What is persistence and what do we learn form it ?

Ivan Dornic

Service de Physique de l'État Condensé
CEA Saclay

- I. D., C. Godrèche, JPA **31**, 5413 (1998).

- A. Baldassarri, J.-P. Bouchaud, I. D., C. Godrèche, *Phys. Rev. E* **59**, R20 (1999).

- A. Baldassarri, H. Chaté, I. D., A. Lemaître, in preparation.

PLAN:

I) What is persistence ?

II) Statistics of the mean local magnetisation

III) An exactly solvable model:
ruin problem for biased Lévy flights

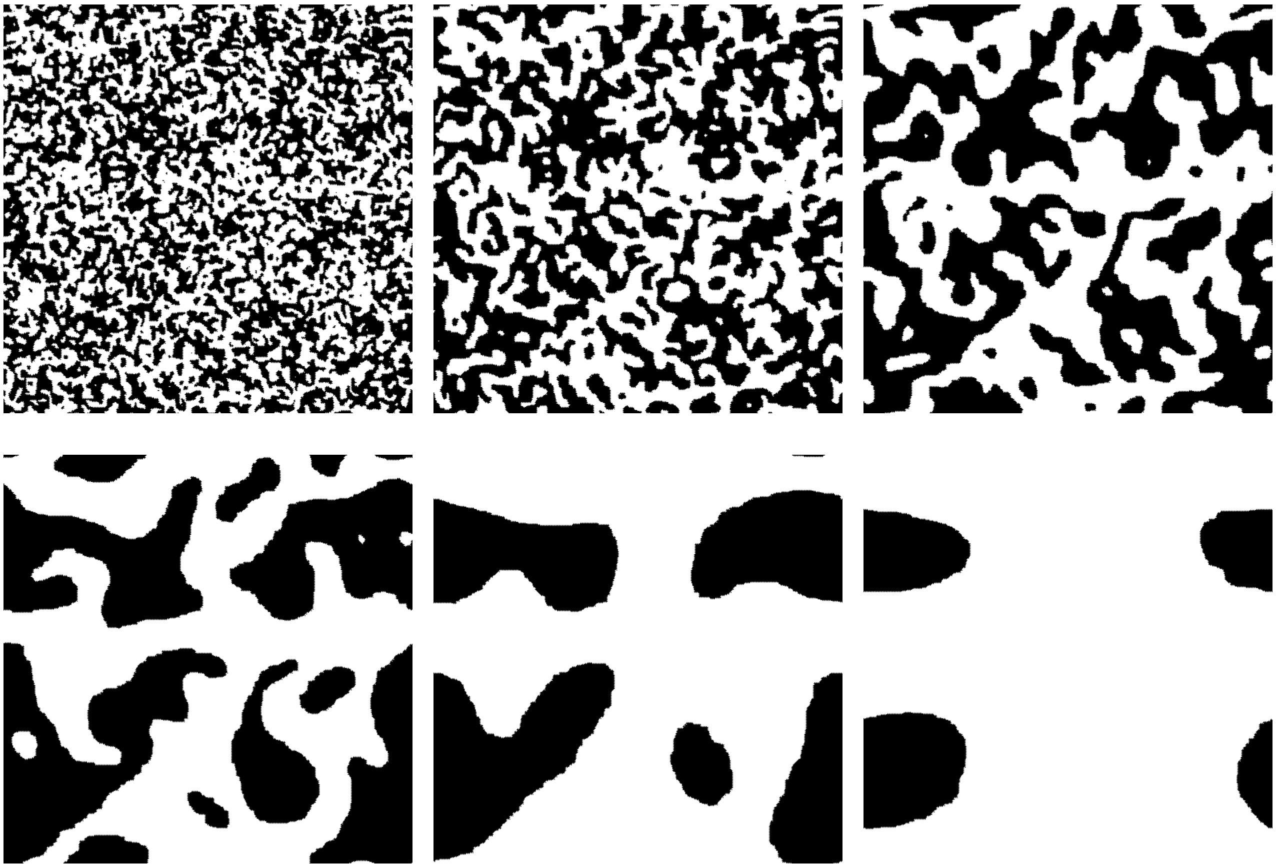
IV) Another exactly solvable model:
constrained Kesten variables

V) Conclusion: what have we learnt ?

I) What is persistence ?

- In coarsening systems

example : quench of a ferromagnetic $2d$ Ising model, $T_i = \infty$, $T_f = 0$, spins $\sigma = \pm 1$



$\rightsquigarrow$ dynamical scaling: $L(t) \sim t^{1/z}$
z dynamical exponent, $z = 2$ for model A
(curvature driven motion : $dL/dt \propto 1/L$)

$\rightsquigarrow$ local equilibrium but global never reached :
$t_{eq} \sim V^{z/d}$ diverges in the the thermodynamic limit

$\rightsquigarrow$ equal-time 2-point correlation function $\left\langle \sigma_{\mathbf{r}'}(t)\sigma_{\mathbf{r}+\mathbf{r}'}(t)\right\rangle$ depends only on $r/L(t)$

$\rightsquigarrow$ two-time (auto)correlation function $A(t_1,t_2) \equiv \langle \sigma_{\mathbf{r}}(t_1)\sigma_{\mathbf{r}}(t_2)\rangle$ depends only on $L(t_1)/L(t_2)$ — or t_1/t_2 if $L(t) \sim t^{1/z}$ — (statistical self-similarity)

Note: TTI recovered w.r.t "fictitious" time $\ell = \ln t$.

- **But**, if $t_2 \gg t_1$, then

$$A(t_1,t_2) \sim \left(\frac{L(t_1)}{L(t_2)}\right)^{\tilde{\lambda}} \sim \left(\frac{t_1}{t_2}\right)^{\lambda}$$

$\lambda = \tilde{\lambda}/z$ Fisher-Huse exponent

$\rightsquigarrow$ non-trivial : Ising $1d$: $\lambda = 1/2$, Ising $2d$: $\lambda \approx 0.62$ even if $z = 2$ for both.

