Induced Representations of Quantum Groups and Non-commutative Geometry

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1. Generalities

Why geometry?

A classical Lie group G is a manifold with multiplication and inverse.

K is a closed subgroup, then G/K is a homogeneous space.

Study of Lie groups requires understanding their geometry.

The same is true for quantum groups.

What is a quantum space?

Non-commutative geometry goes through a back door.

It is known that the (commutative) algebra of functions on a manifold determines the manifold itself.

This is taken as the starting point of non-commutative geometry.

A non-commutative space is defined by its non-commutative algebra $\mathcal{A}$ of functions.

A vector bundle over the non-commutative space is defined by a *projective* $\mathcal{A}$-module Γ, which is regarded as the space of sections. (Projectivity will be explained later.)

To be discussed

quantum groups
quantum homogeneous spaces
quantum homogeneous vector bundles
induced representations
irreducible reps and Borel-Weil.

2. Quantum Groups

<u>$\mathrm{SU}_q(2)$ as an example:</u>

Quantized enveloping algebra $U_q(sl_2)$

generated by $\{e, f, k, k^{-1}\}$,

with relations

$$
\begin{aligned}
[e, f] &= \frac{k^2 - k^{-2}}{q - q^{-1}}, \\
kek^{-1} &= qe, \\
kfk^{-1} &= q^{-1}f, \quad 0 < q \in \mathbb{R}.
\end{aligned}
$$

$U_q(sl_2)$ is a Hopf algebra with

co-multiplication

$$
\begin{aligned}
\Delta(e) &= e \otimes k + k^{-1} \otimes e, \\
\Delta(f) &= f \otimes k + k^{-1} \otimes f, \\
\Delta(k) &= k \otimes k,
\end{aligned}
$$

co-unit

$$\epsilon(e) = \epsilon(f) = \epsilon(k^{\pm 1} - 1) = 0,$$

antipode

$$S(e) = qe, \qquad S(f) = q^{-1}f,$$
$$S(k) = k^{-1}.$$

Δ is not left-right symmetric, e.g.,

$$e \otimes k + k^{-1} \otimes e \neq k \otimes e + e \otimes k^{-1}.$$

A notation: $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$.

The irreducible reps

$$t^{(l)}, \;\; l \in \mathbb{Z}_+,$$

$$\mathrm{dim} t^{(l)} = l + 1.$$

For $x \in U_q(sl_2)$,

$$t^{(l)}(x) = \begin{pmatrix} t^{(l)}_{11}(x) & t^{(l)}_{12}(x) & \ldots & t^{(l)}_{1l+1}(x) \\ t^{(l)}_{21}(x) & t^{(l)}_{22}(x) & \ldots & t^{(l)}_{2l+1}(x) \\ \ldots & \ldots & \ldots & \ldots \\ t^{(l)}_{l+1,1}(x) & t^{(l)}_{l+1,2}(x) & \ldots & t^{(l)}_{l+1,l+1}(x) \end{pmatrix}.$$

Matrix elements $t^{(l)}_{ij} \in (U_q(sl_2))^*$.

Set

$$T_q(sl_2) = \bigoplus_{l \geq 0} \left(\oplus_{i,j=1}^{l+1} \mathbb{C} t^{(l)}_{ij} \right).$$

Theorem: $T_q(sl_2)$ forms a Hopf algebra.

Multiplication is defined by

$$\langle t^{(k)}_{pq} t^{(l)}_{ij},\ x \rangle = \langle t^{(k)}_{pq} \otimes t^{(l)}_{ij},\ \Delta(x) \rangle.$$

Tensor product of irreps:

$$t^{(k)} \otimes t^{(l)} = \bigoplus_{m=|k-l|}^{k+l} t^{(m)}.$$

This implies

$$t^{(k)}_{pq} t^{(l)}_{ij} \in \bigoplus_{m=|k-l|}^{k+l} \left(\oplus_{r,s=1}^{m+1} \mathbb{C} t^{(m)}_{rs} \right).$$

i.e., $T_q(sl_2)$ is closed under multiplication.

$T_q(sl_2)$ (a completion of it) will be taken as the algebra of functions on $\mathrm{SU}_q(2)$.

It is **non-commutative** as Δ is not left-right symmetric.

<u>The general case:</u>

$U_q(\mathfrak{g})$ is the quantum algebra. Set

$T_q(\mathfrak{g})$ = space spanned by the matrix elements of finite dim'l $U_q(\mathfrak{g})$-reps.

