Self-organized criticality (a reminder)

Dynamical systems drive themselves into states characterized by algebraic correlations
(no tuning of parameters is necessary)

critical state is an attractor of the dynamics

$$\text{SOC} \rightarrow \begin{cases} \text{fractals} & \text{(spatial power-laws)} \\ 1/f \text{ noise} & \text{(temporal power-laws)} \end{cases}$$

Separation of timescales:

external driving much slower than internal relaxation processes

typical:

threshold dynamics, existence of metastable states

Sandpiles

1) slow driving by adding grains
2) fast relaxation through avalanches

Real sandpiles:

large sandpiles do not show critical behaviour
too many large avalanches exists due to inertia effects

Ricepiles:

lower density $\rightarrow$ less inertia effects

indications for critical behaviour

Sandpile models:

different variants show critical behaviour
but: $1/f$ noise only in special cases

Evolution

Biological evolution:

Does it take place gradually at a slow, smooth pace or via periods of hectic activity separated by long intervals of tranquility ???

Is the collection of all species moving towards some terminal configuration?

punctuated equilibrium:

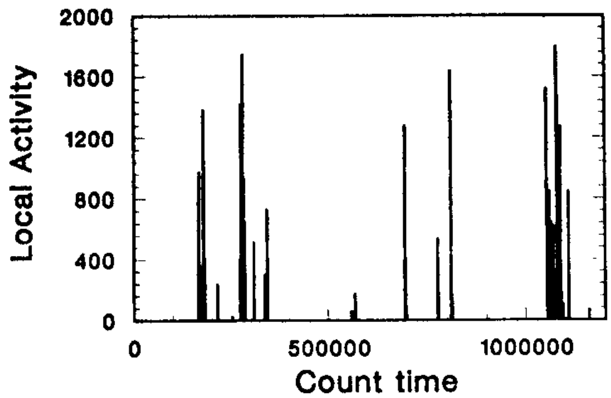


FIG. 4. Punctuated equilibrium behavior. Activity vs time in a local segment of ten consecutive sites is shown for a system of size $N = 512$. Time is measured in units of the number of mutations. In real time, the intermittency is further enhanced by the exponential enlargement of the periods of stasis.

Bak-Sneppen model

Model for coevolution of interacting species

Species interact with each other, e.g. via food chain

fitness landscape is “random”, not given explicitly given as function of internal variables

species are not represented by their genetic code, but by their fitness

assumption: any adaptive move leads to a new random fitness

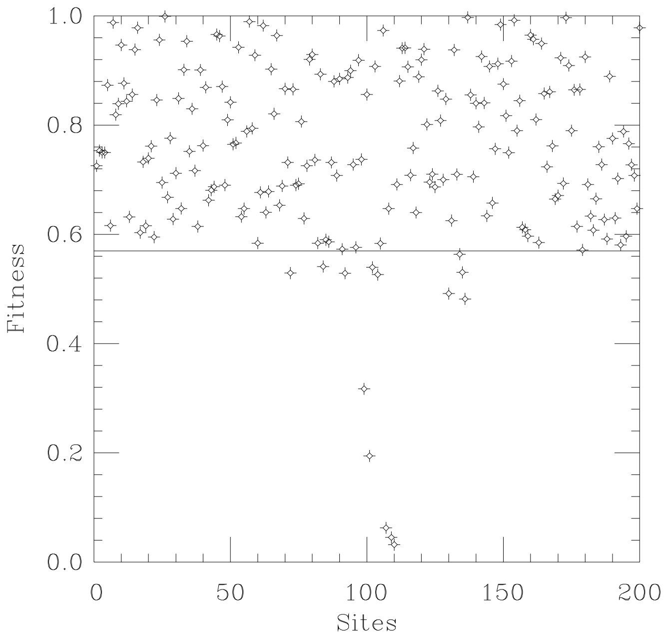
evolution slow → low “temperature” → species with smallest fitness changes next

