Introduction to Self-Organized Criticality

Andreas Schadschneider

Institut für Theoretische Physik
Universität zu Köln
Germany

Overview

- What is self-organized criticality ?
- The basic models:
  - Sandpiles
  - Bak-Sneppen model (Evolution)
  - Forest fires
  - Earthquakes
  - Traffic flow
- What is self-organized criticality ?

What is self-organized criticality ?

What kind of systems are we dealing with ?

spatially extended dynamical systems
with spatial and temporal degrees of freedom
often driven by external force + dissipation
(non-equilibrium)

One observes omnipresence of power-laws

spatial power-laws: fractals
(insert your favourite example here, e.g. land-scapes)

temporal power-laws: $1/f$-noise (flicker noise)
e.g. light from quasars, intensity of sunspots, current through resistors, sand flowing through hourglass, flow of rivers (Nile), stock market
(for more information, see
`http://linkage.rockefeller.edu/wli/1fnoise/`)

Power spectrum

N(t) is a time-dependent signal (e.g. sunspot activity, level of rivers)

Autocorrelation function:

$$G(t) = \langle N(t_0)N(t_0+t)\rangle_{t_0} - \langle N(t_0)\rangle_{t_0}^2$$

Power spectrum:

$$\begin{aligned} S(f) &= \lim_{T\to\infty} \frac{1}{2T}\Big|\int_{-T}^{T} N(t)e^{2\pi i f t}dt\Big|^2 \\ &= 2\int_0^\infty G(t)\cos(2\pi f t)dt \end{aligned}$$

Why is $1/f$ special ??

For $S(f) \propto 1/f$ and $G(\tau) \propto 1/\tau^\alpha$ one finds $\alpha = 0$, i.e. $G(\tau)$ decays logarithmically
$1/f$ power spectra correspond to extremely long temporal correlations

Self-organized criticality (SOC)

dynamical explanation for the existance of so many systems with complex spatial and temporal structure

complex means here: Behaviour is not determined by one time or length scale, no characteristic event size exists

No clear-cut or generally excepted definition of SOC exists !!

Self-organization: the ability of certain non-equilibrium systems to develop patterns and structures without external control or manipulation

criticality: has a precise meaning in equilibrium physics where a general theory exists (partition function, renormalization group etc)
for nonequilibrium we do not have such a general formalism or understanding

What is SOC?

Dynamical systems drive themselves into states characterized by algebraic correlations (unlike equilibrium systems where tuning of parameters is necessary), i.e. the system evolves to the critical state without detailed specification of the initial conditions

(critical state is an attractor of the dynamics)

Typical properties of SOC-systems

Separation of timescales:
external driving much slower than internal relaxation processes
(e.g. earthquakes)

Existence of thresholds and metastable states: system reacts only after certain thresholds have been reached

marginally stable states are special metastable states: small changes in driving force can lead to reactions on all length scales

Example I: Earthquakes

Gutenberg-Richter law:
Number of earthquakes, $N(E)$, for which an energy in excess of E is released

$$N(E) \simeq E^{-B}$$

with $1.8 < B < 2.2$ (depending slightly on region)

Omori law:
Number of aftershocks, $n(t)$, after big earthquakes

$$n(t) \simeq t^{-A}$$

with $1.0 < A < 1.5$

Example II: Zipf's law

Zipf's law:

frequency (size) f versus rank r

$$f(r) = \frac{a}{r^\gamma}$$

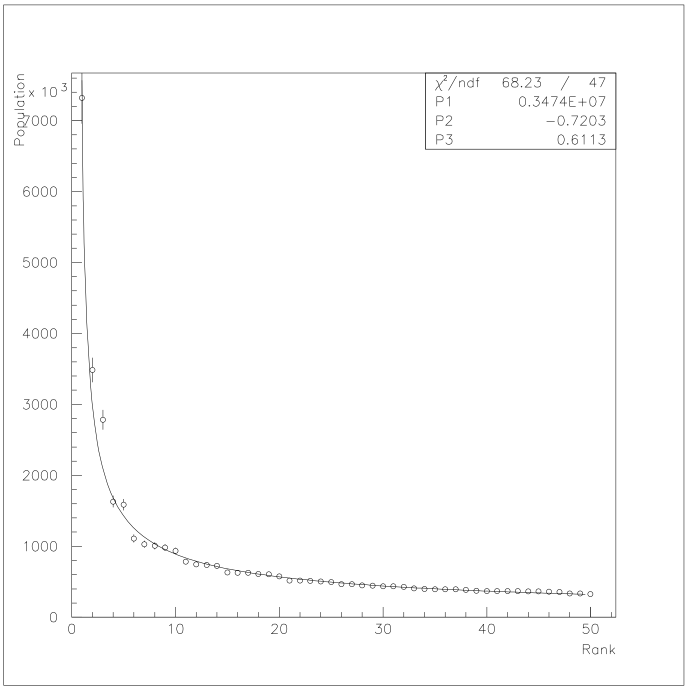
Mandelbrot's generalization:

