INTEGRABLE BOUNDARY KONDO IMPURITIES IN CORRELATED ELECTRON MODELS

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Brief Historical Overview

- Andrei and Johanesson 1984
  Integrable spin s impurity in the XXX chain.
- Schmitteckert, Schwab and Eckern 1995
  Eckle, Punnoose and Römer 1997
  Integrable impurities in Heisenberg models introduced by way of inhomogeneities in the spectral parameters of local L-operators.
- Bares 1994 (unpublished)
  Spectral parameter inhomogeneities in $t-J$ model.
- Bedürftig, Essler and Frahm 1997
  Schlottmann and Zvyagin 1997
  Links and Foerster 1998
  Abad and Ríos 1998
  Bulk impurities in $t-J$ models introduced by way of changing space of states for local L-operators.

All the above works employ the Quantum Inverse Scattering Method to construct a periodic model which is integrable. Generally there are two problems associated with this approach

- Such models do not exhibit backscattering at the impurity sites.
- The local Hamiltonians at the impurity involve three site interactions which are generally complicated.

An alternative is to consider open chain models with impurities on the boundaries. Open boundary conditions in the $t-J$ model have been studied

- González-Ruiz 1994
  Essler 1996
  Integrable boundary fields introduced using Sklyanin's open chain formulation of the Quantum Inverse Scattering Method.
- Wang, Dai, Hu and Pu 1997
  Hu, Pu and Wang 1998
  Boundary spin 1/2 magnetic (or Kondo) impurities introduced and solved by the co-ordinate Beth ansatz method.

This talk will describe a reformulation of the work of Wang, Dai, Hu and Pu in terms of the Quantum Inverse Scattering Method (thus establishing integrability) and extends the results to the case of arbitrary spin s impurities.

Integrable $t-J$ models with boundary impurities

In the standard notation we consider Hamiltonians of the following form

$$
\begin{aligned}
H = & -t\sum_{j=1,\sigma}^{L-1}\mathcal{P}(c^{\dagger}_{j\sigma}c_{j+1\sigma}+c^{\dagger}_{j+1\sigma}c_{j\sigma})\mathcal{P} \\
& +\sum_{j=1}^{L-1}(J\mathbf{S}_j\cdot\mathbf{S}_{j+1}+Vn_jn_{j+1}) \\
& +J_a\mathbf{S}_1\cdot\mathbf{S}_a+V_an_1+J_b\mathbf{S}_L\cdot\mathbf{S}_b+V_bn_L.
\end{aligned}
$$

Above, the projector $\mathcal{P}=\Pi_{j=1}^{L}(1-n_{j\uparrow}n_{j\downarrow})$ ensures that double electron occupancy of sites is forbidden. The boundary spin operators can take arbitrary half integer spin values.

For the choice

$$
t=1,\quad J=2,\quad V=-\frac{1}{2}
$$

it is well known that the bulk Hamiltonian acquires $gl(2|1)$ symmetry and is integrable. We will establish integrability of the Hamiltonian for the choice of the boundary parameters

$$
\begin{aligned}
J_\alpha &= \frac{-2}{(c_\alpha-s)(c_\alpha+s+1)}, \\
V_\alpha &= \frac{-c_\alpha^2+s(s+1)}{(c_\alpha-s)(c_\alpha+s+1)}
\end{aligned}
$$

by working in the framework of the QISM.

Another (less known) choice of couplings which leads to a bulk integrable model is given by

$$t = 1, \quad J = -2, \quad V = -\frac{3}{2}.$$

This case corresponds to an $sl(3)$ invariant solution. We can introduce integrable Kondo impurities on the boundary for this coupling by choosing

$$J_\alpha = \frac{2}{(c_\alpha + s)(c_\alpha - s - 1)},$$
$$V_\alpha = \frac{c_\alpha^2 - 1 - s(s+1)}{(c_\alpha + s)(c_\alpha - s - 1)}.$$

Below, we describe how the aforementioned integrable cases are obtained.