Bak-Sneppen model:

one-dimensional ring of L sites (representing species) with periodic boundary conditions
to each site j a random number B_j (fitness or barrier) is assigned ($0 < B_j < 1$)

1) locate species with lowest fitness B_i and mutate $B_i \to B_i'$
2) change fitness of the two neighbours $B_{i\pm 1} \to B'_{i\pm 1}$

Snapshot of Bak-Sneppen model:



Stationary state

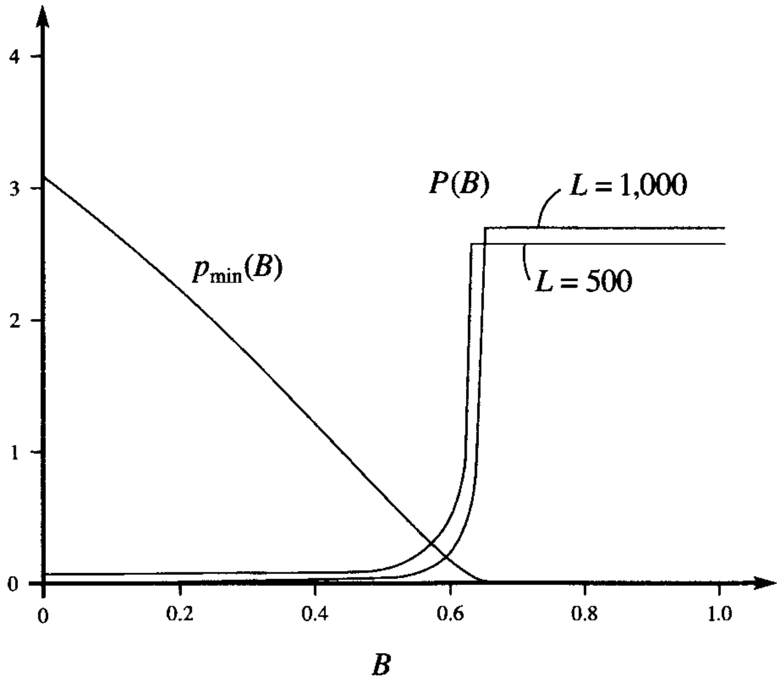


Figure 4.17. Probability density for the fitness barrier in the Bak–Sneppen model. The size dependence of $p_{\min}(B)$ (the distribution of the minimum barriers) is not visible on the scale of the figure.

probability distribution:

$$P(B) = \begin{cases} 0 & \text{for } B < B_c \\ 1/(1 - B_c) & \text{for } B > B_c \end{cases}$$

with $B_c = 0.66702 \pm 0.00003$

Avalanches:

$B_{min}(t) =$ smallest B_j at time t

avalanche of lifetime T:

$B_{min}(t) > B_0$ and $B_{min}(t') < B_0$ for $t < t' < t + T$ and $B_{min}(T+1) > B_0$

for $B_0 = B_c$: $P(T) \propto T^{-\alpha}$ with $\alpha \approx 1$

Distribution of avalanches:

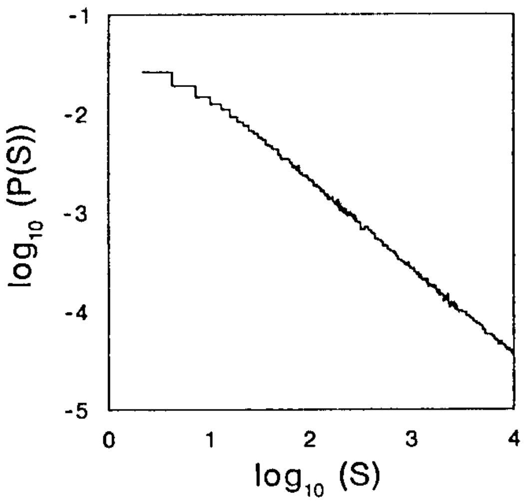


FIG. 5. Distribution of avalanche sizes in the critical state. Here an avalanche is defined by subsequent sequential activity below punctuation of the barrier $B = 0.65$.