$$f(r) = \frac{p_1}{(p_2 + r)^{p}_{3}}$$

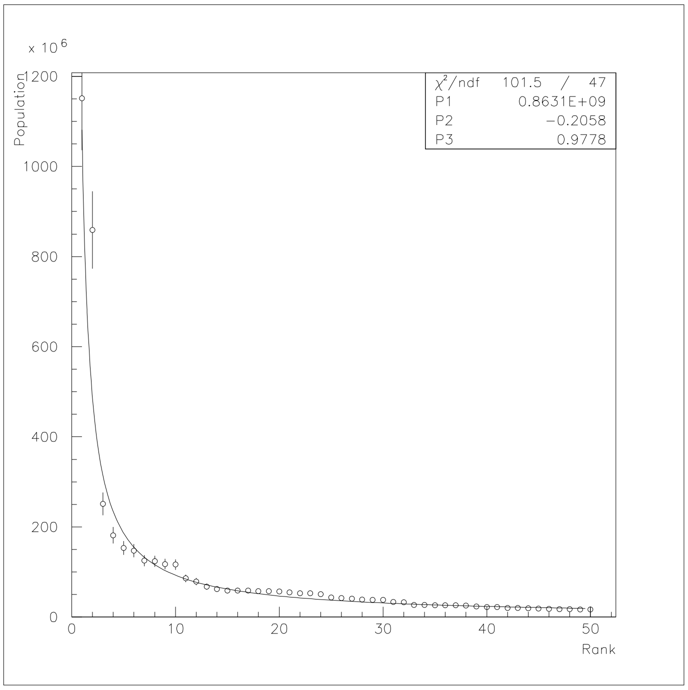
Examples:

city sizes in many countries
word frequency in the bible: $a \approx 0.1$ and $\gamma \approx 1$
citations of scientific papers

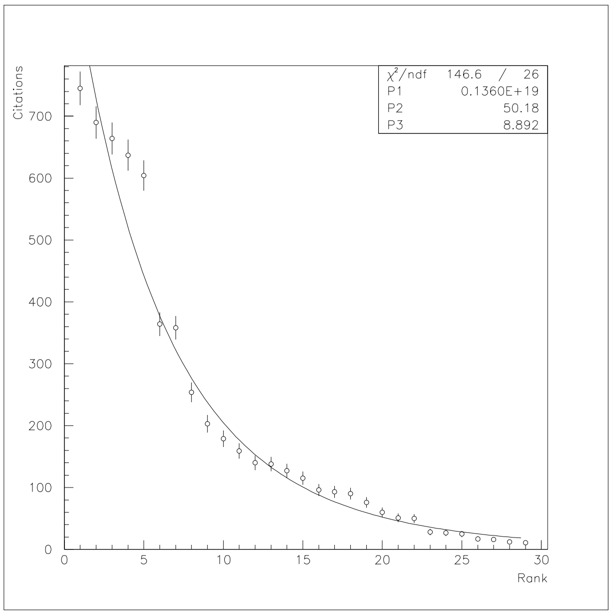
50 largest cities in the USA:



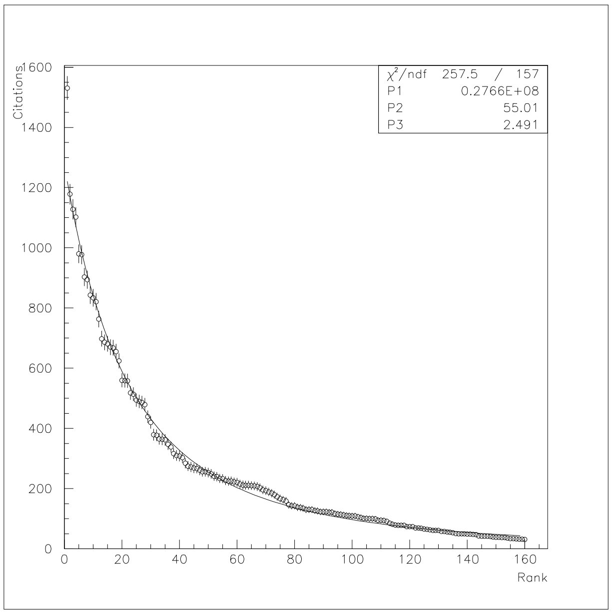
Most populated countries:



(Recent) citations of M. Gell-Mann:



Citations of E. Witten:



Example III: Real sandpiles

Experiment 1: sandpile on round plate; grains are added at a low rate (e.g. 1 grain per second); flow over boundary of the plate is measured using a scale

small piles: distribution of avalanche sizes $P(s) \propto s^{-2.2}$ (for $3 < s < 80$)

large piles: no indiction of power laws; large avalanches occur in regular intervals; angle of sandpile, θ, oscillates between angle of repose, θ_r and maximal angle θ_m

very large piles: behaviour is determined by two different mechanisms
a) large avalanches in regular intervals with oscillatory behaviour
b) small avalanches with $P(s) \propto s^{-\alpha}$ for $2 < s < 20$

Experiment 2: Sand in rotating drum

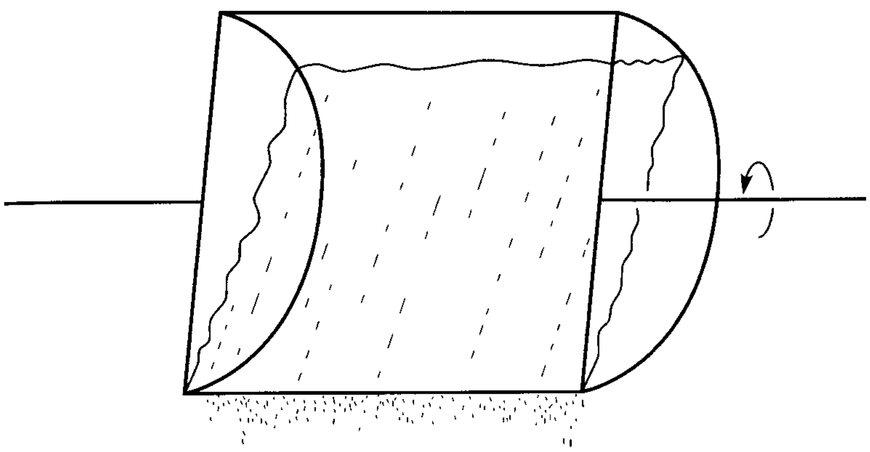


Figure 3.1. The rotating drum experiment

all surface avalanches can be observed
Results similar to experiment 1

Ricepiles

Real sandpiles are dominated by inertia effects; avalanches above a certain size can not be stopped and spread through the whole system

Rice has a smaller density and inertia effects are less important

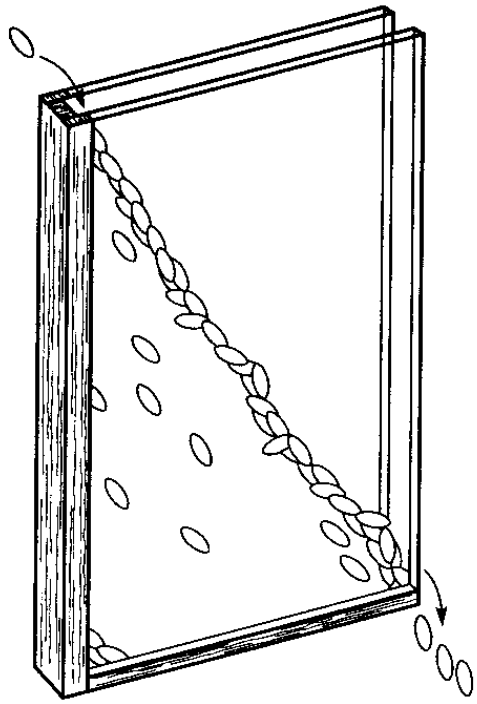


Figure 3.2. The ricepile experiment.