Theorem:

1). $T_q(\mathfrak{g})$ is a $*$-Hopf algebra.

2). The matrix elements $\{t_{ab}^{(\lambda)}\}$ of the irreps form a basis of $T_q(\mathfrak{g})$.

3). The $\{t_{ab}^{(\lambda)}\}$ are orthonormal with respect to the quantum Haar functional

$\int : T_q(\mathfrak{g}) \to \mathbb{C}$.

($\int \mathbf{1} = 1$, $\int t_{ab}^{(\lambda)} = 0$, for nontrivial irrep $t^{(\lambda)}$.

Prof. Scheunert will discuss quantum groups extensively.)

3. Some Notations

<u>Reductive Hopf subalgebras</u>

- Generators of $U_q(\mathfrak{g})$ are $\cup_i\{e_i, f_i, k_i^{\pm 1}\}$.

  E.g., $U_q(sl_3)$ is generated by

$$\{e_1, f_1, k_1^{\pm 1}\} \cup \{e_2, f_2, k_2^{\pm 1}\}$$

$$\begin{aligned} k_1 k_2 &= k_2 k_1, \\ k_i e_j k_i^{-1} &= q^{a_{ij}/2} e_j, \\ k_i f_j k_i^{-1} &= q^{a_{ij}/2} f_j, \\ [e_i, f_j] &= \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q - q^{-1}}, \\ &\text{plus Serre relations.} \end{aligned}$$

- All the $k_i^{\pm 1}$ and some pairs of e_i, f_i generate a reductive Hopf subalgebra $U_q(\mathfrak{k})$.

  E.g., $U_q(sl_2 \oplus gl(1)) \subset U_q(sl_3)$ by

$$\{e_1, f_1, k_1^{\pm 1}, k_2^{\pm 1}\}.$$

Reps of $U_q(\mathfrak{k})$ are completely reducible.

Right 'translation':

$$\begin{aligned} R : U_q(\mathfrak{g}) \otimes T_q(\mathfrak{g}) &\to T_q(\mathfrak{g}), \\ x \otimes f &\mapsto R_x(f), \\ R_x(f)(y) &= f(yx), \\ &\forall \ y \in U_q(\mathfrak{g}). \end{aligned}$$

R defines a left $U_q(\mathfrak{g})$-action on $T_q(\mathfrak{g})$:

$$R_x R_y = R_{xy}.$$

Also, $R_x(fh) = \sum R_{x_{(1)}}(f) R_{x_{(2)}}(h)$.

Left 'translation':

$$\begin{aligned} L : U_q(\mathfrak{g}) &\otimes T_q(\mathfrak{g}) \to T_q(\mathfrak{g}), \\ x \otimes f &\mapsto L_x(f), \\ L_x(f)(y) &= f(S^{-1}(x)y), \\ &\forall \ y \in U_q(\mathfrak{g}). \end{aligned}$$

L defines a left $U_q(\mathfrak{g})$-action on $T_q(\mathfrak{g})$:

$$L_x L_y = L_{xy}.$$

Also, $L_x R_y = R_y L_x$.

4. Induced Representations

Right K-translation invariant $f : G \to \mathbb{C}$,

$$f(gk) = f(g), \quad \forall g \in G,\ k \in K,$$

leads to $\tilde{f} : G/K \to \mathbb{C}$

$$\tilde{f}([g]) = f(g).$$

Functions on G/K are the right K-translation invariant functions on G.

<u>Quantum homogeneous spaces:</u>

$R_{U_q(\mathfrak{k})}$-invariant subspace of $T_q(\mathfrak{g})$

$$\mathcal{A} = \left\{ f \in T_q(\mathfrak{g}) | R_u(f) = \epsilon(u) f, \forall u \in U_q(\mathfrak{k}) \right\}.$$

$\mathcal{A}$ is an algebra

$$R_u(fh) = \sum R_{u_{(1)}}(f) R_{u_{(2)}}(h) = \epsilon(u) fh,$$
$$\forall \ u \in U_q(\mathfrak{k}),\ f, h \in T_q(\mathfrak{g}).$$

$\mathcal{A}$ is regarded as the algebra of functions on a quantum homogeneous space.

Induced modules:

V is a finite dim'l $U_q(\mathfrak{k})$-module.