$\rightsquigarrow$ related to the initial slip exponent

Persistence probability

- fraction of space always in the same phase up to time t
- proba. that a spin has never flipped

$R(t) \equiv \text{Prob}(\sigma(t') = +1, \ \forall t' \le t)$

$\rightsquigarrow$ slow decay :

$$R(t) \sim L(t)^{-\tilde{\theta}} \sim t^{-\theta}$$

$\theta = \tilde{\theta}/z$: persistence exponent

examples : Ising $1d$, $\theta = 3/8$, Ising $2d$, $\theta \approx 0.22$, TDGL $2d$ $\theta \approx 0.19$

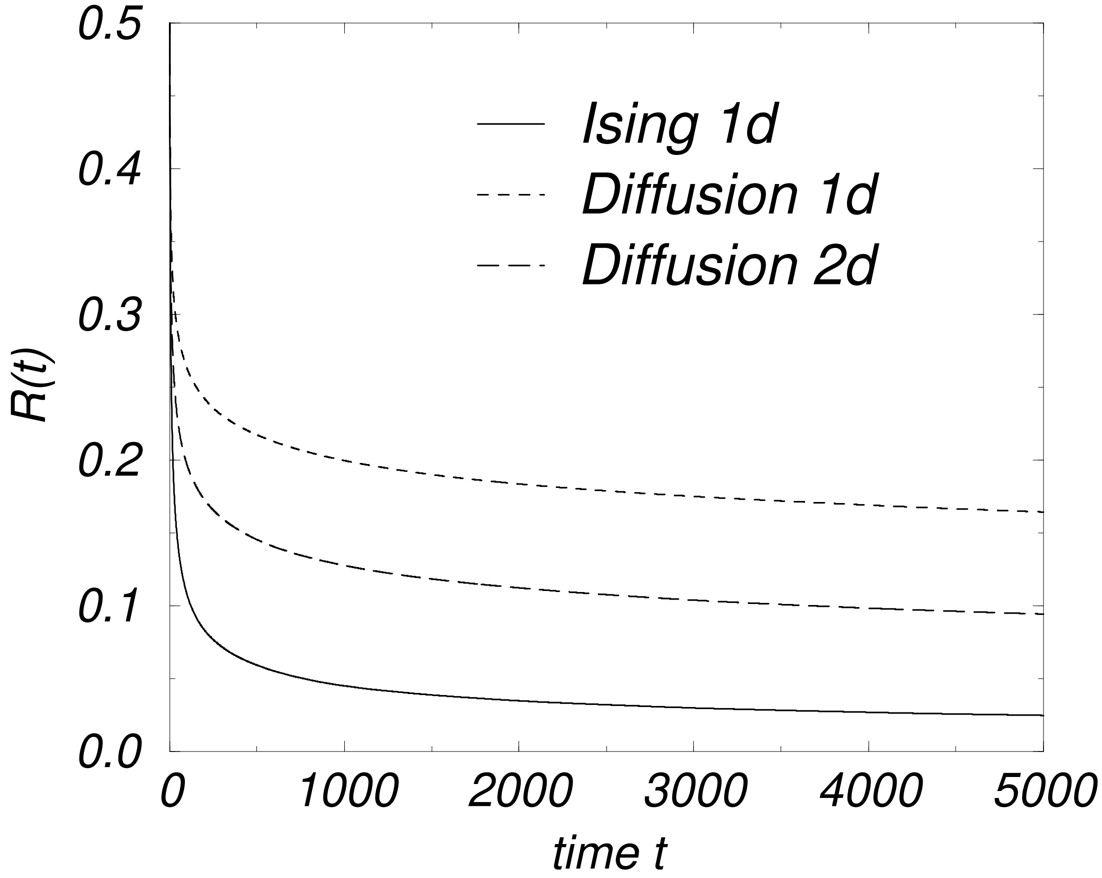
$\rightsquigarrow$ $\theta(T < T_c) = \theta(T = 0)$: Temperature irrelevant parameter

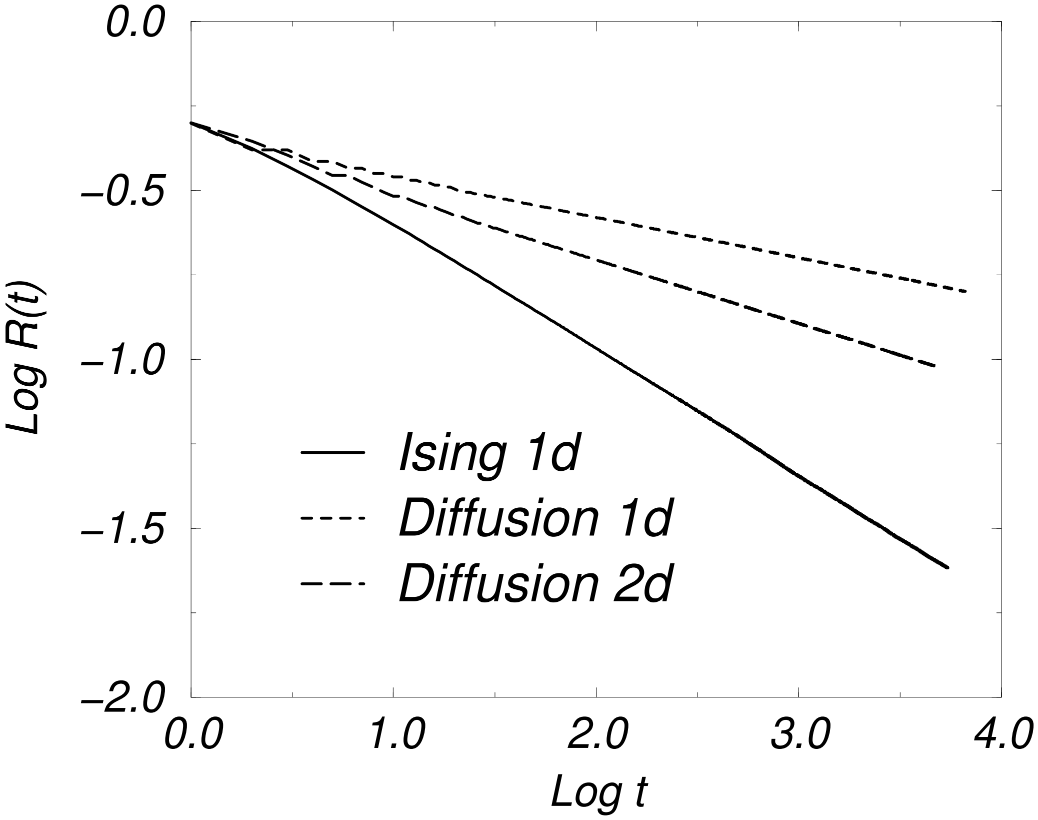
$\rightsquigarrow$ naive guess : $\theta = \lambda$ since
$A(0,t) = \text{Prob}(0 flip) - \text{Prob}(1 flip) + \ldots$

Wrong except for Markovian processes

$\theta =$ first-passage exponent in a strongly correlated system

$$-\frac{dR(t)}{dt} \sim t^{-1-\theta}$$


0.5
0.4
0.3
0.2
0.1
0.0
R(t)
0
1000
2000
3000
4000
5000
time t
Ising 1d
Diffusion 1d
Diffusion 2d


