Quantum groups (M. Scheunert)

Definition of $U_q(sl(2))$, various versions

Choose $q \in \mathbb{K}$, $q \neq 0$, $q^4 \neq 1$. Then $U_q(sl(2))$ is the (universal) associative algebra (with unit element) with generators e, f, k, k^{-1} and relations

$$kk^{-1} = k^{-1}k = 1$$

$$kek^{-1} = q^2 e \quad , \quad kfk^{-1} = q^{-2} f$$

$$ef - fe = \frac{k^2 - k^{-2}}{q^2 - q^{-2}} \, .$$

Coproduct

$$\begin{aligned} \Delta(e) &= e \otimes k^{-1} + k \otimes e \\ \Delta(f) &= f \otimes k^{-1} + k \otimes f \\ \Delta(k^{\pm 1}) &= k^{\pm 1} \otimes k^{\pm 1} \end{aligned}$$

Counit

$$\varepsilon(e) = \varepsilon(f) = 0 \quad , \quad \varepsilon(k^{\pm 1}) = 1$$

Antipode

$$\begin{aligned} S(e) &= -q^{-2} e \\ S(f) &= -q^2 f \\ S(k^{\pm 1}) &= k^{\mp 1} \end{aligned}$$

Endowed with these maps, $U_q(sl(2))$ is a Hopf algebra.

Choose $q \in \mathbb{K}$, $q \neq 0$, $q^2 \neq 1$. Then $\tilde{U}_q(sl(2))$ is the (universal) associative algebra (with unit element) with generators E, F, K, K^{-1} and relations

$$KK^{-1} = K^{-1}K = 1$$

$$KEK^{-1} = q^2 E \quad , \quad KFK^{-1} = q^{-2} F$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}} .$$

Coproduct

$$\begin{aligned} \Delta(E) &= E \otimes 1 + K \otimes E \\ \Delta(F) &= F \otimes K^{-1} + 1 \otimes F \\ \Delta(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1} \end{aligned}$$

Counit

$$\varepsilon(E) = \varepsilon(F) = 0 \quad , \quad \varepsilon(K^{\pm 1}) = 1$$

Antipode

$$\begin{aligned} S(E) &= -K^{-1}E \\ S(F) &= -FK \\ S(K^{\pm 1}) &= K^{\mp 1} \end{aligned}$$

Endowed with these maps, $\tilde{U}_q(sl(2))$ is a Hopf algebra.

Connection between the two versions:
There exists a (unique) Hopf algebra homomorphism

$$\varphi : \tilde{U}_{q^2}(sl(2)) \longrightarrow U_q(sl(2))$$

such that

$$\begin{aligned} \varphi(E) &= ke \\ \varphi(F) &= fk^{-1} \\ \varphi(K^{\pm 1}) &= k^{\pm 2}, \end{aligned}$$

and this homomorphism is injective.

Qualitatively, we obtain $U_q(sl(2))$ from $\tilde{U}_{q^2}(sl(2))$ by adjoining $K^{1/2}$.

Especially in the more recent mathematical literature, $\tilde{U}_q(sl(2))$ is denoted by $U_q(sl(2))$.

Definition mainly used in Physics:
In the definition of $U_q(sl(2))$ given above, write q in place of q^2 and the generators in the form

$$e = S^+ \quad , \quad f = S^- \quad , \quad k^{\pm 1} = q^{\pm S^3}.$$

Relations

$$\begin{aligned} q^{S^3} q^{-S^3} &= q^{-S^3} q^{S^3} = 1 \\ q^{S^3} S^{\pm} q^{-S^3} &= q^{\pm 1} S^{\pm} \\ [S^+, S^-] &= \frac{q^{2S^3} - q^{-2S^3}}{q - q^{-1}} . \end{aligned}$$

Coproduct

$$\begin{aligned} \Delta(S^{\pm}) &= S^{\pm} \otimes q^{-S^3} + q^{S^3} \otimes S^{\pm} \\ \Delta(q^{\pm S^3}) &= q^{\pm S^3} \otimes q^{\pm S^3} \end{aligned}$$

