Covariance of bosonic q–oscillators under q–symplectic transformations

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The classical (undeformed) case

Consider the canonical commutation relations

$$x_i x_j - x_j x_i = C_{ij} \quad , \quad i, j \in \{1, 2, \ldots, 2n\} \ ,$$

where C is a non–degenerate skew–symmetric matrix.

Make a linear transformation of the x_j

$$x_j \longrightarrow x'_j = \sum_i a_{ij} x_i \quad \text{with } a_{ij} \in \mathbb{C} \ .$$

The x'_j satisfy the same commutation relations as the x_j if and only if the matrix

$$A = (a_{ij})$$

satisfies the condition

$$A^t C A = C \ ,$$

i.e., if and only if

$$A \in \mathrm{Sp}(2n; \mathbb{C}) \ .$$

Let $W(n)$ be the (universal) algebra with generators x_i, $1 \leq i \leq 2n$, and relations

$$x_i x_j - x_j x_i = C_{ij}$$

(Weyl algebra). Then the foregoing can be rephrased as follows:
The linear transformation

$$x_j \longrightarrow \sum_i a_{ij} x_i$$

of the generators extends to an automorphism of the Weyl algebra $W(n)$ if and only if $A = (a_{ij})$ belongs to $\mathrm{Sp}(2n)$.

Let V be the vector space spanned by the x_i. Then we have a natural representation of $\mathrm{Sp}(2n)$ on V (the vector representation). Moreover, if $A \in \mathrm{Sp}(2n)$ and if $\tilde{A}$ is the automorphism of $W(n)$ associated to A, then $A \longrightarrow \tilde{A}$ is a representation of $\mathrm{Sp}(2n)$ in $W(n)$.

We summarize:
The map $A \longrightarrow \tilde{A}$ is a representation of $\mathrm{Sp}(2n)$ in $W(n)$, and the representatives $\tilde{A}$ are algebra automorphisms.

This is the result that we want to generalize.

Transcription to the Hopf algebra language

Consider again the vector space

$$V = \sum_i \mathbb{C}\, x_i$$

and the vector representation acting on it:

$$x_j \longrightarrow x'_j = \sum_i x_i a_{ij} \ \ , \quad A = (a_{ij}) \in \mathrm{Sp}(2n) \ .$$

In order to make the dependence of the numbers a_{ij} on the group element A manifest, we introduce the following complex–valued functions on $\mathrm{Sp}(2n)$:

$$t_{ij}(A) = a_{ij} \quad \text{if } A = (a_{ij}) \in \mathrm{Sp}(2n) \ .$$

Then the transformation above takes the form

$$x_j \longrightarrow x'_j = \sum_i x_i\, t_{ij}(A) \ .$$

The functions t_{ij} play a basic role in the following. Let

$$F = \mathrm{Fun}(\mathrm{Sp}(2n))$$

be the algebra of functions on $\mathrm{Sp}(2n)$ generated by these functions. This is the algebra that is going to replace the group $\mathrm{Sp}(2n)$ itself.

The functions t_{ij} are not algebraically independent. In fact, we have

$$C^{-1}T^tC\,T = TC^{-1}T^tC = I \ ,$$

where we have set

$$T = (t_{ij}) \ ,$$

and where I is the unit matrix. Thus $C^{-1}T^tC$ is the inverse of T.

The Hopf algebra structure of F

Recall that the algebra $F \otimes F$ can be regarded as an algebra of functions on $\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$ by setting

$$(f \otimes f')(A, A') = f(A)f'(A') \ .$$

Then there exists a unique algebra homomorphism

$$\Delta : F \longrightarrow F \otimes F \quad \text{(coproduct)}$$

such that

$$(\Delta(f))(A, A') = f(AA') \quad \text{for all } A, A' \in \mathrm{Sp}(2n) \ .$$

Moreover, we define the algebra homomorphism

$$\varepsilon : F \longrightarrow \mathbb{C} \quad \text{(counit)}$$

by

$$\varepsilon(f) = f(I) \ .$$

Finally, we define the map

$$S : F \longrightarrow F \quad \text{(antipode)}$$

by

$$(S(f))(A) = f(A^{-1}) \quad \text{for all } A \in \mathrm{Sp}(2n) \ .$$

Endowed with these maps Δ, ε and S, the algebra F becomes a Hopf algebra. Note that

$$\begin{aligned}
\Delta(t_{ij}) &= \sum_k t_{ik} \otimes t_{kj} \\
\varepsilon(t_{ij}) &= \delta_{ij} \\
S(t_{ij}) &= (C^{-1} T^t C)_{ij} \ .
\end{aligned}$$

Let us now return to the transformation

$$x_j \longrightarrow x'_j = \sum_i x_i \, t_{ij}(A) \ .$$

It suggests to define a linear map

$$\omega : V \longrightarrow V \otimes F$$

by

$$\omega(x_j) = \sum_j x_i \otimes t_{ij} \ .$$

We get the original transformation by evaluating the second tensorial factor at $A \in \mathrm{Sp}(2n)$.