0.0
−0.5
−1.0
−1.5
−2.0
Log R(t)
0.0
1.0
2.0
3.0
4.0
Log t
Ising 1d
Diffusion 1d
Diffusion 2d

- Question not limited to coarsening systems :

$X(t)$ random , $R(t) = \text{Prob}\{X(t') > 0,\ \forall t' \le t\}$

example: diffusion equation with random gaussian initial conditions

$$\frac{\partial}{\partial t}\varphi(\mathbf{r},t) = \nabla^2\varphi(\mathbf{r},t)$$

$X(t) \equiv \varphi(\mathbf{r},t)/\sqrt{\langle\varphi^2(\mathbf{r},t)\rangle}$ still gaussian in time and stationary w.r.t. "fictitious time" $\ell = \ln t$

$C(\ell) \equiv \langle X(\ell_1)X(\ell_2)\rangle = (\cosh\ell/2)^{-d/2} \sim \exp(-\lambda\ell)$

effective spin $\sigma = \text{sign}\varphi = \text{sign}X$

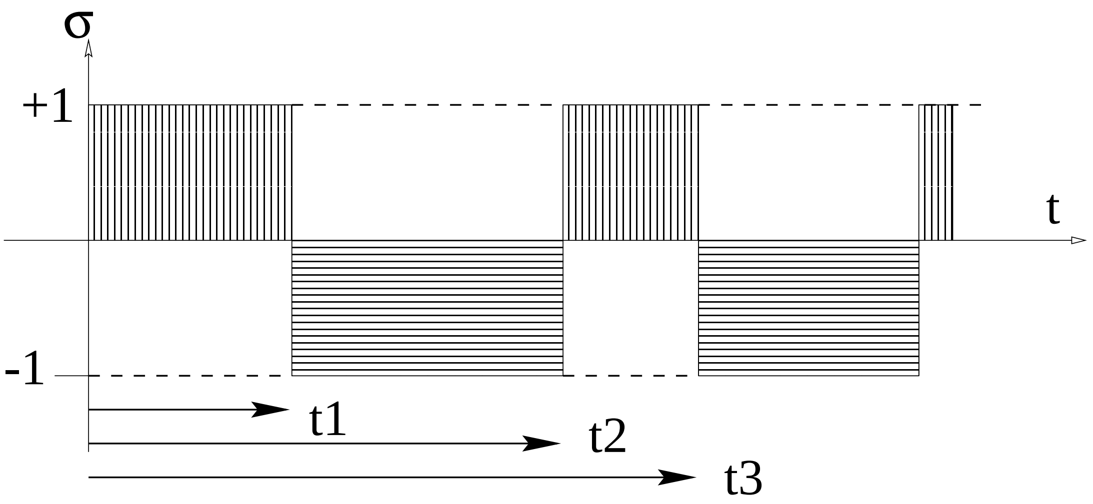
$$\langle\sigma(\ell_1)\sigma(\ell_2)\rangle = \frac{2}{\pi}\sin^{-1}\left[C(\ell_2 - \ell_1)\right]$$

$$R(t) \sim t^{-\theta} \Leftrightarrow R(\ell) \sim \exp(-\theta\ell)$$

$\rightsquigarrow$ $\theta =$ inverse of a decay rate in logarithmic time for a stationary gaussian process.

unsolved problem for 50 years except when ...
X Markovian (Doob's theorem) $\Leftrightarrow \lambda = \theta$

II) Statistics of the mean local magnetisation



For a given spin,

$$M(t) \equiv \frac{1}{t}\int_0^t \mathrm{d}t'\sigma(t') = \frac{T_t^+ - T_t^-}{t}$$

random variable, $\in [-1,1]$, $\langle M(t)\rangle = 0$

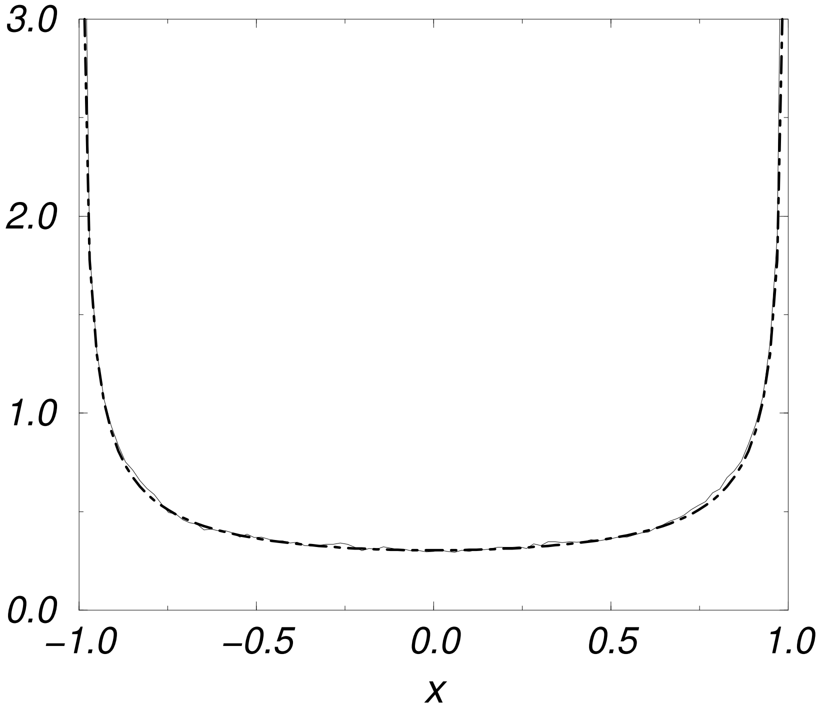
$$P(t,x) \equiv \text{Prob}\,\{M(t) \geq x\} \;\text{ and }\; f_M(x) \equiv -\frac{dP(t,x)}{dx}$$

- behaviour of $P(t,x)$ for $t \gg 1$?