Counit

$$\varepsilon(S^{\pm}) = 0 \quad , \quad \varepsilon(q^{\pm S^3}) = 1$$

Antipode

$$\begin{aligned} S(S^{\pm}) &= -q^{\mp 1} S^{\pm} \\ S(S^{\pm S^3}) &= S^{\mp S^3} \end{aligned}$$

Universal R–matrix

$$R = q^{2S^3 \otimes S^3} \sum_{n \geq 0} \frac{(1-q^2)^n}{[n]!} q^{\frac{1}{2}n(n-1)} (q^{-S^3} S^-)^n \otimes (q^{S^3} S^+)^n$$

The $U_q(sl(2))$–invariant spin chain

We set

$$V = \mathbb{C}^2 \,.$$

The state space is equal to $V^{\otimes N}$.
Representation of $U_q(sl(2))$ in V :

$$\begin{aligned} \varrho(S^{\pm}) &= \sigma^{\pm} \\ \varrho(q^{S^3}) &= q^{\frac{1}{2}\sigma^3} \,, \end{aligned}$$

i.e.,

$$\varrho(S^3) = \tfrac{1}{2}\sigma^3 \,.$$

Invariant operators in $V \otimes V$:

$$I_2 \otimes I_2 \quad \text{and} \quad B = q^{1/2}\,\psi_{V,V} \,.$$

We have

$$(B - q)(B + q^{-1}) = 0 \,.$$

For $q = 1$,

$$B = T = \tfrac{1}{2}\Big(\sum_{r=1}^{3} \sigma^r \otimes \sigma^r + I_2 \otimes I_2\Big)$$

is the twist:

$$T(v \otimes w) = w \otimes v \,.$$

Let us define

$$E = qI_2 \otimes I_2 - B\,.$$

Then we have

$$E = q^{-1}I_2 \otimes I_2 - B^{-1}$$

and

$$E^2 = (q+q^{-1})E\,.$$

Explicitly,

$$\begin{aligned} -E \;=\; & \sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+ \\ & +\tfrac{1}{4}(q+q^{-1})(\sigma^3 \otimes \sigma^3 - I_2 \otimes I_2) \\ & +\tfrac{1}{4}(q-q^{-1})(\sigma^3 \otimes I_2 - I_2 \otimes \sigma^3)\,. \end{aligned}$$

Recall that

$$\sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+ = \tfrac{1}{2}(\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2)\,.$$

Remark

Yet another invariant operator in $V \otimes V$ is $(\varrho \otimes \varrho)\Delta(C')$, where

$$C' = S^-S^+ + (q-q^{-1})^{-2}(q^{1/2}q^{S^3} - q^{-1/2}q^{-S^3})^2 - [\tfrac{1}{2}]^2$$

is a modified quadratic Casimir element, defined such that its eigenvalue in the spin $-j-$representation is equal to $[j][j+1]$. We have

$$(\varrho \otimes \varrho)\Delta(C') = (q+q^{-1})I_2 \otimes I_2 - E\,.$$

Hamiltonian of the spin chain

Set, for $1 \le i \le N-1$,

$$E_i = I_2^{\otimes(i-1)} \otimes E \otimes I_2^{\otimes(N-i-1)}$$

and then

$$H = -J \sum_{i=1}^{N-1} E_i \,.$$

The operators E_i satisfy the defining relations of the so–called Hecke algebra (of type A_{N-1}):

$$\begin{array}{lrcll}
(1) & E_i^2 & = & (q+q^{-1})E_i & \\
(2) & E_iE_jE_i - E_i & = & E_jE_iE_j - E_j & \text{for } |i-j| = 1 \\
(3) & E_iE_j & = & E_jE_i & \text{for } |i-j| \ge 2 \,,
\end{array}$$

where $i, j \in \{1, 2, \ldots, N-1\}$. In fact, if we define

$$B_i = I_2^{\otimes(i-1)} \otimes B \otimes I_2^{\otimes(N-i-1)} \,,$$

the relations (2) and (3) are just the braid group relations which are known to hold for the B_i .