Review of Boundary QISM

We begin with a solution of the supersymmetric Yang-Baxter equation

$$R_{12}(u_1-u_2)R_{13}(u_1)R_{23}(u_2)=R_{23}(u_2)R_{13}(u_1)R_{12}(u_1-u_2).$$

The multiplication rule for any homogeneous elements $x,\ y,\ x',\ y'\in \mathrm{End}\,V$ is given by

$$(x\otimes y)(x'\otimes y')=(-1)^{[y][x']}\,(xx'\otimes yy')$$

where $[x]$ stands for the $\mathbf{Z}_2$-grading of the element x.

We form the monodromy matrix $T(u)$ for a L-site lattice chain by

$$T(u)=R_{0L}(u)\cdots R_{01}(u).$$

$T(u)$ generates a representation of the $\mathbf{Z}_2$-graded Yang-Baxter algebra,

$$R_{12}(u_1-u_2)\overset{1}{T}(u_1)\overset{2}{T}(u_2)=\overset{2}{T}(u_2)\overset{1}{T}(u_1)R_{12}(u_1-u_2)$$

where for notational convenience we have

$$\overset{1}{T}(u)=T_{13}(u),\qquad \overset{2}{T}(u)=T_{23}(u)$$

and the subscript 3 now labels the quantum superspace $W=V^{\otimes L}$.

The R-matrices are assumed to enjoy the unitarity property,

$$R_{12}(u_1-u_2)R_{21}(-u_1+u_2)=1,$$

and crossing unitarity

$$R_{12}^{st_1}(u_1-u_2)R_{21}^{st_1}(-u_1+u_2+\eta)=1.$$

We introduce the associative superalgebras $\mathcal{T}_-$ and $\mathcal{T}_+$ defined by the R-matrix and the relations

$$R_{12}(u_1-u_2)\overset{1}{\mathcal{T}_-}(u_1)R_{21}(u_1+u_2)\overset{2}{\mathcal{T}_-}(u_2)$$

$$=\overset{2}{\mathcal{T}_-}(u_2)R_{12}(u_1+u_2)\overset{1}{\mathcal{T}_-}(u_1)R_{21}(u_1-u_2)$$

$$R_{21}^{st_1\,ist_2}(-u_1+u_2)\overset{1}{\mathcal{T}_+^{st_1}}(u_1)R_{12}(-u_1-u_2+\eta)\overset{2}{\mathcal{T}_+^{ist_2}}(u_2)$$

$$=\overset{2}{\mathcal{T}_+^{ist_2}}(u_2)R_{21}(-u_1-u_2+\eta)\overset{1}{\mathcal{T}_+^{st_1}}(u_1)R_{12}^{st_1\,ist_2}(u_1-u_2).$$

One can obtain a class of realizations of the superalgebras $\mathcal{T}_+$ and $\mathcal{T}_-$ by choosing $\mathcal{T}_\pm(u)$ to be the form

$$\begin{aligned}\mathcal{T}_-(u)&=T_-(u)K_-(u)T_-^{-1}(-u),\\ \mathcal{T}_+^{st}(u)&=T_+^{st}(u)K_+^{st}(u)\left(T_+^{-1}(-u)\right)^{st},\end{aligned}$$

with

$$\begin{aligned}T_-(u)&=R_{0M}(u)\cdots R_{01}(u),\\ T_+(u)&=R_{0L}(u)\cdots R_{0,M+1}(u),\end{aligned}$$

where M is any index between 1 and L, and $K_\pm(u)$, called boundary K-matrices, are representations of $\mathcal{T}_\pm$. In the following, without loss of generality, we shall choose $M=L$ so that $\mathcal{T}_+(u)\equiv K_+(u)$.

One defines the transfer matrix $\tau(u)$ as

$$\begin{aligned}\tau(u) &= str(K_+(u)\mathcal{T}_-(u)) \\ &= str\left(K_+(u)T(u)K_-(u)T^{-1}(-u)\right).\end{aligned}$$

Then it can be shown that

$$[\tau(u_1), \tau(u_2)] = 0$$

so the transfer matrix can be used to construct an integrable model.

