Quantum Projective Superspaces and

Tensor Representations of $\mathbf{SL}_q(m|n)$

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To be discussed:
— Quantum supergroup $SL_q(m|n)$.
— Tensor representations.
— Quantum projective superspaces.
— Realization of representations.

Based on
Commun Math Phys '98;
Rev Math Phys '98 (with A R Gover);
J Math Phys, to appear (with H C Lee).

$\mathbf{U}_q(sl(m|n))$ and $\mathbf{SL}_q(m|n)$

A $\mathbb{Z}_2$-graded Hopf algebra generated by

$$e_a,\ f_a,\ k_a^{\pm 1}, \quad a \in \mathbf{I}',$$
$$\mathbf{I}' = \{1, 2, .., m+n-1\}.$$

The e_m, f_m are odd, the rest even.

$$\underbrace{\bigcirc - - \cdots - - \bigcirc}_{m-1} - - \otimes - - \underbrace{\bigcirc - - \cdots - - \bigcirc}_{n-1}$$

$$\begin{array}{cc} e_i,\ f_i,\ k_i^{\pm 1}, & e_\mu,\ f_\mu,\ k_\mu^{\pm 1}, \\ 1 \le i < m, & m < \mu < m+n, \\ \text{generate } U_q(gl(m)); & \text{generate } U_q(gl(n)). \end{array}$$

$e_m, f_m, k_m^{\pm 1}$ generate $\mathsf{U}_q(sl(1|1))$:

$$\begin{aligned} (e_m)^2 = (f_m)^2 &= 0, \\ e_m f_m + f_m e_m &= \frac{k_m - k_m^{-1}}{q - q^{-1}}, \\ k_m e_m k_m^{-1} &= e_m, \\ k_m f_m k_m^{-1} &= f_m. \end{aligned}$$

Three pieces are connected by relations such as

$$\begin{aligned} (e_{m\pm 1})^2 e_m &- (q + q^{-1}) e_{m\pm 1} e_m e_{m\pm 1} \\ &+ e_m (e_{m\pm 1})^2 = 0, \\ e_m E &+ E e_m = 0, \quad etc., \\ \text{where} \quad E &= (e_{m-1} e_m - q^{-1} e_m e_{m-1}) e_{m+1} \\ & \quad - q e_{m+1} (e_{m-1} e_m - q^{-1} e_m e_{m-1}). \end{aligned}$$

Assume $0 < q \in \mathbb{R}$.

Remark: Finite dim'l reps are **not** completely reducible even at generic q.

Exist two classes of 'good' reps.
Vector rep π:

$$\begin{aligned} \pi(e_a) &= e_{aa+1}, \\ \pi(f_a) &= e_{aa-1}, \\ \pi(k_m) &= 1 + (q-1)(e_{mm} + e_{m+1,m+1}), \\ \pi(k_b) &= 1 + (q-1)e_{bb} + (q^{-1}-1)e_{b+1,b+1}, \\ & \quad b \neq m. \end{aligned}$$

$\pi^{\otimes k}$, $\forall k$, decompose into direct sums of contravariant tensor irreps.

$\bar{\pi}$ is the dual irrep of π. $\bar{\pi}^{\otimes k}$, $\forall k$, decompose into direct sums of covariant tensor irreps.

RemarkS: $\pi^{\otimes k}$ and $\bar{\pi}^{\otimes k}$ do not contain 1-dim'l irrep if $k > 0$.

$SL_q(m|n)$ $[\mathfrak{g} = sl(m|n)]$

The dual $(\mathsf{U}_q(\mathfrak{g}))^*$ of $\mathsf{U}_q(\mathfrak{g})$ is a superalgebra.
Matrix elements t_{ab} of π:

$$\begin{aligned} & t_{ab} \in (\mathsf{U}_q(\mathfrak{g}))^*, \qquad a, b \in \mathbf{I}, \qquad \mathbf{I} = \mathbf{I}' \cup \{m+n\}, \\ & t_{ab}(x) = \pi(x)_{ab}, \qquad \forall x \in \mathsf{U}_q(\mathfrak{g}), \end{aligned}$$

— satisfy RTT type relations;
— generate subalgebra T of $(\mathsf{U}_q(\mathfrak{g}))^*$.

T forms a bi-superalgebra, but admits **no** antipode. The matrix elements of the contravariant tensor irreps form a basis of T [similar to Peter-Weyl bases for compact Lie groups].

Matrix elements $\bar{t}_{ab}$ generate bi-superalgebra $\bar{T}$.

Definition:
$\mathcal{T}(\mathfrak{g})$ = subalgebra of $(\mathsf{U}_q(\mathfrak{g}))^0$ generated by $\{t_{ab},\ \bar{t}_{ab}\}$.

Theorem 1 *$\mathcal{T}(\mathfrak{g})$ forms a *-Hopf superalgebra, and separates points of $U_q(\mathfrak{g})$.*

[Antipode maps T to $\bar{T}$.]