Actually, instead of (2) the E_i satisfy the stronger relations

$$(2') \qquad E_iE_jE_i = E_i \quad \text{for } |i-j| = 1 \,.$$

The relations (1),(2’),(3) define the so–called Temperley–Lieb algebra.

Irreducible representations of $\tilde{U}_q(sl(2))$ when q is a root of unity

(See D. Arnaudon, Commun. Math. Phys. 159 (1994) 175 and the literature cited there.)

We consider the Hopf algebra $\tilde{U}_q(sl(2))$.

The so–called deformed quadratic Casimir element

$$C = FE + (q - q^{-1})^{-2}(qK + q^{-1}K^{-1})$$

belongs to the center of $\tilde{U}_q(sl(2))$. If q is not a root of unity, the center is generated by C. However, if q *is* a root of unity, the center is much larger. More precisely:

Let q be a *primitive* nth root of unity. Set

$$\begin{aligned} m &= n \quad \text{if } n \text{ is odd} \\ m &= \tfrac{1}{2}n \text{ if } n \text{ is even}. \end{aligned}$$

Then the elements E^m, F^m and $K^{\pm m}$ also belong to the center and, together with C, generate the center of $\tilde{U}_q(sl(2))$. Moreover, these elements satisfy a relation of the form

$$P_m(C) = E^m F^m + q^m \frac{K^m + K^{-m}}{(q - q^{-1})^{2m}} \ ,$$

where P_m is a polynomial of degree m (depending on q).

For an irreducible representation of $\tilde{U}_q(sl(2))$ the representatives of the elements

$$C \quad , \quad E^m \quad , \quad F^m \quad , \quad K^m$$

are scalar multiples of the unit operator, we denote the corresponding scalar factors by

$$c \quad , \quad x \quad , \quad y \quad , \quad z\,.$$

They satisfy the relation

$$P_m(c) = xy + q^m \, \frac{z + z^{-1}}{(q - q^{-1})^{2m}} \quad .$$

The irreducible representations fall into two families.

1) Deformed classical representations

These are characterized by an integer or half–integer 'spin' j with $0 \leq 2j+1 \leq m$ and a discrete parameter $\omega = \pm 1$. Their dimension is equal to $2j+1$, and for a suitable choice of the basis $(e_1, e_2, \ldots, e_{2j})$ of the representation space we have

$$
\begin{aligned}
K \cdot e_r &= \omega q^{2j-2r} e_r && \text{for } \ 0 \leq r \leq 2j \\
F \cdot e_r &= e_{r+1} && \text{for } \ 0 \leq r \leq 2j-1 \\
F \cdot e_{2j} &= 0 \\
E \cdot e_r &= \omega [r][2j-r+1] e_{r-1} && \text{for } \ 1 \leq r \leq 2j \\
E \cdot e_0 &= 0
\end{aligned}
$$

(modul notation). For these representations we have

$$
x = y = 0 \quad , \quad z = (\omega q^{2j})^m = \pm 1
$$

$$
c = \omega (q - q^{-1})^{-2} (q^{2j+1} + q^{-2j-1}) \, .
$$

(If q is not a root of unity, then all irreducible representations have this form.)