$R_{U_q(\mathfrak{k})}$-covariant subspace of $V \otimes T_q(\mathfrak{g})$

$$\Gamma(V) = \left\{ \begin{array}{c|l} & (id \otimes R_u)\zeta \\ \zeta \in V \otimes T_q(\mathfrak{g}) & = (S(u) \otimes id)\zeta, \\ & \forall u \in U_q(\mathfrak{k}) \end{array} \right\}.$$

Explicitly, for

$$u \in U_q(\mathfrak{k}), \qquad \zeta = \sum v_r \otimes f_r \in \Gamma(V),$$
$$\begin{aligned}(id \otimes R_u)\zeta &= \sum v_r \otimes R_u(f_r) \\ &= \sum S(u)v_r \otimes f_r.\end{aligned}$$

Lemma: $\Gamma(V)$ forms a $U_q(\mathfrak{g})$-module

$$U_q(\mathfrak{g}) \otimes \Gamma(V) \to \Gamma(V),$$
$$x \otimes \zeta \mapsto (id \otimes L_x)\zeta.$$

$(id \otimes L_x)\zeta \in \Gamma(V)$ as L_x and R_y commute.

$\Gamma(V)$ is infinite dim'l, and so is $\mathcal{A}$.

Problem: Obtain irreducibles from $\Gamma(V)$.

<u>Geometry of Induced Modules</u>

i). $\Gamma(V)$ forms a 2-sided $\mathcal{A}$-module.

$$\begin{aligned} a &\in \mathcal{A}, \\ \zeta &= \sum v_r \otimes f_r \in \Gamma(V), \\ a\zeta &= \sum v_r \otimes af_r, \\ \zeta a &= \sum v_r \otimes f_r a. \end{aligned}$$

$$\begin{aligned} u &\in U_q(\mathfrak{k}), \\ (id \otimes R_u)(a\zeta) &= \sum v_r \otimes R_u(af_r) \\ &= \sum v_r \otimes aR_u(f_r) \\ &= (S(u) \otimes id)a\zeta. \end{aligned}$$

Similarly for the right module structure.

ii). Always exists $U_q(\mathfrak{k})$-module $V^{\perp}$ such that $W = V \oplus V^{\perp}$ forms a $U_q(\mathfrak{g})$-module.

- Example: $U_q(sl_2 \oplus gl(1)) \subset U_q(sl_3)$

$U_q(sl_2 \oplus gl(1))$-module V

$$V = \mathbb{C}v_1 \oplus \mathbb{C}v_2$$

being the standard vector module over $U_q(sl_2)$ with the k_2-action

$$k_2 v_1 = v_1,\ k_2 v_2 = q^{1/2} v_2.$$

$V^\perp = \mathbb{C}v_3$ being a trivial $U_q(sl_2)$-module with the k_2-action

$$k_2 v_3 = q^{-1/2} v_3.$$

Then $V \oplus V^\perp$ extends to the standard vector module over $U_q(sl_3)$.

iii). For a $U_q(\mathfrak{g})$-module W (restricts naturally to a $U_q(\mathfrak{k})$-module), $\Gamma(W)$ is free either as a left or right $\mathcal{A}$-module (This requires considerable proof.):

$$\Gamma(W) \cong W \otimes_{\mathbb{C}} \mathcal{A}.$$

iv). $\Gamma(V) \oplus \Gamma(V^\perp) \cong (V \oplus V^\perp) \otimes_{\mathbb{C}} \mathcal{A}.$

That is, $\Gamma(V)$ is projective either as a left or right $\mathcal{A}$-module.

$\Gamma(V)$ can be regarded as the space of sections of some noncommutative vector bundle over the quantum homogeneous space. This bundle is the quantum analogue of a homogeneous vector bundle.

(Aspects of differential geometry of the bundles have been developed.)

5. Irreducible Representations

V_μ is a finite dim'l *irreducible* $U_q(\mathfrak{k})$-module with highest weight μ.

'Holomorphic' sections of $\Gamma(V_\mu)$:

$$\mathcal{O}(\mu) = \left\{ \begin{array}{ll} \zeta \in \Gamma(V_\mu) & \mid (id \otimes R_{f_i})\zeta = 0, \\ & \mid \forall f_i \notin U_q(\mathfrak{k}) \end{array} \right\}.$$

Theorem (quantum Borel-Weil):

1). If μ is integral dominant, $\mathcal{O}(\mu)$ forms an irreducible $U_q(\mathfrak{g})$-module with highest weight μ;

2). otherwise, $\mathcal{O}(\mu) = 0$.

5. Work in progress

Relates cohomology groups of the bundles to $\mathrm{U}_q(\mathfrak{g})$-modules, developing a Borel-Weil-Bott theory for quantum groups....