$\rightsquigarrow$ take first $x = 1$:

$$P(t,1) = \text{Prob}\,\{M(t) = 1\} \equiv R(t) \sim t^{-\theta}$$

$\rightsquigarrow$ if $x < 1$, allows for spin flips example : histogram of $M(t)$ for Ising $1d$:



• broad limit law: $\langle M^2 \rangle > 0$ i.e. $M(t)$ not a self-averaging quantity : $f_M(x) \neq \delta(x)$

generic as soon as dynamical scaling holds :

$$\langle M^2 \rangle = \frac{2!}{t^2} \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 A(t_1, t_2)$$

• singular when $x \to \pm 1$ with the persistence exponent

$$f_M(x) \sim (1 - x^2)^{\theta - 1}, \; x \to \pm 1$$

$\rightsquigarrow$ stationary definition of a dynamic exponent

$\rightsquigarrow$ if $\theta < 1$, mean not the most probable value

$\rightsquigarrow$ $f_M(x)$ very closeto but different from a "pure" beta law $f_{beta}(x) = B^{-1}(1/2, \theta)(1 - x^2)^{\theta - 1}$

for the diffusion equation, assuming independence of the $\ell_k - \ell_{k-1} = \ln(t_k/t_{k-1})$, then as $x \to \pm 1$

$$f_M(x) = 0.99 f_{beta}(x) + \text{less singular terms}$$

(Kesten variable)

$f_M(x) = f_{beta}(x)$ only for Markovian processes

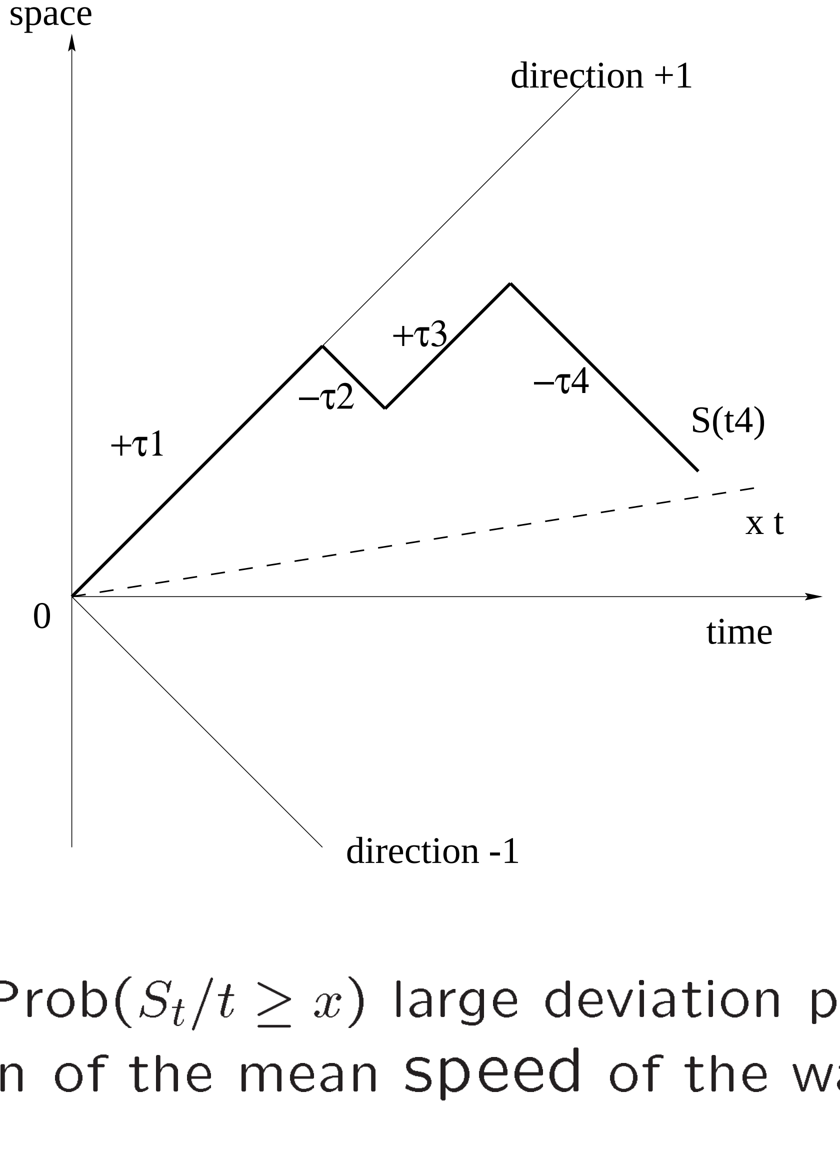
$\rightsquigarrow$ generalisation of arcsine law for brownian motion ($\theta = 1/2$)

$\rightsquigarrow$ if $0 < T_f < T_c$, then $f_M(x)$ singular at $x = \pm\mathcal{M}_0(T)$, the equilibrium magnetisation.

- Interpretation: fictitious random walker

$$M(t) = \frac{1}{t}(\tau_1 - \tau_2 + \tau_3 + \ldots) \equiv \frac{S(t)}{t}$$

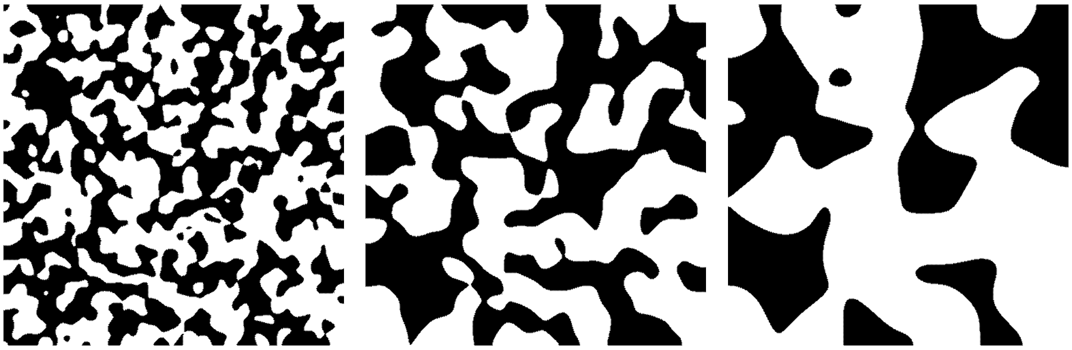
with (correlated) steps of length $\tau_k = t_k - t_{k-1}$



