpha = \beta.$$

Asymptotics for R_L

$$\begin{aligned} R_L(a) &\sim L^{-3/2}, && \text{for } a < 1 \\ &\sim L^{-1/2}, && \text{for } a = 1 \\ &\sim \left(\frac{p^2(1-\alpha)}{\alpha(p-\alpha)}\right)^L, && \text{for } a > 1. \end{aligned}$$

Phase diagram

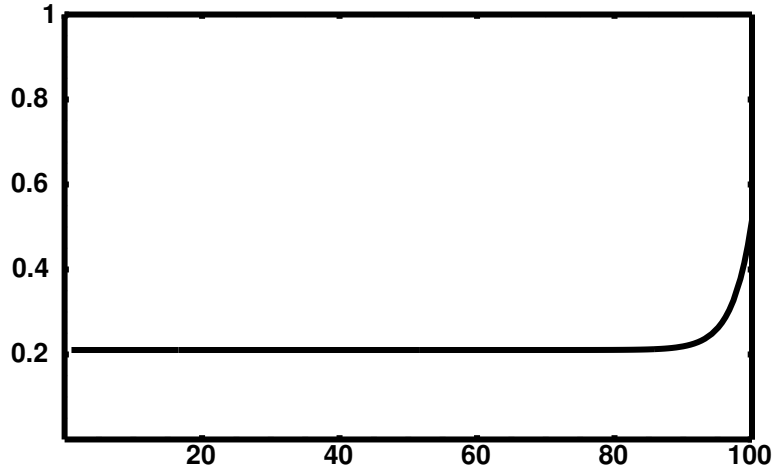


Phase diagram in the $\alpha - \beta$ plane. Phase boundaries are at $\alpha, \beta = 1 - \sqrt{1-p}$ and at $\alpha = \beta$ where the transition is discontinuous in the density. On the dashed line, given by $1 - p = (1-\alpha)(1-\beta)$, the mean field solution is exact.

The current may take three different values depending on the parameters α and β. These values correspond to a Low Density, a High Density and a so called Maximum Current phase. The density profile will give rise to a further discrimination of phases within the low density and high density phase.

- **Low density phase LD_{I}**

  Density profile along the road:



  This phase is chararcterized by the values

$$\alpha < \beta < 1 - \sqrt{1-p}.$$

  The current and bulk density ρ in this phase is given by

$$J_\alpha = \frac{\alpha(p-\alpha)}{p-\alpha^2}, \qquad \rho = 1 - J/\alpha.$$

The slope of the density profile is given by (with $r = L - i$),

$$t_L(i) \sim \mathrm{e}^{-r/\xi},$$

where

$$\xi^{-1} = \xi_\alpha^{-1} - \xi_\beta^{-1},$$

with

$$\xi_\alpha^{-1} = -\log\left(\frac{J_\alpha}{1-J_\alpha}\frac{1-J_{\max}}{J_{\max}}\right).$$

- **Transition line from $\mathrm{LD_I}$ to $\mathrm{LD_{II}}$**

  On this line, where $\beta = 1 - \sqrt{1-p}$, ξ_β diverges while ξ_α remains finite. The slope of the density profile now becomes

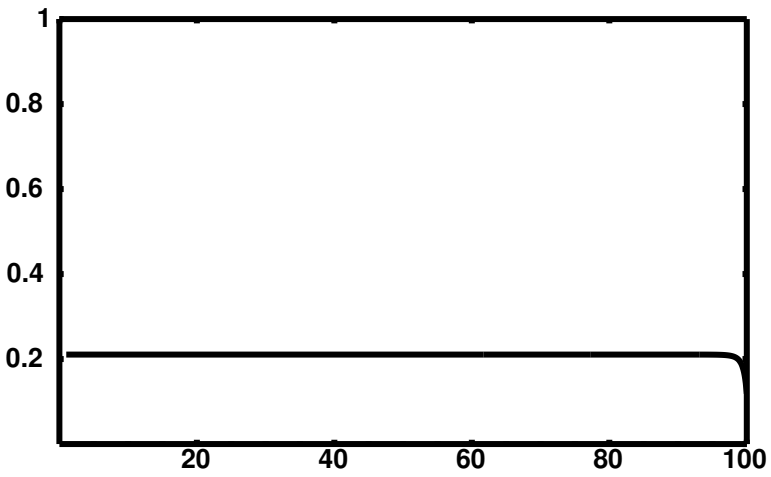
$$t_L(i) \sim r^{-1/2}\mathrm{e}^{-(r-1)/\xi_\alpha}.$$

- **Low density phase $\mathrm{LD}_{\mathrm{II}}$**

  Slope has power law correction to the exponential:

$$t_L(i) \sim (a - b^{-1}) r^{-3/2} \mathrm{e}^{-r/\xi_\alpha}.$$

  The slope changes sign at the curve $a = b^{-1}$ or $1 - p = (1 - \alpha)(1 - \beta)$. This is the curve on which the mean field solution is exact: On this line the stationary state completely factorizes and the density profile is flat.