2) Representations without a classical analogue
These depend on three parameters x, y, z and a choice of the eigenvalue c among the (in general) m solutions $c_1, c_2, \ldots, c_m$ of the equation above. All of these representations are m–dimensional. Let λ be an m th root of z. Then, with a suitable basis $(e_1, e_2, \ldots, e_m)$ of the representation space, the representation takes the form

$$\begin{aligned} K \cdot e_r &= \lambda q^{-2r} e_r \ \text{ for } \ 0 \le r \le m-1 \\ F \cdot e_r &= b_r e_{r+1} \quad \text{for } \ 0 \le r \le m-2 \\ F \cdot e_{m-1} &= b_{m-1} e_0 \\ E \cdot e_r &= a_r e_{r-1} \quad \text{for } \ 1 \le r \le m-1 \\ E \cdot e_0 &= a_0 e_{m-1} \end{aligned}$$

with some coefficients a_r, b_r. Obviously, we have

$$a_0 a_1 \ldots a_{m-1} = x \quad , \quad b_0 b_1 \ldots b_{m-1} = y \, .$$

The representation is irreducibe provided at least one of the following conditions is satisfied:

$$x \neq 0 \, , \; y \neq 0 \, , \; z \neq \pm 1 \, , \; c \neq \frac{2\omega}{(q-q^{-1})^2} \, ,$$

where $\omega = \pm 1$.

The representations with

$$x = y = 0 \ , \ \ z = \pm 1 \ , \ \ c = \frac{2\omega}{(q - q^{-1})^2}$$

belong to the first family and are excluded here.

Representations with $x \neq 0$, $y \neq 0$ are called *cyclic*, those for which exactly one of the numbers x, y is different from zero are called *semi cyclic*. Finally, the representations of the type above with $x = y = 0$ are called *nilpotent*, they still depend on one parameter.

The quantum group $SL_q(2)$

See Yu.I. Manin, Quantum groups and non–commutative geometry, Centre de Recherches Mathématiques, Université de Montréal (1988)

We choose $q \in \mathbb{K}$, $q \neq 0$.

Definition

The coordinate algebra of the quantum plane is the universal algebra $A_q(2|0)$ with generators x_1, x_2 and the relation

$$x_1 x_2 = q x_2 x_1 \ .$$

Definition

The coordinate algebra of the quantum superplane is the universal algebra $A_q(0|2)$ with generators ξ_1, ξ_2 and the relations

$$\xi_1^2 = \xi_2^2 = \xi_1 \xi_2 + q^{-1} \xi_2 \xi_1 = 0 \ .$$

Lemma

In the following, $i, j \in \{1, 2\}$.

Let a_{ij} be four elements of some associative algebra A (with unit element). Consider the following five statements:

1) There exists a (unique) algebra homomorphism

$$\omega : A_q(2|0) \longrightarrow A \otimes A_q(2|0)$$

such that

$$\omega(x_i) = \sum_j a_{ij} \otimes x_j \ .$$

1') There exists a (unique) algebra homomorphism

$$\omega' : A_q(2|0) \longrightarrow A_q(2|0) \otimes A$$

such that

$$\omega'(x_i) = \sum_j x_j \otimes a_{ji} \ .$$

2) There exists a (unique) algebra homomorphism

$$\rho : A_q(0|2) \longrightarrow A \otimes A_q(0|2)$$

such that

$$\rho(\xi_i) = \sum_j a_{ij} \otimes \xi_j \ .$$

2') There exists a (unique) algebra homomorphism

$$\rho' : A_q(0|2) \longrightarrow A_q(0|2) \otimes A$$

such that

$$\rho'(\xi_i) = \sum_j \xi_j \otimes a_{ji} \ .$$

3) The a_{ij} satisfy the following relations:

$$a_{11}a_{12} = q a_{12}a_{11} \quad , \quad a_{11}a_{21} = q a_{21}a_{11}$$

$$a_{12}a_{22} = q a_{22}a_{12} \quad , \quad a_{21}a_{22} = q a_{22}a_{21}$$

$$a_{12}a_{21} = a_{21}a_{12}$$

$$a_{11}a_{22} - a_{22}a_{11} = (q - q^{-1})a_{12}a_{21} \; .$$

Then we have:

α) 1) is equivalent to 2')

β) 2) is equivalent to 1')

γ) 3) implies all of the other statements.

δ) 1) and 2) together imply 3) provided that $q^2 \neq -1$.

Now let $M_q(2)$ be the universal algebra with generators x_{ij} ; $i, j \in \{1, 2\}$, and the relations of statement 3) of the lemma, with a_{ij} replaced by x_{ij} .