The fact that ω stems from a representation of $\mathrm{Sp}(2n)$ is contained in the following relations:

$$(\omega \otimes \mathrm{id}_F) \circ \omega = (\mathrm{id}_V \otimes \Delta) \circ \omega$$

(these are maps from V into $V \otimes F \otimes F$) and

$$(\mathrm{id}_V \otimes \varepsilon) \circ \omega = \mathrm{id}_V$$

(these are maps from V into $V \otimes \mathbb{C} \simeq V$).

A vector space endowed with a map ω of this type is called a *right F–comodule.*

As we have seen, the vector representation of $\mathrm{Sp}(2n)$ in V extends uniquely to a representation of $\mathrm{Sp}(2n)$ in the Weyl algebra $W(n)$, acting by algebra automorphisms. Transcribed into the present language, this means that there exists a unique *algebra homomorphism*

$$\varpi : W(n) \longrightarrow W(n) \otimes F$$

which extends ω, and that ϖ converts $W(n)$ into a right *F–comodule.*

As stressed above, the action of a group element$A \in \mathrm{Sp}(2n)$ can be recovered by evaluating the second tensorial factor at A. This evaluation

$$\delta_A : F \longrightarrow \mathbb{C} \quad , \quad \delta_A(f) = f(A)$$

is just a special linear form on F. In fact, δ_A is a character of the algebra F, and every character of F has this form. In this way, we can recover the group $\mathrm{Sp}(2n)$ from the algebra F.

The given characterization of covariant algebras makes sense if F is replaced by an arbitrary Hopf algebra H. Of course, if H is not commutative we definitely loose the interpretation in terms of groups (in fact, the sole character of a Hopf algebra may be its counit).

The q–deformed case

Our goal is to find a suitable deformation of the algebra Fun(Sp($2n$)) which will then be regarded as an "algebra of functions on the quantum group $\mathrm{Sp}_q(2n)$".

A natural ansatz emerges if the q–deformed enveloping algebra (quantum algebra) $U_q(\mathrm{sp}(2n))$ is already known (at present it is sufficient to know that q is a complex deformation parameter). This is a quasitriangular Hopf algebra which should be dual (in an appropriate sense) to the function algebra we are looking for.

The function algebra is expected to be non–commutative, and this corresponds to the fact that $U_q(\mathrm{sp}(2n))$ is not cocommutative. As is well–known, the non–cocommutativity of $U_q(\mathrm{sp}(2n))$ is controlled by the universal R–matrix $\mathcal{R}$. In fact, the corresponding relation for $\mathcal{R}$ is

$$\mathcal{R}\Delta(h) = \Delta'(h)\mathcal{R} \quad \text{for all } h \in U_q(\mathrm{sp}(2n)) \ ,$$

where

$$\Delta' = \tau \circ \Delta \ ,$$

and where τ is the so–called twist,

$$\tau(h \otimes h') = h' \otimes h \ .$$

If q is not a root of unity, the representation theory of $U_q(\mathrm{sp}(2n))$ is almost completely analogous to that of $\mathrm{sp}(2n)$.

In particular, there is a vector representation (of dimension $2n$). Call it t^q and set

$$t^q(h) = (t^q_{ij}((h)) \quad \text{for all } h \in U_q(\mathrm{sp}(2n)) \ .$$

Then the t^q_{ij} are elements of the finite dual $U_q(\mathrm{sp}(2n))^\circ$ of $U_q(\mathrm{sp}(2n))$. Applying $t^q \otimes t^q$ to the relation for $\mathcal{R}$ we obtain

$$R_q \, T^q_1 \, T^q_2 = T^q_2 \, T^q_1 \, R_q \ ,$$

where

$$R_q = (t^q \otimes t^q)(\mathcal{R}) \ ,$$

$$T^q = (t^q_{ij}) \quad , \quad T^q_1 = T^q \otimes I \quad , \quad T^q_2 = I \otimes T^q \ .$$