Many works in the literature have studied the case when the entries of the K matrices are scalar functions. It is of interest however to consider cases when the elements of the K matrices are operators on some Hilbert space which may be interpreted as a boundary impurity space of states. This is precisely the approach we must adopt to show that the aforementioned Hamiltonians are integrable.

It is importatnt to note that the solutions we will consider are not regular in the sense that cannot be obtained by dressing scalar solutions; i.e. they are not of the form

$$K_-(u) = L(u)k_-(u)L^{-1}(-u)$$

where $k_-(u)$ is a scalar solution and $L(u)$ is a representation of the Yang-Baxter algebra.

Construction of the K matrices for the $gl(2|1)$ case

Recall that the local Hamiltonian of the supersymmetric $t-J$ model is derived from the following R-matrix which satisfies the Yang-Baxter equation

$$
\begin{aligned}
R(u) &= uI + P \\
&= \begin{pmatrix}
u-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & u & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & u & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & u-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & u & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & u & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & u & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u+1
\end{pmatrix}.
\end{aligned}
$$

We adopt the $\mathbf{Z}_2$-grading $[1] = [2] = 1$, $[3] = 0$ on the indices labelling the basis vectors.

We now solve the reflection equations for $K_+(u)$ and $K_-(u)$ beginning with the ansatz

$$
K_-(u) = \begin{pmatrix}
A_-(u) & B_-(u) & 0 \\
C_-(u) & D_-(u) & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

Substitution into the reflection equation yields 22 equations of which two are automatically satisfied.

Assuming further that

$$A(u) = \alpha(u) + \beta(u)\mathbf{S}^z, \quad B(u) = \beta(u)\mathbf{S}^-,$$
$$C(u) = \beta(u)\mathbf{S}^+, \quad D(u) = \alpha(u) - \beta(u)\mathbf{S}^z$$

then reduces the problem to the following 3 equations

$$\begin{aligned}
&u_+(\alpha(u_1) - \alpha(u_2)) \\
&= u_-(-1 + \alpha(u_1)\alpha(u_2) + s(s+1)\beta(u_1)\beta(u_2)), \\
&u_+(\beta(u_1) - \beta(u_2)) \\
&= u_-(\alpha(u_1)\beta(u_2) + \alpha(u_2)\beta_(u_1) - \beta(u_1)\beta(u_2)), \\
&u_+(\alpha(u_2)\beta(u_1) - \alpha(u_1)\beta(u_2)) \\
&= u_-(\alpha(u_1)\beta(u_2) + \alpha(u_2)\beta(u_1)) + u_-(u_+ - 1)\beta(u_1)\beta(u_2))
\end{aligned}$$

with $u_+ = u_1 + u_2$, $u_- = u_1 - u_2$. These equations may be solved yielding the solutions

$$\begin{aligned}
A_-(u) &= \frac{c_a(c_a+1) - u^2 + u - s_a(s_a+1) + 2u\mathbf{S}_a^z}{(c_a + u - s_a)(c_a + u + s_a + 1)}, \\
B_-(u) &= \frac{2u\mathbf{S}_a^-}{(c_a + u - s_a)(c_a + u + s_a + 1)}, \\
C_-(u) &= \frac{2u\mathbf{S}_a^+}{(c_a + u - s_a)(c_a + u + s_a + 1)}, \\
D_-(u) &= \frac{c_a(c_a+1) - u^2 + u - s_a(s_a+1) - 2u\mathbf{S}_a^z}{(c_a + u - s_a)(c_a + u + s_a + 1)}.
\end{aligned}$$