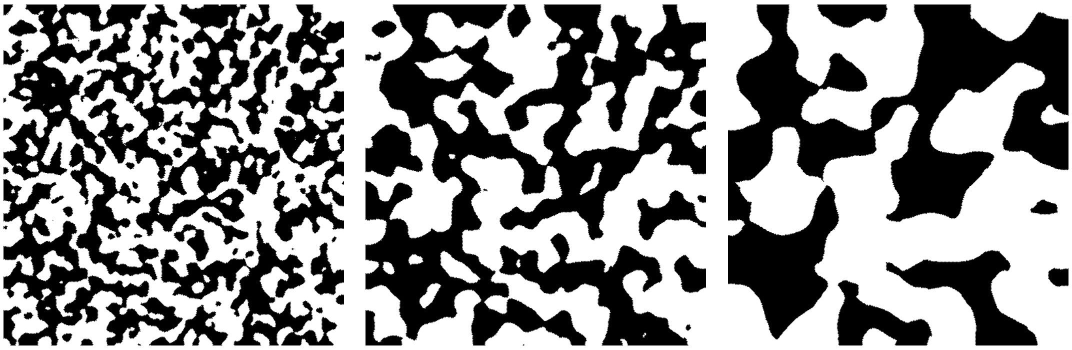
$P(t,x) = \text{Prob}(S_t/t \geq x)$ large deviation probability distribution of the mean speed of the walker

Illustrations: Diffusion equation in 2D

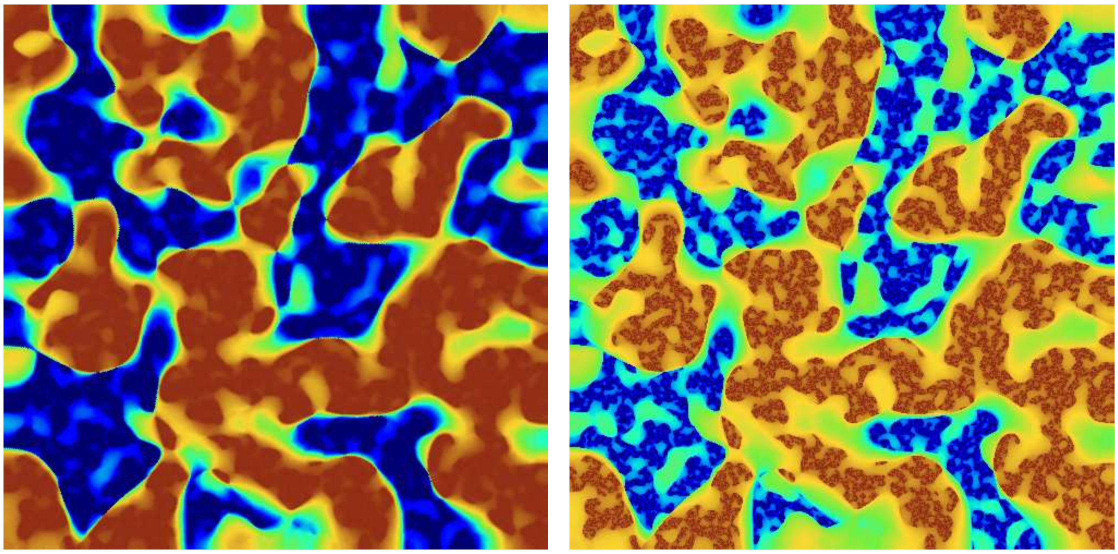
snapshots of sign$\varphi(\mathbf{r}, t)$



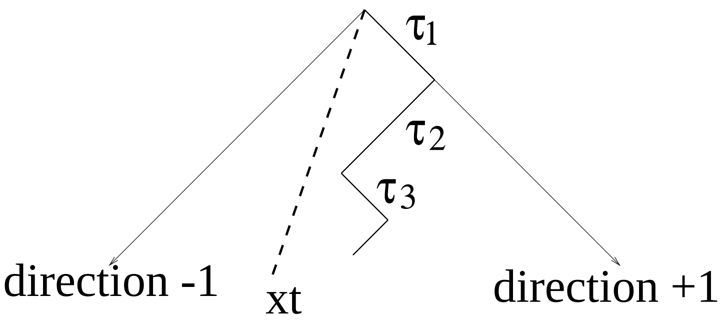
snapshots of sign$M(t)$



"coloured" histogram of $M(t)$



Persistent large deviations



$$R(t,x) \equiv \text{Prob}\left\{M(t') \geq x,\ \forall t' \leq t\right\}$$

proba. that a spin has always been at least a fraction $(1+x)/2$ of time in the + phase.

$\leadsto$ $R(t,x)$ = proba that the random walker has always been in the angular sector $(\widehat{x,1})$

$\leadsto$ $x = 1$: gives "usual" persistence

$$R(t,1) = \text{Prob}\left\{M(t') \geq 1,\ \forall t' \leq t\right\} = R(t)$$

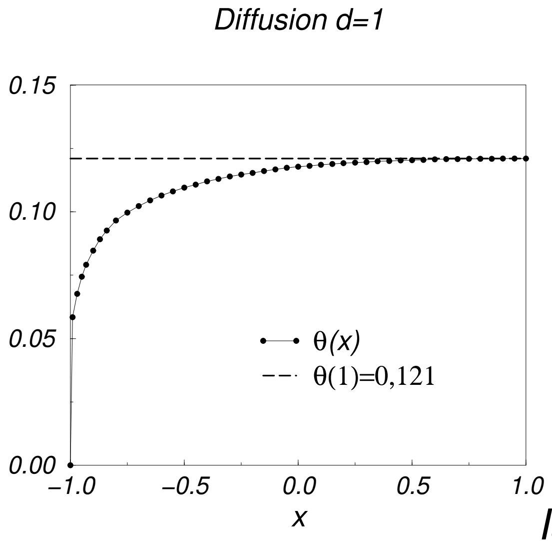
$\leadsto$ also persistence for effective spin $\sigma(t,x) = \text{sign}[M(t) - x]$:

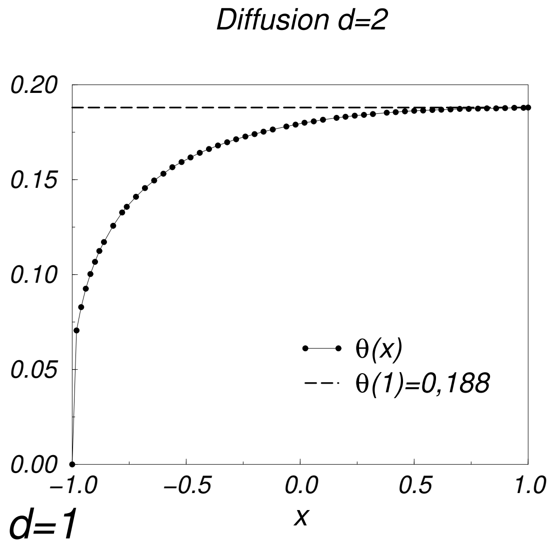
$$R(t,x) = \text{Prob}\left\{\sigma(t',x) = +1,\ \forall t' \leq t\right\}$$