All factors in this equation are $(2n)^2 \times (2n)^2$–matrices. For example, R_q may be written in the form

$$R_q = \Sigma \, R^q_{ij,kl} E_{ik} \otimes E_{j\ell} \ ,$$

where the E_{ij} are the familiar basic $2n \times 2n$–matrices. Using this notation, we have

$$(T^q_1)_{ij,k\ell} = t^q_{ik} \, \delta_{j\ell} \quad , \quad (T^q_2)_{ij,k\ell} = \delta_{ik} \, t^q_{j\ell} \ .$$

On the other hand, we expect that there exists a non–degenerate bilinear form on $V \simeq \mathbb{C}^{2n}$ which is invariant under the vector representation. This is indeed the case, and if C_q is its matrix, the corresponding invariance relation reads (as in the classical case)

$$C_q^{-1} (T^q)^t C_q T^q = I \ ,$$

which implies (since $U_q(\mathrm{sp}(2n))^\circ$ is a Hopf algebra, and hence $(T^q)^{-1} = S(T^q)$ exists)

$$T^q C_q^{-1} (T^q)^t C_q = I \; .$$

The R–matrix R_q and the q–symplectic form C_q are known. Hence we are led to the following definition, where we have adopted the notation of RTF.

The quantum group $\mathrm{Sp}_q(2n)$

Definition according to

N.Yu. Reshetikhin, L.A. Takhtadzhyan, L.D. Faddeev
Quantization of Lie groups and Lie algebras
Leningrad Math. J. 1(1) (1990) 193–225

Notation

$$n \in \{1, 2, 3, \dots\}$$
$$J = \{1, 2, \dots, 2n\}$$

In the following, all indices belong to J.

$$\varepsilon_i = \begin{cases} 1 & \text{for } i \in \{1, 2, \dots, n\} \\ -1 & \text{for } i \in \{n+1, n+2, \dots, 2n\} \end{cases}$$

$$(\rho_1, \rho_2, \dots, \rho_{2n}) = (n, n-1, \dots, 1, -1, -2, \dots, -n)$$

$$E_{ij} = \text{basic } J \times J\text{–matrices} \quad , \quad (E_{ik})_{k\ell} = \delta_{ik}\delta_{j\ell}$$

Define

$$i' = 2n + 1 - i \quad \text{for every } i \in J\,.$$

Choose $q \in \mathbb{C} - \{0\}$. Then the R–matrix corresponding to $\mathrm{Sp}_q(2n)$ is given by

$$\begin{aligned} R_q \;=\; & q \sum_{i \in J} E_{ii} \otimes E_{ii} \\ & + \sum_{\substack{i,j \in J \\ i \neq j, j'}} E_{ii} \otimes E_{jj} \\ & + q^{-1} \sum_{i \in J} E_{i'i'} \otimes E_{ii} \\ & + (q - q^{-1}) \sum_{\substack{i,j \in J \\ i > j}} E_{ij} \otimes E_{ji} \\ & - (q - q^{-1}) \sum_{\substack{i,j \in J \\ i > j}} q^{\rho_i - \rho_j} \varepsilon_i \varepsilon_j E_{ij} \otimes E_{i'j'} \; . \end{aligned}$$

Properties

R_q satisfies the Yang–Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \; .$$

Define the $J \times J$–matrix C by

$$C_{ij} = \varepsilon_i q^{\rho_j} \delta_{i'j} \quad (q\text{–deformed symplectic form}).$$

Then we have $C^2 = -I$ (unit operator) and

$$R_q = C_1 (R_q^{t_1})^{-1} C_1^{-1} = C_2 (R_q^{-1})^{t_2} C_2^{-1} \; ,$$

where t_1 resp. t_2 denotes transposition with respect to the first resp. second tensorial factor.