Proposition

a) There exists a unique algebra homomorphism

$$\Delta : M_q(2) \longrightarrow M_q(2) \otimes M_q(2)$$

such that

$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$$

and a unique algebra homomorphism

$$\varepsilon : M_q(2) \longrightarrow \mathbb{K}$$

such that

$$\varepsilon(x_{ij}) = \delta_{ij} \ .$$

Endowed with these maps $M_q(2)$ becomes a coalgebra and hence a bialgebra.

b) Choose in the lemma $A = M_q(2)$ and $a_{ij} = x_{ij}$. Then the homomorphisms ω, ω' and ρ, ρ' define on $A_q(2|0)$ resp. $A_q(0|2)$ the structure of a left or right $M_q(2)$–module.

The bialgebra $M_q(2)$ is called the coordinate algebra of the quantum semigroup of quantum 2×2–matrices.

It is easy to see that

$$\rho(\xi_1\xi_2) = D \otimes \xi_1\xi_2 \quad , \quad \rho'(\xi_1\xi_2) = \xi_1\xi_2 \otimes D \ ,$$

with

$$D = x_{11}x_{22} - qx_{12}x_{21} = x_{22}x_{11} - q^{-1}x_{21}x_{12} \ .$$

The comodule properties of $A_q(0|2)$ then imply that

$$\Delta(D) = D \otimes D \ ,$$

moreover, we have

$$\varepsilon(D) = 1 \ .$$

Finally, it is easy to check that D is a central element of $M_q(2)$. The element D is called the quantum determinant of $M_q(2)$.

The quantum group $SL_q(2)$, or rather its coordinate algebra $\mathcal{K}(SL_q(2))$, can now be constructed, as follows. Let J be the ideal of $M_q(2)$ generated by $D - 1$. Then the foregoing implies that

$$\Delta(J) \subset M_q(2) \otimes J + J \otimes M_q(2)$$

$$\varepsilon(J) = \{0\} \ ,$$

i.e., that J is a so–called biideal of $M_q(2)$. Consequently, the quotient space

$$\mathcal{K}(SL_q(2)) = M_q(2)/J$$

inherits from $M_q(2)$ the structure of a bialgebra. Actually, $\mathcal{K}(SL_q(2))$ is a Hopf algebra, which can be described as follows:

$\mathcal{K}(SL_q(2))$ is the universal algebra with generators x_{ij}, $i,j \in \{1,2\}$ and relations

$$x_{11}x_{12} = qx_{12}x_{11} \quad , \quad x_{11}x_{21} = qx_{21}x_{11}$$

$$x_{12}x_{22} = qx_{22}x_{12} \quad , \quad x_{21}x_{22} = qx_{22}x_{21}$$

$$x_{12}x_{21} = x_{21}x_{12}$$

$$x_{11}x_{22} - qx_{12}x_{21} = x_{22}x_{11} - q^{-1}x_{21}x_{12} = 1 \ .$$

Coproduct

$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$$

Counit

$$\varepsilon(x_{ij}) = \delta_{ij}$$

Antipode

$$\begin{aligned} S(x_{11}) &= x_{22} \quad , & S(x_{12}) &= -q^{-1}x_{12} \\ S(x_{21}) &= -qx_{21} \quad , & S(x_{22}) &= x_{11} \ . \end{aligned}$$

Note that the matrix $(S(x_{ij}))$ is the inverse of the matrix (x_{ij}).

The finite dual of an algebra

Let A be an associative algebra with a unit element, and let A^* denote the dual vector space of A. If A is infinite–dimensional, the algebra structure of A does not, in general, induce a coalgebra structure on A^*. The following construction provides a substitute for this.

Define A° to be the set of all elements $a' \in A^*$ such that the kernel of a' contains a (twosided) ideal of A of finite codimension. It is easy to see that A° is a subspace of A^*.