Type A: unpolished, ratio length l to width b:
$l/b = 3.8$

Type B: polished, $l/b = 2.0$

Type C: polished, $l/b = 2.6$

Results for type B:

$P(E) \propto \exp\left[-(E/E_0)^{\gamma}\right]$ (with $\gamma \approx 0.45$)

behaviour similar to sand

Results for types A and C:

$P(E) \propto s^{-2.04}$

(also for larger systems)

behaviour dominated by friction, not inertia

One-dimensional sandpiles

h_j = number of sand grains over position j (height)

$z_j = h_{j+1} - h_j$ = slope

Dynamics:

- adding a grain to the pile (at site j):
  $h_j \to h_j + 1$
  $z_{j-1} \to z_{j-1} + 1$ and $z_j \to z_j - 1$

- relaxing the slope of the pile whenever local gradient exceeds stability threshold z_c:
  $h_j \to h_j - 1$ and $h_{j+1} \to h_{j+1} + 1$
  $z_j \to z_j - 2$, $z_{j\pm1} \to z_{j\pm1} + 1$

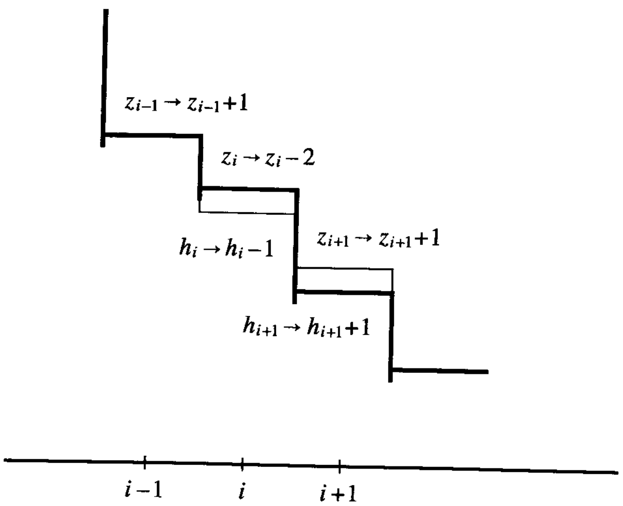


Figure 4.1. The one-dimensional BTW sandpile cellular automaton.

Wall at the left end: $z_0 = 0$

right end is open: $z_{N-1} \to z_{N-1} + 1$ and $z_N \to z_N - 1$ whenever $z_N > z_c$
grains can leave the system

This is the height model.

The slope model works similarly with just $z_j \to z_j + 1$

During relaxation process (avalanche):

$P(s)$ = probability that s sites are involved in relaxation process

$P(t)$ = probability that avalanche last t timesteps

$$P(s) = P(t) = \frac{1}{N+1}$$

for $s, t \in \{1, \ldots, N+1\}$

attractor of the dynamics consists of only one state, the marginally stable state with $z_i = z_c$ for all i.

d-dimensional sandpiles

d-dimensional hypercubic lattice, L^d sites $\mathbf{i}$
dynamic variables $E(\mathbf{i})$ = "energy" at site $\mathbf{i}$
threshold E_c

A configuration is stable, if $E(\mathbf{i}) < E_c$ for all $\mathbf{i}$

R1: Adding energy (to stable configuration)

choose site $\mathbf{i}$ at random
increase $E(\mathbf{i})$ by δE

R2: Relaxation rule (for unstable config.)

If $E(\mathbf{i}) \geq E_c$ then

$$E(\mathbf{i}) \to E(\mathbf{i}) - \sum_{\mathbf{e}} \Delta E(\mathbf{e})$$
$$E(\mathbf{i} + e) \to E(\mathbf{i} + \mathbf{e}) + \Delta E(\mathbf{e})$$

where $\mathbf{e}$ is a set of unit vectors from site $\mathbf{i}$ to *some* neighbours
relaxation vector $\Delta E = (E_N, E_E, E_S, E_W)$ in 2d

open boundaries: energy removed from system

δE, E_c are irrelevant for scaling behaviour

Important variants

Bak-Tang-Wiesenfeld model:

$E_c = 4$

$\delta E = 1$

$\Delta E = (1, 1, 1, 1)$

Zhang model (continuous):