The matrix $K_+(u)$ can be obtained from the isomorphism of the superalgebras $\mathcal{T}_-$ and $\mathcal{T}_+$. Given a solution $K_-(u)$ of the first reflection equation, then $K_+(u)$ defined by

$$K_+^{st}(u) = K_-(-u + \frac{1}{2})$$

is a solution of the second reflection equation. (Here $\eta = 1$). Therefore, we choose the boundary matrix $K_+(u)$ as

$$K_+(u) = \begin{pmatrix} A_+(u) & B_+(u) & 0 \\ C_+(u) & D_+(u) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$A_+(u) = \frac{c_b^2 - u^2 - s_b(s_b+1) + (2u-1)\mathbf{S}_b^z}{(c_b + u + s_b)(c_b + u - s_b - 1)},$$

$$B_+(u) = \frac{(2u-1)\mathbf{S}_b^-}{(c_b + u + s_b)(c_b + u - s_b - 1)},$$

$$C_+(u) = \frac{(2u-1)\mathbf{S}_b^+}{(c_b + u + s_b)(c_b + u - s_b - 1)},$$

$$D_+(u) = \frac{c_b^2 - u^2 - s_b(s_b+1) - (2u-1)\mathbf{S}_b^z}{(c_b + u + s_b)(c_b + u - s_b - 1)}.$$

Taking the Hamiltonian to be negative the logarithmic derivative of the transfer matrix $\tau(u)$ evaluated at $u = 0$ gives

$$-H = \sum_{j=1}^{L-1} h_{j,j+1} + \frac{1}{2} \overset{1}{K'}_-(0) + \frac{str_0 K_+(0) H_{L0}}{str_0 K_+(0)},$$

with

$$h = \frac{d}{du} PR(u).$$

Algebraic Bethe ansatz solution

Introduce the 'doubled' monodromy matrix $U(u)$,

$$
\begin{aligned}
U(u) &= T(u)K_-(u)T^{-1}(u) \\
&= \begin{pmatrix} \mathcal{A}_{11}(u) & \mathcal{A}_{12}(u) & \mathcal{B}_1(u) \\ \mathcal{A}_{21}(u) & \mathcal{A}_{22}(u) & \mathcal{B}_2(u) \\ \mathcal{C}_1(u) & \mathcal{C}_2(u) & \mathcal{D}(u) \end{pmatrix}
\end{aligned}
$$

Letting $|\Psi\rangle$ denote the pseudovacuum we find

$$
\begin{aligned}
\mathcal{D}(u)|\Psi\rangle &= (-\frac{u+1}{u-1})^L|\Psi\rangle, \\
\mathcal{B}_d(u)|\Psi\rangle &= 0, \qquad \mathcal{C}_d(u)|\Psi\rangle \neq 0, \\
\mathcal{A}_{db}(u)|\Psi\rangle &= (-\frac{u^2}{u^2-1})^L K_-(u)_{db}|\Psi\rangle, \\
\mathcal{A}_{dd}(u)|\Psi\rangle &= (-\frac{u^2}{u^2-1})^L (K_-(u)_{dd} - \frac{1}{2u+1})|\Psi\rangle \\
&\quad + \frac{1}{2u+1}(-\frac{u+1}{u-1})^L|\Psi\rangle
\end{aligned}
$$

which motivates us to set

$$
\check{\mathcal{A}}_{bd}(u) = \mathcal{A}_{bd}(u) - \frac{1}{2u+1}\delta_{bd}\mathcal{D}(u).
$$

We choose the Bethe states $|\Omega\rangle$ to have the form

$$
|\Omega\rangle = \mathcal{C}_{i_1}(u_1)\cdots\mathcal{C}_{i_N}(u_N)|\Psi\rangle F^{i_1\cdots i_N}. \tag{1}
$$

¿From the reflection equation we find the relations

$$
\begin{aligned}
&\check{\mathcal{A}}_{bd}(u_1)\mathcal{C}_c(u_2)\\
&=\frac{(u_- - 1)(u_+)}{(u_-)(u_+ + 1)} r(u_+ + 1)^{eb}_{gh} r(u_-)^{ih}_{cd} \mathcal{C}_e(u_2)\check{\mathcal{A}}_{gi}(u_1)\\
&-\frac{4u_1u_2}{(u_+ + 1)(2u_1 + 1)(2u_2 + 1)} r(2u_1 + 1)^{gb}_{cd} \mathcal{C}_g(u_1)\mathcal{D}(u_2)\\
&+\frac{2u_1}{(u_-)(2u_1 + 1)} r(2u_1 + 1)^{gb}_{id} \mathcal{C}_g(u_1)\check{\mathcal{A}}_{ic}(u_2),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{D}(u_1)\mathcal{C}_b(u_2)\\
&=\frac{(u_- - 1)(u_+)}{(u_-)(u_+ + 1)} \mathcal{C}_b(u_2)\mathcal{D}(u_1) + \frac{2u_2}{(u_-)(2u_2 + 1)} \mathcal{C}_b(u_1)\mathcal{D}(u_2)\\
&-\frac{1}{u_+ + 1} \mathcal{C}_d(u_1)\check{\mathcal{A}}_{db}(u_2).
\end{aligned}
$$