Define

$$\widehat{R}_q = PR_q \ ,$$

where P is the permutation operator (twist)

$$P = \sum_{i,j \in J} E_{ij} \otimes E_{ji} \ .$$

The Yang–Baxter equation for R_q is equivalent to the braid relation for $\widehat{R}_q$

$$(\widehat{R}_q)_{12}(\widehat{R}_q)_{23}(\widehat{R}_q)_{12} = (\widehat{R}_q)_{23}(\widehat{R}_q)_{12}(\widehat{R}_q)_{23} \ ,$$

furthermore, we have

$$(\widehat{R}_q - qI)(\widehat{R}_q + q^{-1}I)(\widehat{R}_q + q^{-2n-1}I) = 0 \ .$$

Suppose that the eigenvalues q, $-q^{-1}$, and $-q^{-2n-1}$ are different. Then we have the following spectral decomposition of $\widehat{R}_q$

$$\widehat{R}_q = qP_q^{(+)} - q^{-1}P_q^{(-)} - q^{-2n-1}P_q^{(0)}$$

$$I = P_q^{(+)} + P_q^{(-)} + P_q^{(0)} \ ,$$

where $P_q^{(+,-,0)}$ are the spectral projectors and

$$\begin{aligned} \text{rank } P_q^{(+)} &= \tfrac{1}{2}\, 2n(2n+1) \\ \text{rank } P_q^{(-)} &= \tfrac{1}{2}\, 2n(2n-1) - 1 \\ \text{rank } P_q^{(0)} &= 1 \ . \end{aligned}$$

Note that $P^{+,-,0} = p^{+,-,0}(\widehat{R}_q)$ with quadratic polynomials $p^{+,-,0}$.

Definition of $\mathrm{Sp}_q(2n)$

The universal algebra (associative, with unit element) with generators t_{ij} , $i, j \in J$ and relations

$$RT_1T_2 = T_2T_1R$$

$$TCT^tC^{-1} = CT^tC^{-1}T = I$$

(with $T = (t_{ij})_{i,j\in J}$) is called the *algebra of functions on the symplectic quantum group* $\mathrm{Sp}_q(2n)$ and will be denoted by $\mathrm{Fun}(\mathrm{Sp}_q(2n))$.

In terms of $\widehat{R}_q$ the RTT–relation can be written in the form

$$\widehat{R}_q(T \otimes T) = (T \otimes T)\widehat{R}_q \quad , \quad \text{with } T \otimes T = T_1T_2 \ .$$

Theorem

The algebra $\mathrm{Fun}(\mathrm{Sp}_q(2n))$ is a Hopf algebra with coproduct Δ

$$\Delta(T) = T \otimes T \ ,$$

counit ε

$$\varepsilon(T) = I \ ,$$

and antipode S

$$S(T) = CT^tC^{-1} \ ,$$

which satisfies

$$S^2(T) = (CC^t)T(CC^t)^{-1} \ .$$

q–deformed bosonic oscillators

in the sense of Biedenharn and Macfarlane

Generators:

$$a_i \ , \ a_i^+ \ , \ N_i \ , \quad i \in \{1, 2, \ldots, n\}$$

Relations:

$$[N_i, N_j] = 0$$

$$[N_i, a_j] = -\delta_{ij} a_j \quad , \quad [N_i, a_j^+] = \delta_{ij} a_j^+$$

$$[a_i, a_j] = 0 \quad , \quad [a_i^+, a_j^+] = 0$$

$$[a_i, a_j^+] = 0 \quad \text{if } i \neq j$$

$$a_i a_i^+ - q a_i^+ a_i = q^{-N_i}$$

$$a_i a_i^+ - q^{-1} a_i^+ a_i = q^{N_i} \ .$$

For $q^2 \neq 1$ the last two relations are equivalent to

$$a_i^+ a_i = \frac{q_i^N - q^{-N_i}}{q - q^{-1}} \quad , \quad a_i a_i^+ = \frac{q^{N_i+1} - q^{-N_i-1}}{q - q^{-1}} \ .$$

q–deformed bosonic oscillators

in the sense of Pusz, Woronowicz

Generators:

$$A_i\ ,\ A_i^+\ ,\ \ i \in \{1, 2, \ldots, n\}$$

Relations:

$$A_i A_j = q A_j A_i \quad , \quad A_i^+ A_j^+ = q^{-1} A_j^+ A_i^+ \quad \text{if } i < j$$

$$A_i A_j^+ = q A_j^+ A_i \quad \text{if } i \neq j$$

$$A_i A_i^+ - q^2 A_i^+ A_i = 1 + (q^2 - 1) \sum_{k<i} A_k^+ A_k\ .$$

Connection with the oscillators of Biedenharn and Macfarlane:

If we define

$$\begin{aligned} A_i &= q^{N_1+N_2+\ldots+N_{i-1}+\frac{1}{2}N_i}\, a_i \\ A_i^+ &= a_i^+\, q^{N_1+N_2+\ldots+N_{i-1}+\frac{1}{2}N_i}\ , \end{aligned}$$

the elements $A_i\,, A_i^+$ satisfy the Pusz, Woronowicz relations.