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Forest fires

Original model:

3-state stochastic cellular automaton on a $d-$dimensional hypercubic lattice with L^d sites

each site can be empty, occupied by a tree or a burning tree

dynamical rules (parallel updating):
(1) burning tree $\rightarrow$ empty site
(2) tree $\rightarrow$ burning tree
(3) empty site $\rightarrow$ tree with probability p

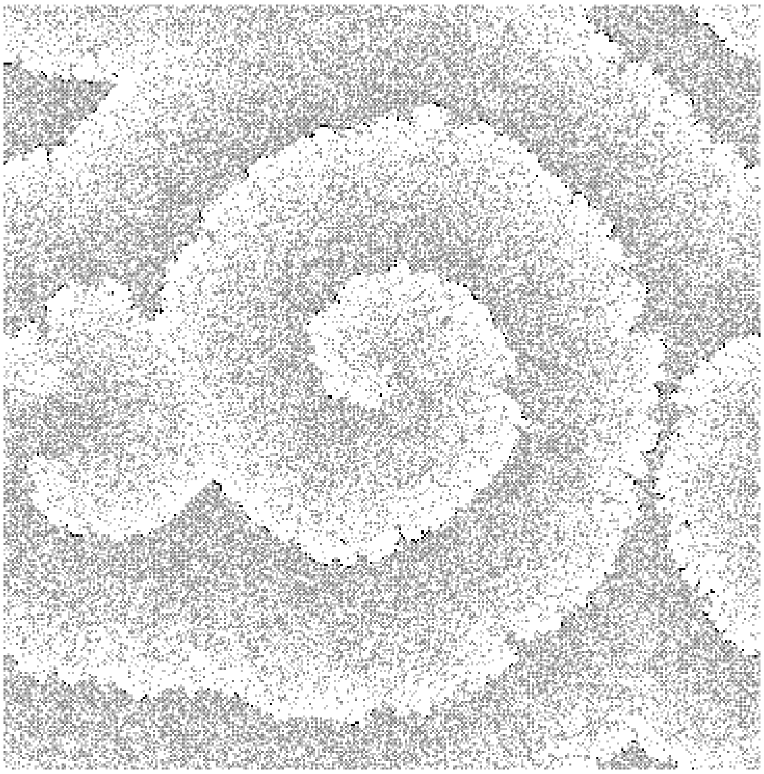
Other applications:
excitable media, (oscillating) chemical reactions, turbulence

small p: fire burns itself out
large p: concentration of trees on fire reaches stationary state

Is this model critical ? No!

Time + length scales proportional to $1/p$ exist!

Spiral waves:



cluster size stable:

burning trees N_b = new trees N_n (at boundary)

$N_n \approx pN_p$ (N_p = # sites at boundary)

since $N_b = O(1)$: $N_p \propto 1/p$

Generalization:

immunity g and lightning probability f

dynamical rules (parallel updating):

(1) burning tree $\to$ empty site

(2) tree $\to$ burning tree with probability $1 - g$
if at least one neighbour is burning