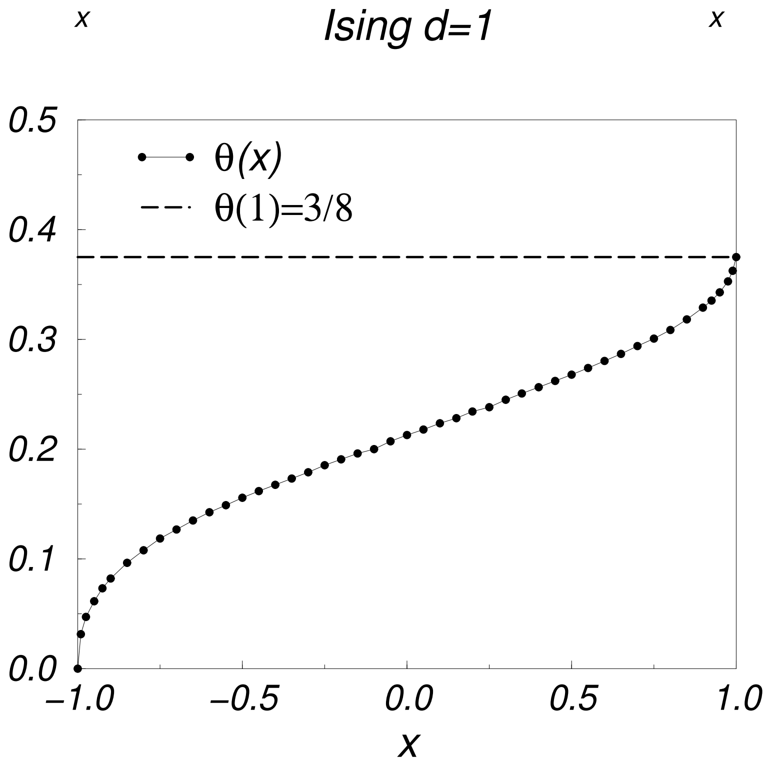
- algebraic decay of $R(t,x)$

$\rightsquigarrow$ $\theta(x)$ continuously-varying exponent with x

$\rightsquigarrow$ $\lim_{x\to 1}\theta(x) = \theta(1) \equiv \theta$

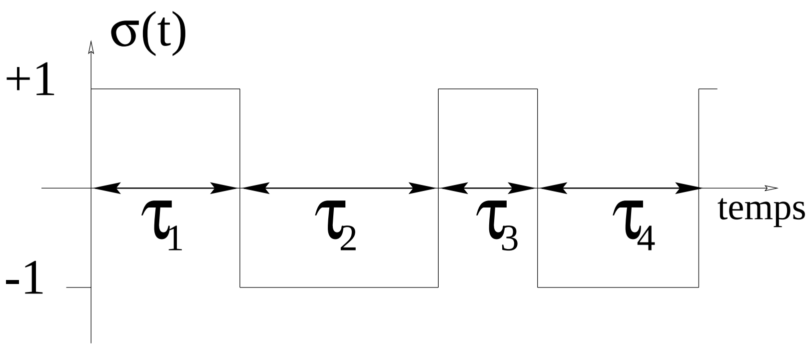






Note : for $x < 1$, $z(x) = z = 2$

III) An exactly soluble model: Levy flights



- If $\theta < 1$, the law $f(\tau_1)$ of the first flip is broad :

$$R(t) = \text{Prob}\,\{\tau_1 \geq t\} \sim t^{-\theta}$$

$\Rightarrow f(\tau_1) \propto 1/\tau_1^{1+\theta}$: Lévy law of index θ

- Assume that all the $\tau_k = t_k - t_{k-1}$ have the same distribution, without correlations :

$$f(\tau_1, \tau_2, \ldots, \tau_n) = \prod_{k=1}^{n} f(\tau_k), \;\; \tau_k = t_k - t_{k-1}$$

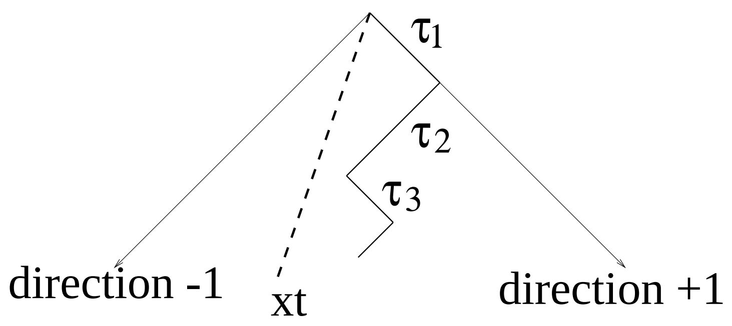
$\neq$ diffusion equation where the t_k/t_{k-1} are independent

θ only parameter of the model

- non-stationarity of two-time correlations :

$$\langle \sigma(t_1)\sigma(t_2)\rangle = \frac{\sin(\pi\theta)}{\pi}\int_0^{t_1/t_2} \mathrm{d}w\; w^{\theta-1}(1-w)^{-\theta}$$

- steps of the fictitious random walker are now uncorrelated



$$\begin{aligned} R(t,x) &= \mathrm{Prob}\left\{M(t') \geq xt', \forall t' \leq t\right\} \sim t^{-\theta(x)} \\ R(n,x) &= \mathrm{Prob}\left\{M(k) \geq xt_k, \forall 1 \leq k \leq n\right\} \sim n^{-\phi(x)} \end{aligned}$$

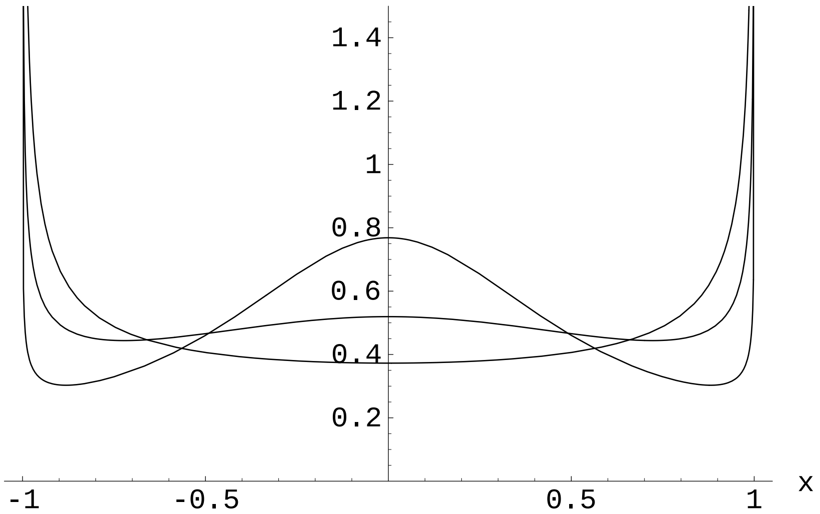
- Limit law

when the number of steps n or the time t goes to infinity :

$$f_M(x) = \frac{2\sin\theta\pi}{\pi} \frac{(1-x^2)^{\theta-1}}{(1+x)^{2\theta} + 2(1-x^2)^{\theta} + (1-x)^{2\theta}}$$