$E_c = 1$

$0 < \delta E < 1$

$\Delta E = (b, b, b, b)$ where $b = E(\mathbf{i})/4$

Manna model (random relaxation):

$E_c = 2$

$\delta E = 1$

energy is distributed to two random neighbours

Office version of BTW

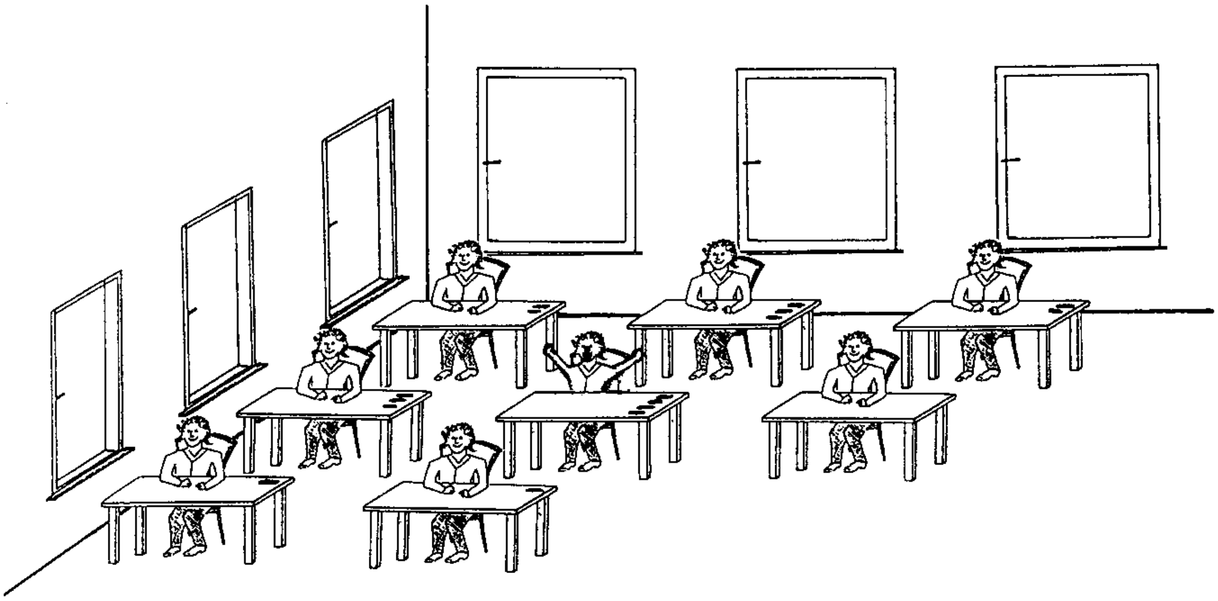


Figure 13. Office version of the sandpile model. At regular intervals a piece of paper lands on the desk of a random bureaucrat. When a bureaucrat finds four or more sheets of paper on his desk he sends one sheet to each of his neighbors, or out the window. (Courtesy of Peter Grassberger.)

Classification of models

Deterministic vs. stochastic rules:

A model is called deterministic if the toppling rules are deterministic

Abelian symmetry:

A model is called abelian if the configuration after an avalanche is independent of the order in which the relaxation of the active sites was performed

Rotational symmetry:

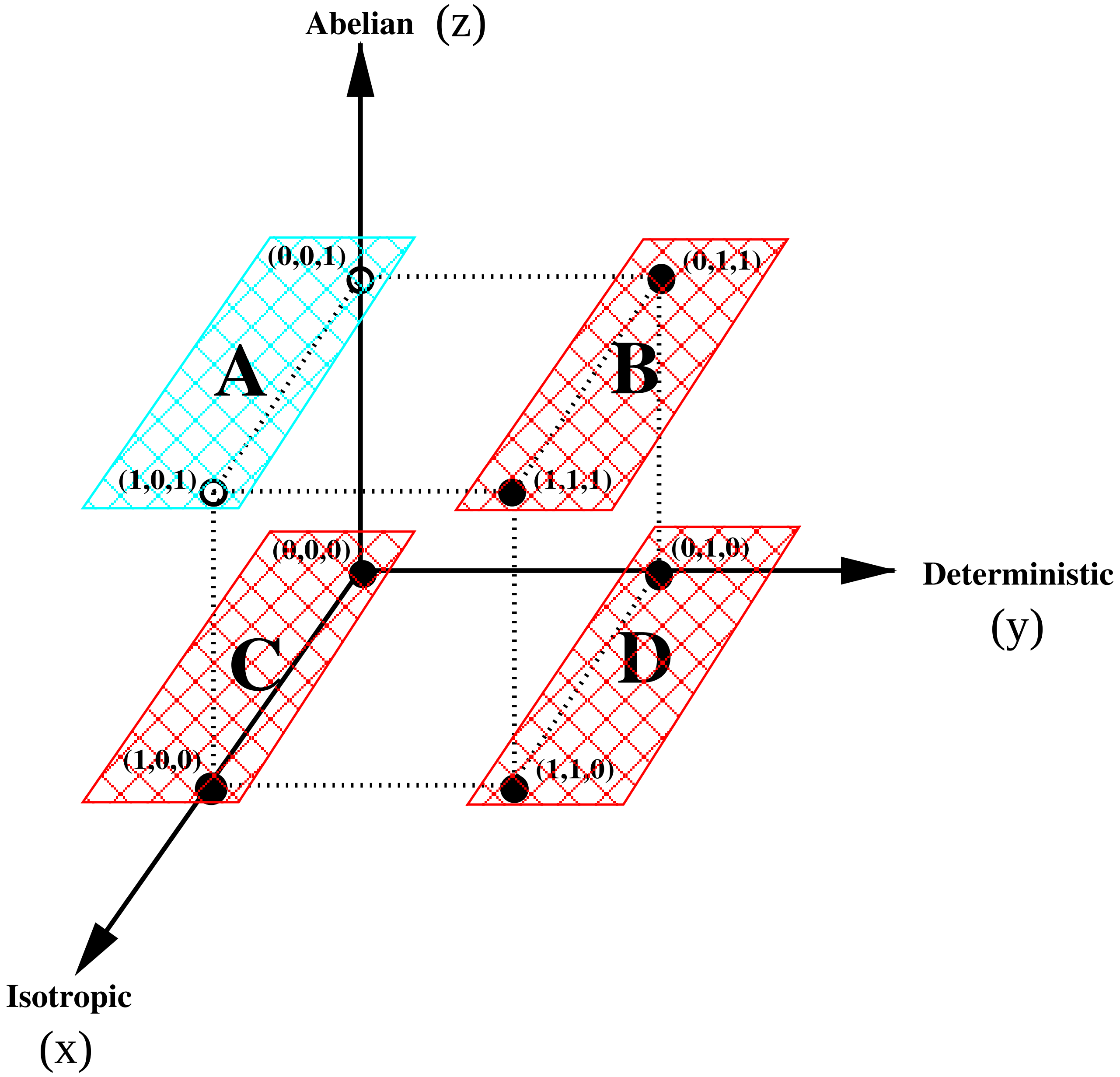
A model is called isotropic if in each toppling the energy is divided equally between nearest neighbours

BTW: abelian, isotropic, deterministic

Zhang: non-abelian, isotropic, deterministic

Manna: non-abelian (?), non-isotropic, non-deterministic

other class: conservative vs. non-conservative



Abelian (z)
(0,0,1)
(0,1,1)
A
B
(1,0,1)
(1,1,1)
(0,0,0)
(0,1,0)
Deterministic
(y)
C
D
(1,0,0)
(1,1,0)
Isotropic
(x)

Fig. 1

Results

Distribution exponents:

size s of an avalanche: total number of relaxation events

area a of an avalanche: number of sites where relaxations occured

one time step: relaxation of all sites which exceed the threshold at a given time
lifetime t of an avalanche: number of such time steps

Geometrical exponents:

linear size d: maximal distance between the origin of the avalanche an all sites of the avalanche cluster

radius of gyration of the cluster

perimeter of an avalanche: a site belongs to the perimeter if it has a nearest neighbour where no relaxation took place

All distributions obey power laws, i.e. the system is critical:

$$P(x) \propto x^{1-\gamma_x}$$

for $x \in \{s, a, t, r, d, p\}$

Power spectrum:

1) total flow down the slope of the pile when grains are added at random positions:
$S(f) \propto 1/f^2$

no $1/f$ noise !!!!