(3) tree $\to$ burning tree with probability f
if no neighbour is burning

(4) empty site $\to$ tree

(3) allows to set small clusters on fire

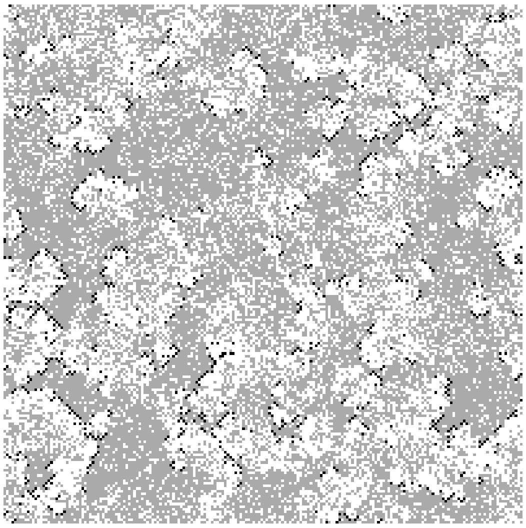
SOC in forest fires

critical immunity $g_c(p)$

for $g > g_c(p)$: fire density becomes 0 and forest density 1

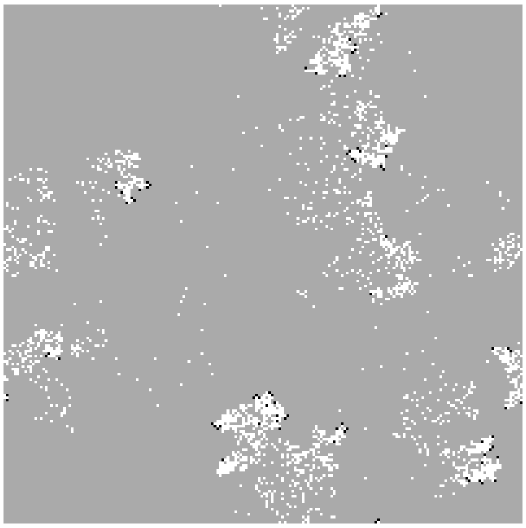
transition: directed percolation

for $g \ll g_c(p)$:



(grey: trees; white: empty; black: burning)

for $g \approx g_c(p)$:



SOC behaviour only for lightning probability $f > 0$

in the following: $g = 0$

p/f measures number of growing trees between two lightning strokes and thus the mean number of trees destroyed per lightning

for $f \ll p$ large forest clusters and correlations over long distances exist

SOC exists when tree growth is so slow that fire can burn down even large clusters before new trees grow at their edge: $p \ll T^{-1}(s_{max})$ ($T(s_{max})$ = time a fire needs to burn down large clusters)

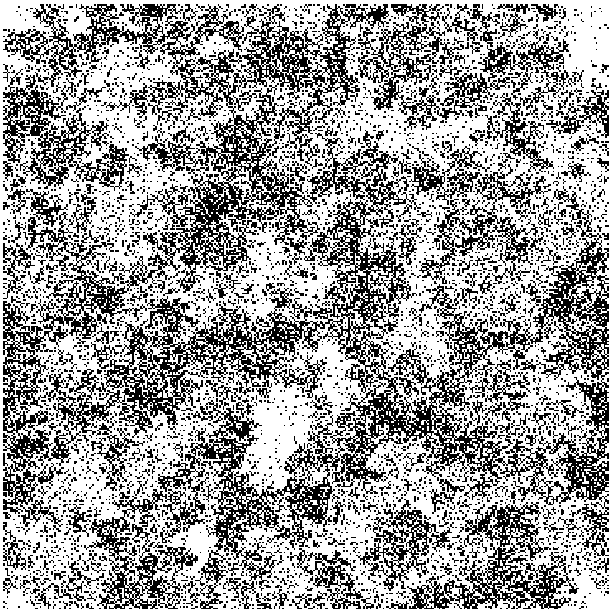
for $f/p \to 0$: $\quad T(s_{max}) \propto (f/p)^{-\nu'}$

(dynamics depends only on f/p)

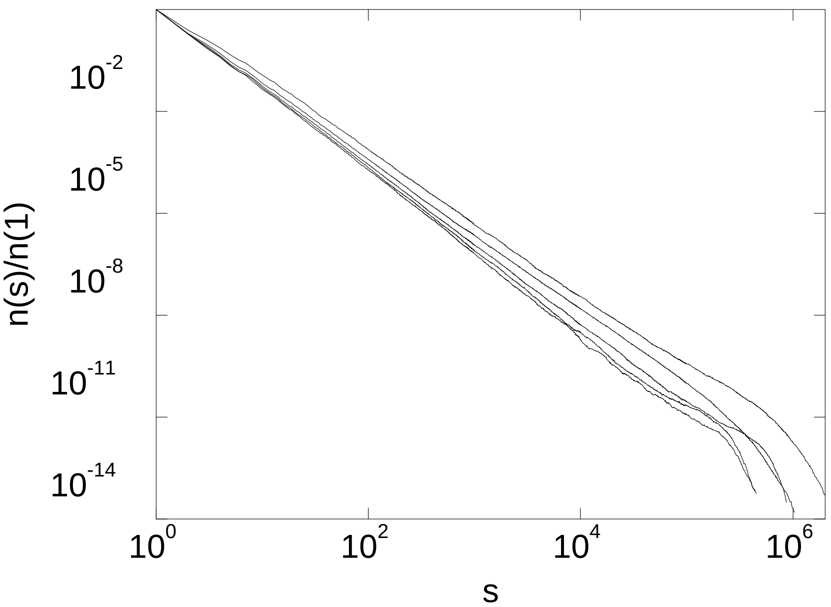
double separation of time scales:

$$\left(\frac{f}{p}\right)^{-\nu'} \ll \frac{1}{p} \ll \frac{1}{f}$$

Snapshot of the critical state ($f/p = 1/500$):



Cluster-size distribution:



Mean-field approach:

neglect correlations: random-neighbour model

$$\begin{aligned} \rho_f &= p\rho_e \\ \rho_f &= \rho_t\left(f + (1-f)(1-(1-\rho_f)^{2d})\right) \\ 1 &= \rho_e + \rho_t + \rho_f \end{aligned}$$

where $\rho_\alpha =$ density of state $\alpha = e, f, t$

The first and third equation are exact, since they involve no nearest-neighbour interaction. In the second equation it is assumed that the probability that one or more neighbours of a given site are burning is given by $1-(1-\rho_f)^{2d}$

upper critical dimension: $d_c = 6$

SOC in traffic flow

$1/f$ noise in real traffic:

measurements on Japanese expressway
power spectrum of transient times of cars passing underneath a bridge spanning the motorway exhibits $1/f$ behaviour

similar behaviour is observed in granular flow

Nagel-Schreckenberg model:

- **Acceleration:**
  increase v by 1 (if possible)

- **Braking:**
  if the distance d to the next car is less than $v + 1$, then reduce velocity to $v = d - 1$.

- **Randomization**:
  $v \rightarrow \max(v - 1, 0)$ with probab. p.

- **Driving:** $x \rightarrow x + v$

$p = 0$: continuous phase transition at
$\rho_c = 1/(v_{max} + 1)$

$p > 0$: crossover at $\rho^* < \rho_c$

Cruise-control limit

cruise-control limit:

apply randomization rule only to cars which move with $v < v_{max}$

periodic boundary conditions:

low densities: laminar flow, cars move deterministically

higher densities: jams exist

mega-jam configuration:

left end: infinitely large jam
right end: open

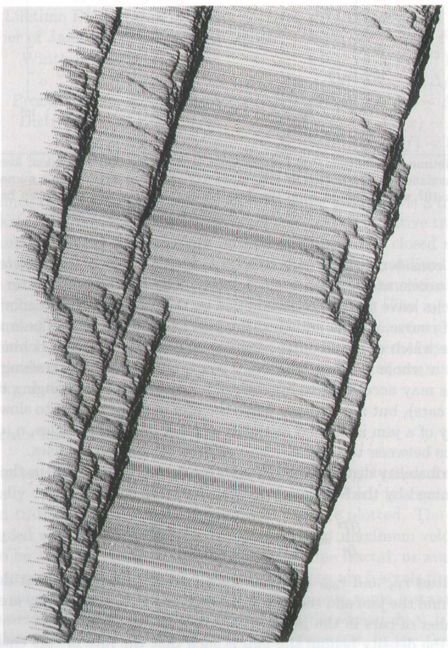


Figure 2: Outflow from a dense region; only the front, or interface, from the dense region is shown as the structure on the left hand side (see text). In the outflow region, a phantom jam is triggered by a small disturbance. This is the structure on the right hand side.

outflow from jam = maximal current J_{max}

outflow pattern is critical !