Lemma

Let A be an algebra, let $m : A \otimes A \longrightarrow A$ denote the product map of A and let $m^t : A^* \longrightarrow (A \otimes A)^*$ be its transpose. If $a' \in A^*$, the following statements are equivalent:

1) $a' \in A^\circ$.
2) $m^t(a') \in A^\circ \otimes A^\circ$.
3) $m^t(a') \in A^* \otimes A^*$.
4) There exists a representation ϱ of A in a finite–dimensional vector space V and elements $x \in V$, $x' \in V^*$ such that

$$\langle a', a \rangle = \langle x', \varrho(a)x \rangle \quad \text{for all } a \in A \,.$$

Recall that the coalgebra structure of A induces by transposition an algebra structure on A^* (the unit element of A^* is the counit of A).

Proposition

1) We use the notation introduced above. Let

$$\Delta_{A^\circ} : A^\circ \longrightarrow A^\circ \otimes A^\circ$$

be the map induced by m^t (see part 2) of the lemma) and let

$$\varepsilon_{A^\circ} : A^\circ \longrightarrow \mathbb{K}$$

be the linear form defined by

$$\varepsilon_{A^\circ}(a') = \langle a', 1_A \rangle \quad \text{for all } a \in A \ .$$

Then the vector space A°, endowed with the mappings Δ_{A° and ε_{A° , is a coalgebra.

2) Suppose that A is a bialgebra. Then A° is a subalgebra of A^* which, when endowed with the coalgebra structure given in part 1), becomes a bialgebra.

3) If A is a Hopf algebra with antipode S, we have

$$S^t(A^\circ) \subset A^\circ \ ,$$

and the map

$$S_{A^\circ} : A^\circ \longrightarrow A^\circ$$

induced by S^t is an antipode of the bialgebra specified in part 2).

Dual pairings of Hopf algebras

Let A and H be two bialgebras and let

$$\varphi : A \times H \longrightarrow \mathbb{K}$$

be a bilinear form on $A \times H$. Then there exists a unique bilinear form

$$\varphi^{\otimes 2} : (A \otimes A) \times (H \otimes H) \longrightarrow \mathbb{K}$$

such that

$$\varphi^{\otimes 2}(a \otimes a', h \otimes h') = \varphi(a, h)\, \varphi(a', h')$$

for all $a, a' \in A$ and $h, h' \in H$.

The form φ is called a *dual pairing* of the bialgebras A and H if the following relations are satisfied:

$$\varphi^{\otimes 2}(\Delta_A(a), h \otimes h') = \varphi(a, hh')$$

$$\varphi^{\otimes 2}(a \otimes a', \Delta_H(h)) = \varphi(aa', h)$$

$$\varphi(a, 1_H) = \varepsilon_A(a) \quad , \quad \varphi(1_A, h) = \varepsilon_H(h)$$

for all $a, a' \in A$ and all $h, h' \in H$.

The prototype of dual pairings of bialgebras is the bilinear form on $A^\circ \times A$ induced by the canonical bilinear form on $A^* \times A$, where A is a bialgebra. Conversely, these special pairings can be used to characterize the general ones, as follows.

Let us use the notation introduced above. It is well–known that, for any bilinear form φ on $A \times H$, there exist two linear mappings

$$\varphi_\ell : A \longrightarrow H^*$$

$$\varphi_r : H \longrightarrow A^*$$

such that

$$\langle \varphi_\ell(a), h \rangle = \langle \varphi_r(h), a \rangle = \varphi(a, h)$$

for all $a \in A$ and $h \in H$ (obviously, they are uniquely determined by this equation).

Proposition
We use the notation introduced above. Then the following statements are equivalent:
1) The bilinear form φ is a Hopf pairing of A and H.
2) We have $\varphi_\ell(A) \subset H^\circ$, and the map of A into H° induced by φ_ℓ is a bialgebra homomorphism.
3) We have $\varphi_r(H) \subset A^\circ$, and the map of H into A° induced by φ_r is a bialgebra homomorphism.