The R-matrix $r(u)$ for the nesting has entries

$$
\begin{aligned}
r^{bb}_{bb}(u) &= 1, \qquad r^{bd}_{bd}(u) = -\frac{1}{u-1},\\
r^{bd}_{db}(u) &= \frac{u}{u-1}, (b \neq d, b, d = 1, 2)
\end{aligned}
$$

and the matrix

$$
K^{(1)}_{-}(u)_{bd} = K_{-}(u)_{bd} - \frac{\delta_{bd}}{2u+1}
$$

satisfies the reflection equation for $r(u)$ with appropriate change of variables. Thus the nesting problem is that of an inhomogeneous open XXX chain with arbitrary impurity spins on the boundaries.

With these considerations, we obtain the Bethe ansatz solution (after a shift of the parameters $u_j \to u_j - \frac{1}{2}$)

$$
\begin{aligned}
(\frac{u_j + \frac{1}{2}}{u_j - \frac{1}{2}})^{2L} \prod_{\alpha=a,b} \frac{u_j + c_\alpha + s_\alpha + \frac{1}{2}}{u_j - c_\alpha - s_\alpha - \frac{1}{2}} \\
&= \prod_{m=1}^{M} \frac{(u_j - v_m + \frac{1}{2})}{(u_j - v_m - \frac{1}{2})} \frac{(u_j + v_m + \frac{1}{2})}{(u_j + v_m - \frac{1}{2})}, \\
\prod_{k \neq m}^{M} \frac{(v_m - v_k - 1)}{(v_m - v_k + 1)} \frac{(v_m + v_k - 1)}{(v_m + v_k + 1)} \\
&= \prod_{\alpha=a,b} \frac{(v_m - c_\alpha - s_\alpha)}{(v_m - c_\alpha + s_\alpha)} \frac{(v_m + c_\alpha - s_\alpha)}{(v_m + c_\alpha + s_\alpha)} \\
&\quad \times \prod_{j=1}^{N} \frac{(v_m - u_j - \frac{1}{2})}{(v_m - u_j + \frac{1}{2})} \frac{(v_m + u_j - \frac{1}{2})}{(v_m + u_j + \frac{1}{2})}
\end{aligned}
\tag{2}
$$

with the corresponding energy eigenvalue E of the model

$$
E \propto \sum_{j=1}^{N} \frac{1}{u_j^2 - \frac{1}{4}}. \tag{3}
$$

Results for sl(3) case

Here we begin with the R-matrix

$$R(u)=\begin{pmatrix} -u-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -u & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -u-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u+1 \end{pmatrix}.$$

Proceeding as before, it can be shown that there exists the following solution of the reflection equation

$$K_-(u)=\begin{pmatrix} A_-(u) & B_-(u) & 0 \\ C_-(u) & D_-(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{aligned} A_-(u) &= \frac{c_a(c_a-1)-u^2-u-s_a(s_a+1)-2u\mathbf{S}_a^z}{(c_a-u+s_a)(c_a-u-s_a-1)}, \\ B_-(u) &= \frac{-2u\mathbf{S}_a^-}{(c_a-u+s_a)(c_a-u-s_a-1)}, \\ C_-(u) &= \frac{-2u\mathbf{S}_a^+}{(c_a-u+s_a)(c_a-u-s_a-1)}, \\ D_-(u) &= \frac{c_a(c_a-1)-u^2-u-s_a(s_a+1)+2u\mathbf{S}_a^z}{(c_a-u+s_a)(c_a-u-s_a-1)}. \end{aligned} \tag{4}$$