- **Transition line from $\mathrm{LD}_{\mathrm{II}}$ to the maximal current phase MC**

  The slope of the density profile on this transition line is given by

  $$t_L(i) \sim -(i/L)^{-1/2} r^{-3/2}.$$

  Near the right boundary the slope decays algebraically as $r^{-3/2}$. Near the left boundary the slope of the density profile decays algebraically with a power of $1/2$ but the amplitude is of order $1/L$.

- **The maximal current phase MC**

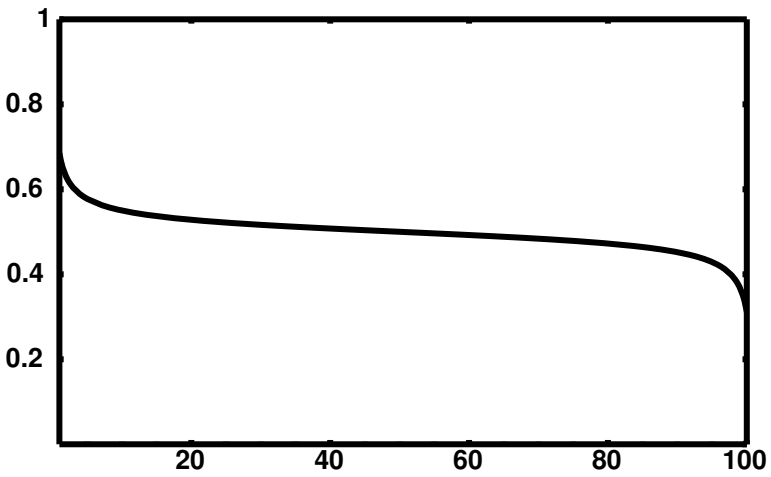
  The current attains it maximum value which is first reached on its phase boundaries. Its value and the bulk value of the density are given by

  $$J_{\max} = \frac{1}{2}(1 - \sqrt{1-p}), \quad \rho = \frac{1}{2}.$$

The slope of the density profile in this phase is given by

$$t_L(i) \sim -i^{-3/2}(r/L)^{-3/2}.$$

Since its slope is negative the density approaches its bulk value of $\rho = 1/2$ from above as $i^{-1/2}$ near the left boundary and from below as $r^{-1/2}$ near the right boundary.



- **The high density phases $\mathrm{HD_I}$ and $\mathrm{HD_{II}}$**

  The behaviour of the density profile in the high density phases and on their phase boundaries can be obtained from those of the low density phase by the particle hole symmetry.

$$\begin{aligned} \tau_{i-1} &\rightarrow 1-\tau_r, \\ \alpha &\leftrightarrow \beta. \end{aligned}$$

- **Coexistence line**

  This line is characterized by $\alpha=\beta<1-\sqrt{1-p}$. The length $\xi_\alpha=\xi_\beta$ remains finite but ξ diverges. On this line one finds a linear profile with a positive slope,

$$t_L(i)=\frac{p-2\alpha+\alpha^2}{(p-\alpha^2)L}.$$

  This is consistent with an instantaneous shock profile.

In the limit of small rates, i.e. $\alpha = p\tilde{\alpha}$ and $\beta = p\tilde{\beta}$ and $p \to 0$, the results for the ASEP with random sequential update are recovered.

Similar results were obtained by Evans et al.

In all phases and phase boundaries the current and bulk density ρ satisfy the following relation which defines the fundamental diagram

$$J = \frac{1}{2}\left(1 - \sqrt{1 - 4p\rho(1-\rho)}\right).$$

Conclusions & Outlook

- Phase diagram is robust

- $v_{\max} > 1$

  $v_{\max} = 1$ is very special. All sorts of Mean Field methods give very good approximations.

- Full Dynamics

  We have only found the groundstate. Can we solve the complete dynamics? Relation with integrable models?