$\rightsquigarrow$ singular as $(1-x^2)^{\theta-1}$ when $x \to \pm 1$

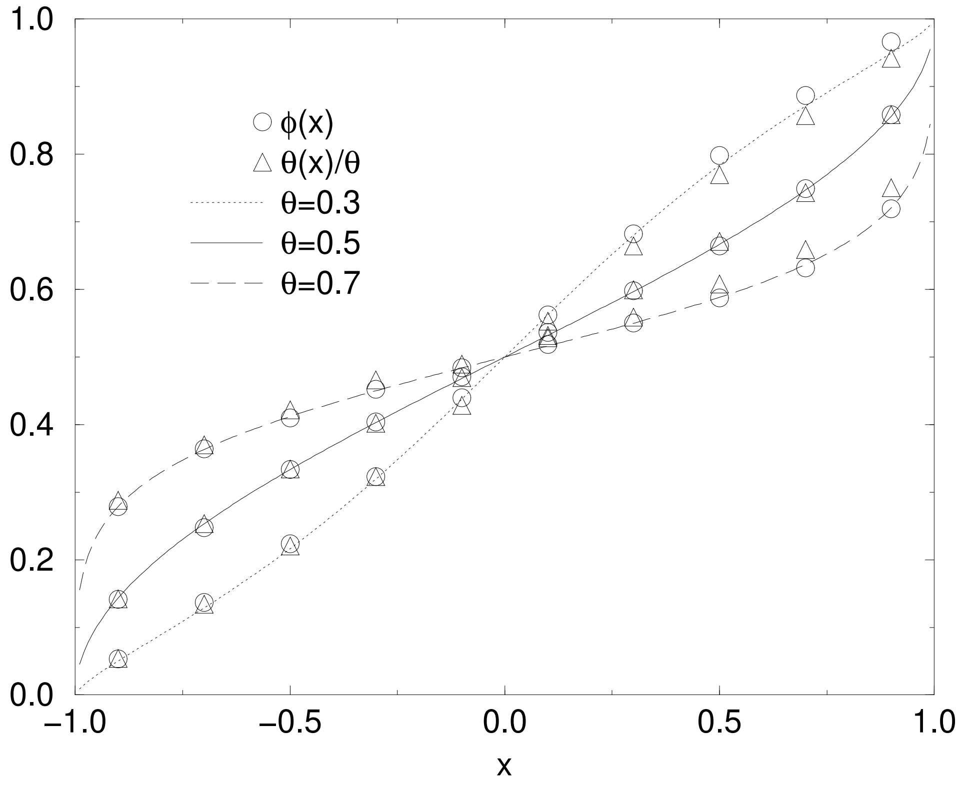
$\rightsquigarrow$ if $0,5946 < \theta < 1$, the mean value is a local maximum



- ## Persistent large deviations

Using Sparre Andersen's results,

$$\phi(x) = \theta(x)/\theta = \int_{-1}^{x} du f_M(u)$$



exponent given by the repartition function of the limit law !

Note: for strict Lévy laws,

$$R(2k,x) = R(2k+1,x) = \frac{\Gamma(k+1-\phi(x))}{\Gamma(k+1)\Gamma(1-\phi(x))}$$

- Sparre Andersen theorem (1953)

Let X_i i.i.d random variables, and consider their partial sums:

$$S_1 = X_1, S_2 = X_1 + X_2, \dots$$

Define $p_n \equiv \mathrm{Prob}\{S_1 > 0, S_2, \dots, S_n > 0\}$ then

$$\sum_{n \geq 0} \frac{p_n s^n}{n!} = \exp\left(\sum_{n \geq 1} \frac{s_n}{n} \mathrm{Prob}\{S_n > 0\}\right)$$

Note that for symmetric continuous variables, universal result ! :

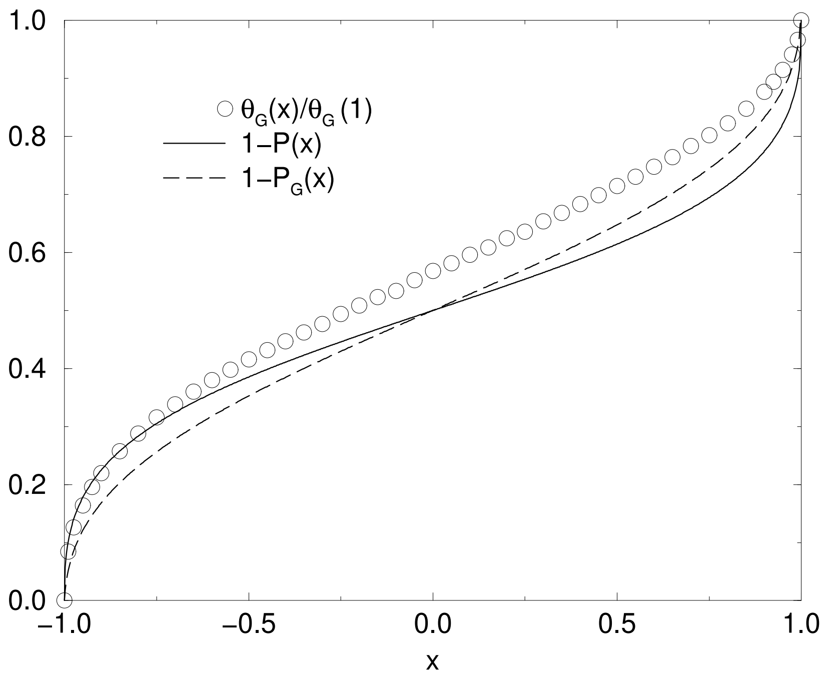
$$p_n = \frac{(2n)!}{2^{2n}(n!)^2} \simeq \frac{1}{\sqrt{\pi n}}$$

For our model, take $X_i = (1-x)\tau_{2i-1} - (1+x)\tau_{2i}$. Stability of Lévy variables implies

$$\mathrm{Prob}\{S_n > 0\} = \mathrm{Prob}\{S_1 > 0\} = 1 - \phi(x)$$

- Real (correlated) systems

$\rightsquigarrow$ good for Ising $1d$ with $\theta = 3/8$



$\rightsquigarrow$ bad for diffusion (= regular (differentiable) process)