Disturb car in free-flow region:
lifetime distribution of jams (exponent 3/2):

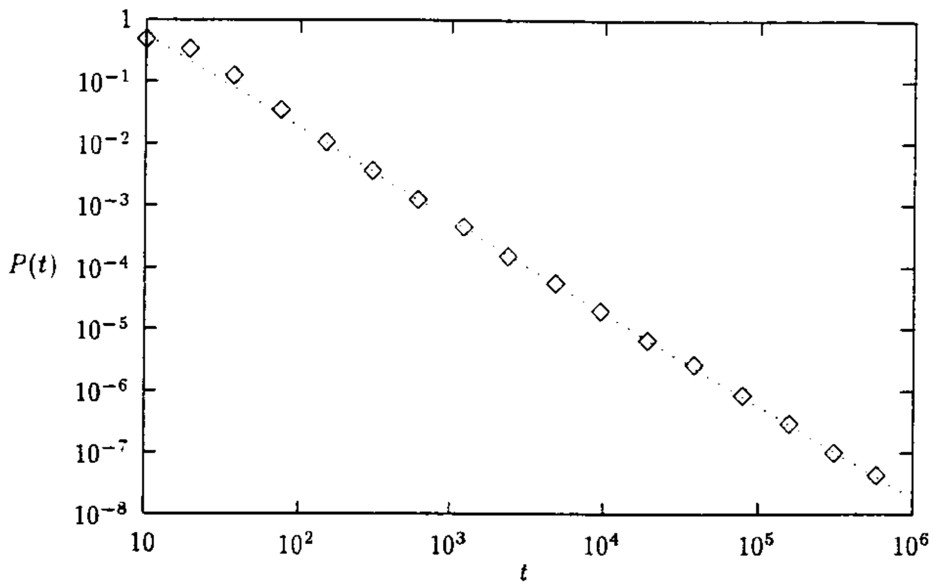


Figure 3: Lifetime distribution $P(t)$ for emergent jams in the outflow region; average over mo than 65 000 avalanches. The dotted line has slope 3/2. All jams larger than a numerically impos cutoff at $t = 10^6$ were removed from the data base. This imposed cutoff can be made infinit which indicates that the outflow is precisely critical.

$1/f$ noise in power spectrum of $N_l(t)$ = number of cars in a small segment of length l (in closed system)