Will regard $\mathcal{T}(\mathfrak{g})$ as the superalgebra determining $SL_q(m|n)$.

[In the classical case, $\mathcal{T}(\mathfrak{g})$ captures the essential information about the sheaf of graded algebras on a Lie group in Kostant's theory.]

Super q-determinant:

$$\begin{aligned} A &= (t_{ij})_{i,j=1}^{m}, \\ \bar{D} &= (\bar{t}_{\mu\nu})_{\mu,\nu=m+1}^{m+n}; \\ Sdet_q &= Det_q A \ Det_q \bar{D}, \\ \Delta(Sdet_q) &= Sdet_q \otimes Sdet_q. \end{aligned}$$

On $SL_q(m|n)$, $\quad Sdet_q = \epsilon$ (identity of $\mathcal{T}(\mathfrak{g})$).

Quantum projective superspaces

Notations: $\quad \mathbf{\Theta} \subset \mathbf{I}'$.
$\mathrm{U}_q(\mathfrak{k})$: generated by $k_a^{\pm 1}, \ e_c, \ f_c, \ c \in \mathbf{\Theta}, \ a \in \mathbf{I}'$,
$\mathrm{U}_q(\mathfrak{p})$: generated by $k_a^{\pm 1}, \ f_a, \ e_c, \ c \in \mathbf{\Theta}, \ a \in \mathbf{I}'$.
$\mathrm{U}_q(\mathfrak{k}) \subset \mathrm{U}_q(\mathfrak{p}) \subset \mathrm{U}_q(\mathfrak{g})$ as Hopf algebras.

Left and right translations: $\quad x \in \mathrm{U}_q(\mathfrak{g}), \ f \in \mathcal{T}(\mathfrak{g})$,

$$\begin{aligned} R_x(f) &= \sum_{(f)} (-1)^{[x]([f]+[x])} \ f_{(1)} \langle f_{(2)}, \ x \rangle \\ L_x(f) &= \sum_{(f)} \langle f_{(1)}, \ S^{-1}(x) \rangle f_{(2)}. \end{aligned}$$

Graded-commute, $R_x L_y = (-1)^{[x][y]} L_y R_x$.

• Quantum homogeneous superspace

$R_{\mathsf{U}_q(\mathfrak{k})}$-invariant functions on $SL_q(m|n)$

$$\mathcal{A}^{\mathfrak{k}} = \{f \in \mathcal{T}(\mathfrak{g}) | R_y(f) = \epsilon(y) f, \ \forall y \in \mathsf{U}_q(\mathfrak{k})\}.$$

$\mathcal{A}^{\mathfrak{k}}$ will be regarded as the superalgebra of functions on a 'quantum homogeneous superspace'.

• Quantum projective spaces

$\mathbf{\Theta} = \mathbf{I}' \backslash \{m+n-1\}, \quad \mathsf{U}_q(\mathfrak{k}) = \mathsf{U}_q(sl(m|n-1) \oplus gl(1)).$

The associated quantum homogeneous superspace is the quantum projective superspace

$$\mathcal{A}^{\mathfrak{k}} = \mathbb{C}P_q^{m|n-1}.$$

• Description of $\mathbb{C}P_q^{m|n-1}$:

$$\begin{array}{ll} \text{Set} & z_a = t_{a\ m+n}, \quad \bar{z}_a = \bar{t}_{a\ m+n}, \quad a \in \mathbf{I}. \\ \text{Then} & *(z_a) = \bar{z}_a. \end{array}$$

They satisfy relations

$$\begin{aligned} (z_c)^2 &= (\bar{z}_c)^2 = 0, \quad c \leq m; \\ z_a\, z_b &= (-1)^{[z_a][z_b]}\, q\, z_b\, z_a, \\ \bar{z}_a \bar{z}_b &= (-1)^{[\bar{z}_a][\bar{z}_b]} q^{-1}\, \bar{z}_b \bar{z}_a, \quad a < b, \\ \bar{z}_a\, z_b &= q(-1)^{[\bar{z}_a][z_b]} z_b\, \bar{z}_a + \delta_{a\,b} \Big\{ (1 - q_a^{-1}) \bar{z}_a\, z_a \\ & \quad - (-1)^{[\bar{z}_a]} (q - q^{-1}) \sum_{c<a} \bar{z}_c\, z_c \Big\}, \quad \forall a, b \in \mathbf{I}, \\ \sum_{c \in \mathbf{I}} \bar{z}_c\, z_c &= 1. \end{aligned}$$

• These define the quantum supersphere $\mathbb{S}_q^{m|n-1}$.