The $\mathrm{Sp}_q(2n)$–invariant bosonic oscillators

We use the notation introduced in the discussion of $\mathrm{Sp}_q(2n)$ and assume that $q^{2n} \neq 1$.

Let $W_q'(n)$ be the universal algebra with generators

$$x_i \ , \ i \in I$$

and relations

$$(x \otimes x) P_q^{(-)} = 0 \ ,$$

where $x \otimes x$ is regarded as a $(2n)^2$ row vector. These relations are equivalent to the following ones:

$$x_i x_j = q x_j x_i \quad \text{if } i < j \text{ but } i \neq j'$$

$$\text{(I)} \ \ x_i x_{i'} - q^2 x_{i'} x_i \ = \ (q^2 - 1) \sum_{j<i} q^{-\rho_i + \rho_j} x_{j'} x_j + q^{2n-\rho_i} \, d$$

$$\text{(II)} \ \ x_i x_{i'} - q^2 x_{i'} x_i \ = \ (q^2 - 1) \sum_{j<i} q^{\rho_i - \rho_j} x_j x_{j'} + q^{\rho_i} \, d \ ,$$

where in both cases $1 \leq i \leq n$ and where

$$d = \frac{q^2 - 1}{q^{2n} - 1} \, x^t C x \ .$$

According to RTF there exists a unique algebra homomorphism

$$\varpi' : W_q'(n) \longrightarrow W_q'(n) \otimes \mathrm{Sp}_q(2n)$$

such that

$$\varpi'(x_j) = \sum_i x_i \otimes t_{ij} \ ,$$

and $\overline{\omega}'$ defines on $W_q'(n)$ the structure of a right $\mathrm{Sp}_q(2n)$–comodule.

It can be shown, without using the definition of d, that the system (I) is equivalent to the system (II), moreover, each of these systems implies that

$$x^t C x = \frac{q^{2n}-1}{q^2-1}\, d \, .$$

Let us now introduce an additional relation for the x_i by requiring that d takes an arbitrary scalar value, and let us set

$$c = q^{2n} d \, .$$

Then the relations for the x_i take the form (we choose, for example, the system (I))

$$x_i x_j = q x_j x_i \quad \text{if } 1 \le i < j \le 2n \text{ but } i \ne j'$$

$$x_i x_{i'} - q^2 x_{i'} x_i = (q^2-1) \sum_{j<i} q^{\rho_{i'}-\rho_{j'}} x_{j'} x_j + q^{\rho_{i'}}\, c$$
$$\text{for } 1 \le i \le n \, .$$

We denote the corresponding algebra by $W_q(n)$, for $c \ne 0$ it is the q–analogue of the Weyl algebra $W(n)$.

Next we rescale the x_i by setting

$$x_i = c_i A_i \quad \text{for } 1 \le i \le 2n \, ,$$

where the constants $c_i \ne 0$ are chosen such that

$$c_i c_{i'} = q^{\rho_{i'}} \quad \text{for } 1 \le i \le n \, .$$

Defining

$$A_i^+ = A_{i'} \quad \text{for } 1 \leq i \leq n \ ,$$

the relations for the A_i , A_i^+, $1 \leq i \leq n$ take the form

$$A_i A_j = q A_j A_i \quad , \quad A_i^+ A_j^+ = q^{-1} A_j^+ A_i^+ \quad \text{if } i < j$$

$$A_i A_j^+ = q A_j^+ A_i \quad \text{if } i \neq j$$

$$A_i A_i^+ - q^2 A_i^+ A_i = c + (q^2 - 1) \sum_{k<i} A_k^+ A_k \ .$$

If $c \neq 0$, we can rescale the A_i , A_i^+ with a common scalar factor such that the new elements satisfy the foregoing relations with $c = 1$, i.e., the Pusz, Woronowicz relations.

It remains to be shown that the additional relation $d = q^{-2n} c$ is compatible with the comodule structure, i.e., that $\overline{\omega}'$ induces a map

$$\overline{\omega} : W_q(n) \longrightarrow W_q(n) \otimes \mathrm{Sp}_q(2n) \ .$$

The required relation reads

$$x^t C x = \frac{q^{2n} - 1}{q^2 - 1} q^{-2n} c \ ,$$

and our claim follows from the fact that $x^t C x$ is invariant in the sense that

$$\overline{\omega}'(x^t C x) = x^t C x \otimes 1 \ .$$