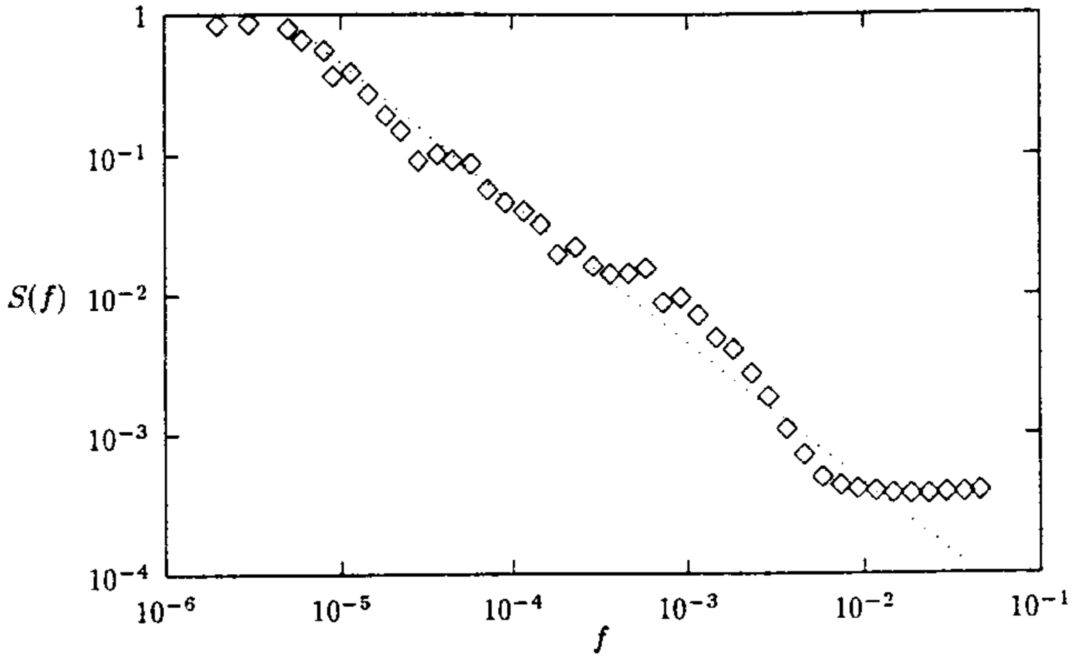


Figure 6: Power spectrum, $S(f)$, smoothed by averaging, for a closed system of len
with $p_{free} = 0.00005$ and $v_{max} = 5$. Dotted line has slope -1.

Is this relevant for practical applications ??

Increasing the flow using traffic control methods can drive open systems to a critical point with large fluctuations where traffic planning and control becomes extremly difficult

What is self-organized criticality?

- Is SOC a well-defined distinct phenomenon different from any other category of behaviour ?

- Do we have a theory of SOC ?

- Has SOC taught us anything that we did not know before ?

- Is there any predictive power in SOC ?
  Do we have necessary and sufficient conditions for the occurence of SOC ?
  Does it help us, if we know that a systems exhibits SOC ?

Is SOC a well-defined phenomenon ?

Two different types of definition seem to be plausible:

phenomenological definition:

Systems that exhibit power-laws without any apparent tuning
but: What is tuning?
driving rate $\rightarrow 0$

constructive definition:

slowly driven, interaction-dominated threshold systems
mutual interaction between degrees of freedom dominate over intrinsic dynamics of the degrees of freedom
thresholds allow for existence of metastable states
due to slow driving the system is dominated by internal interactions and relaxes from one metastable to another one

Do we have a theory of SOC ?

No general formalism like one has for equilibrium statistical mechanics
only small steps towards formalism: mean-field, exact solutions, dynamical driven RG

What has SOC taught us ?

importance of fluctuations, in particular for slowly driven systems
stimulated interest in threshold systems with strongly fluctuating avalanche-like temporal evolution

What do we learn if we know that a system is SOC ?

????

Systems exhibiting SOC are likely to be slowly driven, interaction-dominated threshold systems!?

My personal summary

A lot of problems remain:

Lack of precise definition
Many experimental data are interpreted very optimistically
Even in model systems it is not easy to find proper SOC behaviour
often not very robust against change of rules
What is tuning?

SOC is not (yet?) the theory of complex systems !

But: The idea has stimulated a lot of work in the field of complex dynamical systems !

Example: Granular materials

Some references

General:
P. Bak, C. Tang, K. Wiesenfeld: Phys. Rev. Lett. **59**, 381 (1987) and Phys. Rev. **A38**, 364 (1988)
P. Bak: *How Nature Works* (Springer, 1998)
H.J. Jensen: *Self-Organized Criticality*, Cambridge Lecture Notes in Physics (CUP 1998)

Zipf's Law:
Z.K. Silagadze: `physics/9901035`

Sandpiles:
V. Frette et al.: Nature **379**, 49 (1996) (Ricepiles)
O. Biham, E. Milshtein, S. Solomon: `cond-mat/9805206`
A. Chessa, H.E. Stanley, A. Vespignani, S. Zapperi: Phys. Rev. **E59**, R12 (1999)
D. Dhar: `cond-mat/9808047`

Evolution:
P. Bak, K. Sneppen: Phys. Rev. Lett. **71**, 4083 (1993)
N.-N. Pang: Int. J. Mod. Phys. **B11**, 1411 (1997)

Forest Fires:
S. Clar, B. Drossel, F. Schwabl: J. Phys. **CM8**, 6803 (1996)

Traffic:
M. Paczuski, K. Nagel: `cond-mat/9602011`

Landscape paradigm (remark by Theo Nieuwenhuizen):

H. Frauenfelder et al. (Ed.): Physica **D107** (1997)