2) when grains are added only along the closed walls: $S(f) \propto 1/f$

Mean-Field approach

Neglect spatial correlations by giving up structure of the lattice:
in each step q_c arbitrary sites are choosen randomly as neighbours

generalization: introduce new parameter α

$$q_c = \alpha z_c \qquad \text{with } \alpha \in (0,1]$$

$$\begin{cases} \alpha = 1 & \text{conservative} \\ 0 < \alpha < 1 & \text{dissipative} \end{cases}$$

Avalanches have interpretation as branching processes: A toppling site can induce $0, 1, \ldots, q_c$ new topplings

For branching ratio σ the probability that a tree in a branching process develops s nodes before it dies is

$$P(s) \propto s^{-3/2} \cdot \exp\left(-s/s_o(\sigma)\right)$$

with $s_0(\sigma) \propto (1-\sigma)^{-2}$ for $\sigma \to 1^-$

It can be shown that $\sigma = \alpha$.

Nonconservative case ($\alpha < 1$):
exponential cutoff $s_0(\alpha)$ is independent of system size and one finds noncritical behaviour

Conservative limit ($\alpha \to 1$):
$P(s) \propto s^{-\tau}$ with $\tau = 3/2$,
in agreement with simulations for $d = 4$ and $d = 5$, i.e. critical dimension is $d_c = 4$.

Abelian sandpiles

lattice of N sites, $i = 1, \ldots N$ with integer height variables z_i and threshold values z_{ic}

If $z_i < z_{ic}$, for all i, the pile is stable

Adding a particle:

Pick site i with probability p_i and add a grain ($\sum_i p_i = 1$), i.e. $z_i \to z_i + 1$

Toppling:

If $z_i \geq z_{ic}$, then site i is unstable and it topples:

Toppling matrix Δ: $N \times N$ integer matrix

$$z_j \to z_j - \Delta_{ij}$$

for $j = 1, \ldots N$

Properties of Δ:

i) $\Delta_{ii} > 0$, for all i
(height at toppled site decreases)

ii) $\Delta_{ij} \leq 0$, for all $i \neq j$
(height at other sites increases)

iii) $\sum_j \Delta_{ij} \geq 0$, for all i
(no creation of sand during toppling)

At boundaries sand is removed from the system.

Without loss of generality: $z_{ic} = \Delta_{ii}$
site i is stable for $0 \leq z_i < z_{ic}$

Define operator a_i that acts on the space of stable configurations.
If C is any stable configuration, $a_i C$ is the stable configuration obtained by adding a particle at i, and relaxing the system.

Easy to check: if there are two or more unstable sites, the final configuration does not depend on the order of the topplings and the addition of particles commutes with topplings

$$[a_i, a_j] = 0$$

for all i and j.

Adding Δ_{ii} grains at i causes toppling and is equivalent to adding $-\Delta_{ij}$ grains at all sites $j \neq i$:

$$\prod_{j=1}^{N} a_j^{\Delta_{ij}} = 1$$

for all i.

$\mathcal{R}$ the space of recurrent configurations:
$\mathcal{R}$ is closed under multiplication by $\{a_i\}$

Burning algorithm: Test whether a state C is in $\mathcal{R}$ or not

scan the lattice and 'burn' recursively all sites for which $z_i \geq \sum_j' (-\Delta_{ji})$, where the primed sum is over all unburnt sites j.
If not all sites are burned away, C is transient.
If all sites are burned away and Δ is symmetric, C is recurrent (for asymmetric Δ: script test).

It can be shown: $|\mathcal{R}| = \det \Delta$

State after addition of t grains to the system ("time" t):

$$|P(t)\rangle = \sum_{C \in \mathcal{R}} P(C,t)|C\rangle$$

Time evolution:

$$|P(t+1)\rangle = \left(\sum_{i=1}^{N} p_i a_i \right) |P(t)\rangle$$

a_i can simultaneously be diagonalized:

$$a_i|\psi\rangle = \exp(i\phi_i)|\psi\rangle$$

with

$$\phi_i = 2\pi \sum_j [\Delta^{-1}]_{ij} n_j$$

Stationary state: $n_i = 0$ for all i

In the stationary state all recurrent configurations are equally probable

$$P(C) = \frac{1}{\det \Delta}$$

Application

G_{ij} = average number of relaxations on site j induced by avalanches starting at site i

$$G_{ij} = \left[\Delta^{-1}\right]_{ij}$$

For BTW:

$$G_r \propto r^{2-d}$$