Notations

$$\begin{array}{rcl}
L &=& (\theta_1, ..., \theta_m; l_1, ..., l_n) \in \{0,\ 1\}^{\otimes m} \otimes \mathbb{Z}_+^{\otimes n}, \\
|L| &=& \sum_{i=1}^{m} \theta_i + \sum_{\mu=1}^{n} l_\mu; \\
Z^L &=& z_1^{\theta_1} ... z_m^{\theta_m} z_{m+1}^{l_1} ... z_{m+n}^{l_n}, \\
\overline{Z}^L &=& \bar{z}_1^{\theta_1} ... \bar{z}_m^{\theta_m} \bar{z}_{m+1}^{l_1} ... \bar{z}_{m+n}^{l_n}.
\end{array}$$

Introduce a $U_q(gl(1)) = \langle K^{\pm 1} \rangle$ action on $\mathbb{S}_q^{m|n-1}$:

$$(K^p) \circ (Z^L \overline{Z}^{L'}) \;=\; q^{p(|L'|-|L|)} Z^L \overline{Z}^{L'}.$$

Quantum projective superspace $\mathbb{C}P_q^{m|n-1}$ is the

$U_q(gl(1))$ invariant subalgebra of $\mathbb{S}_q^{m|n-1}$

$$\mathbb{C}P_q^{m|n-1} \;=\; \left(\mathbb{S}_q^{m|n-1}\right)^{U_q(gl(1))}.$$

Induced representations

V is any finite dim'l $\mathrm{U}_q(\mathfrak{k})$-module. Define

$$\Gamma^{\mathfrak{k}}(V) = \left\{ \zeta \in V \otimes_{\mathbb{C}} \mathcal{T}(\mathfrak{g}) \quad \begin{array}{|l} (id \otimes R_y)\zeta = (S(y) \otimes id)\zeta, \\ \forall\ y \in U_q(\mathfrak{k}). \end{array} \right\}$$

Proposition 1 $\Gamma^{\mathfrak{k}}(V)$ *forms a left* $U_q(\mathfrak{g})$*-module under the action* $id \otimes L$.

[Follows from commutativity of R and L.]

Proposition 2 $\Gamma^{\mathfrak{k}}(V)$ *forms a 2-sided* $\mathcal{A}^{\mathfrak{k}}$*-module under the multiplication of* $\mathcal{T}(\mathfrak{g})$.

$\Gamma^{\mathfrak{k}}(V)$ is the space of sections of a 'supervector bundle' over a quantum homogeneous superspace.

V is a finite dim'l irreducible $\mathrm{U}_q(\mathfrak{p})$-module with the $\mathrm{U}_q(\mathfrak{k})$ highest weight (h.w.) μ.

$$\begin{aligned}
\mathcal{O}^V &= \left\{ \zeta \in \Gamma^{\mathfrak{k}}(V) \;\middle|\; \begin{array}{l} (id \otimes R_y)\zeta = (S(y) \otimes id)\zeta \\ \forall\ y \in \mathrm{U}_q(\mathfrak{p}) \end{array} \right\}, \\
\overline{\mathcal{O}}(\mu) &= \mathcal{O}^V \cap (V \otimes T), \\
\mathcal{O}(\mu) &= \mathcal{O}^V \cap \left(V \otimes \bar{T}\right).
\end{aligned}$$

Theorem 2 *As $U_q(\mathfrak{g})$-modules,*

$$\begin{aligned}
\mathcal{O}(\mu) &\cong \begin{cases} W(\mu), & \text{if } \mu \in \Lambda^{(1)}, \\ \{0\}, & \text{otherwise.} \end{cases} \\
\overline{\mathcal{O}}(\mu) &\cong \begin{cases} W(\mu), & \text{if } \mu \in \Lambda^{(2)}, \\ \{0\}, & \text{otherwise.} \end{cases}
\end{aligned}$$

$W(\mu)$: irreducible $\mathrm{U}_q(\mathfrak{g})$-module with h.w. μ.
$\Lambda^{(1)}$: the set of h.w.'s of the contravariant tensor irreps,
$\Lambda^{(2)}$: the set of h.w.'s of the covariant tensor irreps.

• Explicit realization of a class of irreps

on quantum projective superspace.

$\mathbf{\Theta} = \mathbf{I}' \backslash \{m + n - 1\}$.

$\mathrm{U}_q(\mathfrak{p})$-module

$$V = \mathbb{C}v$$

with

$$\begin{aligned} e_b v &= f_c v = 0, \\ k_b v &= v, \quad b, c \in \mathbf{I}', \ b < m+n-1, \\ k_{m+n-1} v &= q^k v, \quad k \in \mathbb{Z}_+. \end{aligned}$$

Then

$$\begin{aligned} \overline{\mathcal{O}}_k &:= \mathcal{O}^V \cap (V \otimes T) \\ &= \left\{ \zeta \in V \otimes T \quad \begin{array}{|l} R_p(\zeta) = (S(p) \otimes id)\zeta, \\ \forall \ p \in \mathsf{U}_q(\mathfrak{p}) \end{array} \right\} \\ &= \bigoplus_{|L|=k} \mathbb{C}v \otimes Z^L. \end{aligned}$$

The $\overline{\mathcal{O}}_k$ furnishes the rank-k symmetric covariant tensor irrep with the highest weight

$$\lambda = -k\epsilon_{m+n}.$$