- $\theta = 1/2$: unknown exponents for Brownian Motion !

$$\theta(x) = \frac{1}{\pi} \arcsin \sqrt{\frac{1+x}{2}}$$

IV) Another exactly solvable model: constrained Kesten variables

IIA: hypothesis of no correlation for the $\ell_i = \ln t_i - \ln t_{i-1}$, their law $f(\ell)$ determined through

$$\widehat{f}(s) = \frac{1 - \langle \ell \rangle s[1 - s\hat{A}(s)]/2}{1 + \langle \ell \rangle s[1 - s\hat{A}(s)]/2}$$

$$\hat{A}(s) = \int_0^{+\infty} \mathrm{d}\ell e^{-s\ell} \langle \sigma(0)\sigma(\ell) \rangle$$

Works only for smooth processes: $\langle \ell \rangle < \infty$
Gives extraordinary good results for the Diffusion equation, for which $\langle \ell \rangle$ and $A(\ell)$ exactly known.
Rightmost pole of $\widehat{f}(s)$ gives $f(\ell) \propto \exp(-\theta_{IIA}\ell)$, with (in, say, $d = 2$)

$$\theta_{IIA} = 0.1862\ldots \quad vs. \quad \theta_{num} = 0.1875 \pm 0.001$$

Limit law for $M(t)$ within IIA

In term of the number of spin flips,

$$M_2 = \frac{t_1 - (t_2 - t_1)}{t_2} = -1 + 2\frac{t_1}{t_2}$$

$$M_3 = 1 - \frac{t_2}{t_3}(1 - M_2)$$

Defining $W_n = [1 + (-1)^n M_n]/2$, one has a random multiplicative recursion relation

$$W_n = z_{n-1}(1 - W_{n-1}), \quad z_i = t_{i-1}/t_i = \exp(-\ell_i)$$

z_i *independent* with density $f_Z(z) \sim_0 z^{\theta-1}$
When $n \to \infty$, convergence in distribution:

$$p(w) = \int_0^{1-w} \mathrm{d}w' \frac{p(w')}{1-w'} f_Z\left(\frac{w}{1-w'}\right)$$

No-closed form solution except when $f_Z(z) = \theta z^{\theta-1}$, then $p(w) = B^{-1}(\theta, \theta+1)(1-w)^\theta w^{\theta-1}$, and

$$f_M(x) = B^{-1}(1/2, \theta)(1 - x^2)^{\theta-1}$$

- Persistent large deviations

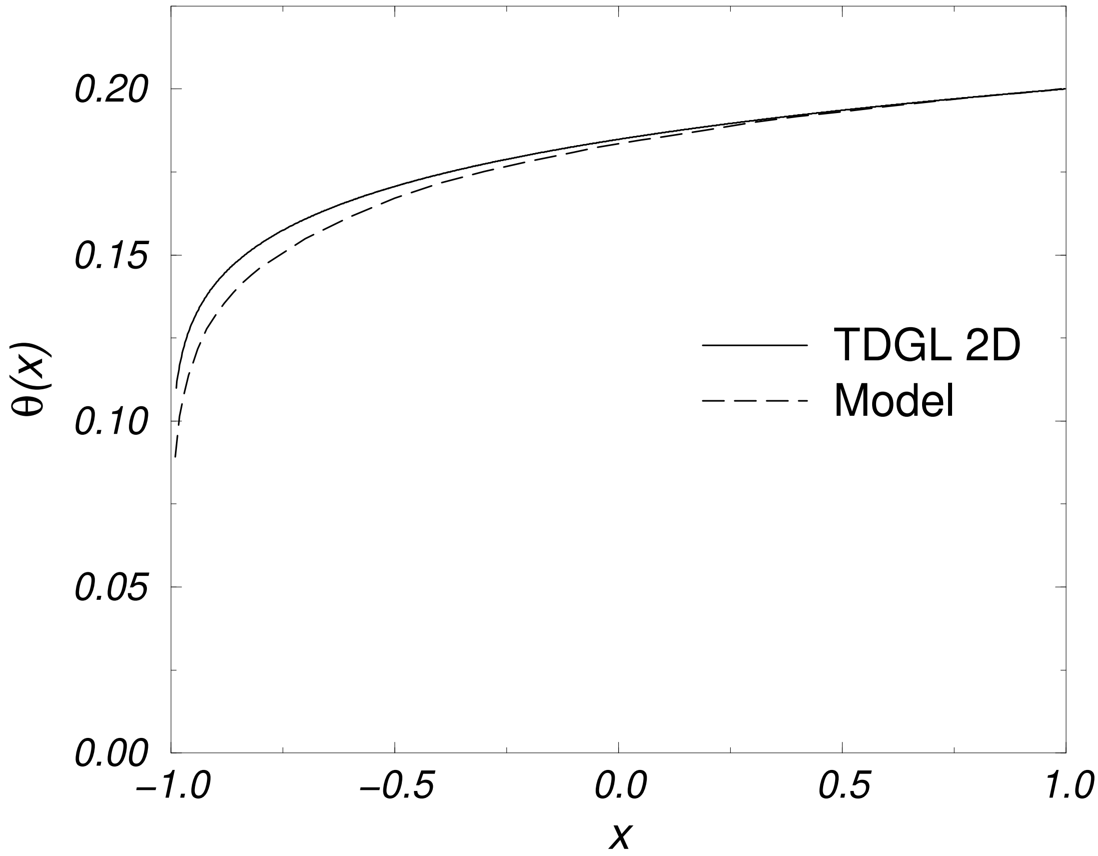
$$R(2n,x) = \text{Prob}\left\{W_{2k} \geq \frac{1+x}{2},\ \forall\ 1 \leq k \leq n\right\}$$

Constrained Kesten variables ...

Exact solution for $f_Z(z) = \theta z^{\theta-1}$:

$$R(2n,x) \sim [\rho(x)]^n,\ \text{with}\ {}_2F_1\left(-\frac{\theta}{\sqrt{\rho}}, \frac{\theta}{\sqrt{\rho}}, \theta; \frac{1-x}{2}\right) = 0$$

Finally, $R(t,x) \sim t^{-\theta(x)}$, with $\theta(x)/\theta = 1 - \rho^2(x)$



V) Conclusion

- Infinite number of non-trivial exponents in these strongly interacting systems.

- The exact solution furnishes a mechanism and shows the intimate relation between the exponents and the limit law

- Universality does not hold anymore, even for Ising 2d vs. TDGL 2d