The only non–trivial part of this proposition is that, for any Hopf pairing, we have

$$\varphi_\ell(A) \subset H^\circ \quad \text{and} \quad \varphi_r(H) \subset A^\circ ,$$

and this follows from the characterization of H° and A° described earlier.

Corollary
We use the notation introduced above, but we suppose in addition that A and H are Hopf algebras. If φ is a Hopf pairing of A and H, then

$$\varphi(S_A(a), h) = \varphi(a, S_H(h))$$

for all $a \in A$ and $h \in H$.

This follows from the well–known fact that a bialgebra homomorphism of a Hopf algebra into another one is, in fact, a Hopf algebra homomorphism, i.e., it intertwines the antipodes.

Example of a dual pairing

Choose a non-zero number $p \in \mathbb{K}$. Let B be the universal algebra (associative, with unit element) with generators g, g^{-1} and X and relations

$$gg^{-1} = g^{-1}g = 1$$

$$gX = pXg \; .$$

Then there exists a Hopf algebra structure on B with coproduct

$$\begin{aligned}\Delta(g) &= g \otimes g \\ \Delta(X) &= X \otimes 1 + g \otimes X\end{aligned}$$

counit

$$\begin{aligned}\varepsilon(g) &= 1 \\ \varepsilon(X) &= 0\end{aligned}$$

antipode

$$\begin{aligned}S(g) &= g^{-1} \\ S(X) &= -g^{-1}X \; .\end{aligned}$$

It is easy to see that the elements $X^m g^n$, $m \in \mathbb{N}$, $n \in \mathbb{Z}$ form a basis of B.

We are going to construct a dual pairing of B with itself. To do so we first look for some finite–dimensional representations of the algebra B.

1) For any non–zero number $c \in \mathbb{K}$, there is a unique one–dimensional representation (character) χ_c such that

$$\chi_c(g) = c \quad , \quad \chi_c(X) = 0 \, .$$

In particular, we define

$$\gamma = \chi_p \quad , \quad \gamma' = \chi_{p^{-1}} \, .$$

2) For every $\alpha \in \mathbb{K}$ there exists a unique two–dimensional matrix representation ϱ such that

$$\varrho(g) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \quad , \quad \varrho(X) = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} \, .$$

For all elements $b \in B$, the matrix $\varrho(b)$ has the form

$$\varrho(b) = \begin{pmatrix} \varepsilon(b) & 0 \\ \xi(b) & \gamma(b) \end{pmatrix} \, .$$

with some linear form ξ on B.

Obviously, γ, γ' and ξ are elements of B°, and one can prove that

$$\gamma\gamma' = \gamma'\gamma = \varepsilon$$
$$\gamma\xi = p\,\xi\gamma \, .$$

Thus there exists a unique algebra homomorphism

$$f : B \longrightarrow B^\circ$$

such that

$$f(g) = \gamma \quad , \quad f(X) = \xi \, ,$$

and it can be shown that f is a Hopf algebra homomorphism.

Consequently, f yields a dual pairing

$$\varphi : B \times B \longrightarrow \mathbb{K}$$

defined by

$$\varphi(b, b') = \langle f(b), b' \rangle \ ,$$

where $\langle \ , \ \rangle$ denotes the canonical pairing of B° and B. Explicitly, we have

$$\varphi(X^r g^s, X^m g^n) = p^{sn} \alpha^r (r)_p! \, \delta_{r,m}$$

for all $r, m \in \mathbb{N}$ and $s, n \in \mathbb{Z}$, with

$$(r)_p! = (1)_p (2)_p \dots (r)_p$$

and

$$(r)_p = \frac{p^r - 1}{p - 1} \ \text{ for all } r \in \mathbb{N} \ .$$

Using this result it is easy to see that φ is non–degenerate if and only if p is not a root of unity.

Construction of the quantum double

Definition

Let H and A be two bialgebras. A *skew pairing* of H and A is a dual pairing of the bialgebras H and A^{op} or, equivalently, of H^{cop} and A.