For this case $K_+(u)$ defined by

$$K_+^{st}(u) = K_-(-u - \frac{3}{2}) \tag{5}$$

is a solution of the second reflection equation (Here $\eta = -3$). Therefore, we may choose the boundary matrix $K_+(u)$ as

$$K_+(u) = \begin{pmatrix} A_+(u) & B_+(u) & 0 \\ C_+(u) & D_+(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6}$$

where

$$A_+(u) = -\frac{c_b^2 - u^2 - 2u - s_b(s_b+1) - 1 - (2u+3)\mathbf{S}_b^z}{(c_b - u + s_b - 1)(c_b - u - s_b - 2)},$$
$$B_+(u) = \frac{(2u+3)\mathbf{S}_b^-}{(c_b - u + s_b - 1)(c_b - u - s_b - 2)},$$
$$C_+(u) = \frac{(2u+3)\mathbf{S}_b^+}{(c_b - u + s_b - 1)(c_b - u - s_b - 2)},$$
$$D_+(u) = -\frac{c_b^2 - u^2 - 2u - s_b(s_b+1) - 1 + (2u+3)\mathbf{S}_b^z}{(c_b - u + s_b - 1)(c_b - u - s_b - 2)}.$$

With these solutions we obtain an integrable model with couplings

$$t = 1, \quad J = -2, \quad V = -\frac{3}{2}.$$
$$J_\alpha = \frac{2}{(c_\alpha + s)(c_\alpha - s - 1)},$$
$$V_\alpha = \frac{c_\alpha^2 - 1 - s(s+1)}{(c_\alpha + s)(c_\alpha - s - 1)}.$$

Again following the method of the algebraic Bethe ansatz we find that the energies of the model are of the form

$$E \propto \sum_{j=1}^{N} \frac{1}{u_j^2 - \frac{1}{4}}$$

for each solution of the Bethe asatz equations

$$(\frac{u_j + \frac{1}{2}}{u_j - \frac{1}{2}})^{2L} \prod_{\alpha=a,b} \frac{u_j - c_\alpha - s_\alpha - \frac{1}{2}}{u_j + c_\alpha + s_\alpha + \frac{1}{2}} \prod_{i \neq j}^{N} \frac{(u_j - u_i - 1)(u_j + u_i - 1)}{(u_j - u_i + 1)(u_j + u_i + 1)}$$
$$= \prod_{m=1}^{M} \frac{(u_j - v_m - \frac{1}{2})}{(u_j - v_m + \frac{1}{2})} \frac{(u_j + v_m - \frac{1}{2})}{(u_j + v_m + \frac{1}{2})},$$

$$\prod_{\alpha=a,b} \frac{(v_m - c_\alpha - s_\alpha)}{(v_m - c_\alpha + s_\alpha)} \frac{(v_m + c_\alpha - s_\alpha)}{(v_m + c_\alpha + s_\alpha)}$$
$$\times \prod_{j=1}^{N} \frac{(v_m - u_j - \frac{1}{2})}{(v_m - u_j + \frac{1}{2})} \frac{(v_m + u_j - \frac{1}{2})}{(v_m + u_j + \frac{1}{2})}$$
$$= \prod_{k \neq m}^{M} \frac{(v_m - v_k - 1)}{(v_m - v_k + 1)} \frac{(v_m + v_k - 1)}{(v_m + v_k + 1)}.$$

Concluding Remarks

1. This procedure can be applied also to four state models giving rise to extended Hubbard models with integrable Kondo boundary impurities (work in progress).

2. Frahm and Slavnov have introduced a projection method to obtain non-regular boundary K-matrices from regular solutions. Their procedure is applicable to reproduce the results presented here and shows a representation theoretic approach to extend the method to more general classes of integrable systems. This projection method gives K-matrices which have a symmetry which is a subalgebra of the symmetry of the bulk model.