More explicitly, a skew pairing of H and A is a bilinear form

$$\sigma : H \times A \longrightarrow \mathbb{K}$$

such that

$$\begin{aligned} \sigma(hh', a) &= \Sigma \sigma(h, a_{(1)}) \sigma(h', a_{(2)}) \\ \sigma(h, aa') &= \Sigma \sigma(h_{(1)}, a') \sigma(h_{(2)}, a) \end{aligned}$$

$$\sigma(h, 1) = \varepsilon(h) \quad , \quad \sigma(1, a) = \varepsilon(a)$$

for all $h, h' \in H$ and $a, a' \in A$.

Definition

A bilinear form

$$\sigma : H \times A \longrightarrow \mathbb{K}$$

is said to be *convolution invertible* (or simply invertible) if the corresponding linear form on $H \otimes A$ is invertible in the convolution algebra $(H \otimes A)^*$ (where the coalgebra structure of $H \otimes A$ is the tensor product of the coalgebra structures of H and A).

More explicitly, σ is invertible if there exists a (necessarily unique) bilinear form

$$\overline{\sigma} : H \times A \longrightarrow \mathbb{K}$$

such that

$$\sum \sigma(h_{(1)}, a_{(1)})\overline{\sigma}(h_{(2)}, a_{(2)}) = \sum \overline{\sigma}(h_{(1)}, a_{(1)})\sigma(h_{(2)}, a_{(2)}) = \varepsilon(h)\varepsilon(a)$$

for all $h \in H$ and $a \in A$.

Let $\sigma : H \times A \longrightarrow \mathbb{K}$ be an invertible bilinear form with inverse $\overline{\sigma}$. Define the bilinear form

$$\overline{\sigma}_{21} : A \times H \longrightarrow \mathbb{K}$$

by

$$\overline{\sigma}_{21}(a, h) = \overline{\sigma}(h, a) \ .$$

Then σ is a skew pairing of H and A if and only if $\overline{\sigma}_{21}$ is a skew pairing of A and H.

A skew pairing $\sigma : H \times A \longrightarrow \mathbb{K}$ is always invertible if H is a Hopf algebra, or if A is a Hopf algebra with an invertible antipode: In this case we have

$$\overline{\sigma}(h, a) = \sigma(S(h), a) \quad \text{or} \quad \overline{\sigma}(h, a) = \sigma(h, S^{-1}(a) \ ,$$

respectively.

Theorem
Consider two bialgebras H and A and an invertible skew pairing $\sigma : H \times A \longrightarrow \mathbb{K}$.
1) The vector space $A \otimes H$, endowed with the product map defined by

$$(a \otimes h)(a' \otimes h') = \Sigma\, aa'_{(2)} \otimes h_{(2)} h' \,\overline{\sigma}(h_{(1)}, a'_{(1)}) \sigma(h_{(3)}, a'_{(3)}) \ ,$$

for all $a, a' \in A$ and $h, h' \in H$, is an associative algebra with the unit element $1 \otimes 1$.
2) Endowed with the tensor product coalgebra structure, this algebra becomes a bialgebra.
3) The canonical injections

$$A \longrightarrow A \otimes H \quad , \quad a \longrightarrow a \otimes 1$$

$$H \longrightarrow A \otimes H \quad , \quad h \longrightarrow 1 \otimes h$$

are homomorphisms of bialgebras.
4) If H and A are Hopf algebras, then $A \otimes H$ is a Hopf algebra as well, and its antipode is given by

$$S(a \otimes h) = (1 \otimes S(h))(S(a) \otimes 1) \ ,$$

for all $a \in A$ and $h \in H$.

The bialgebra (resp. Hopf algebra) defined in the theorem is called the generalized *quantum double* of H and A with respect to σ and will be denoted by $\mathcal{D}(H, A; \sigma)$.

Recall that if σ is an invertible skew pairing of H and A, then $\overline{\sigma}_{21}$ is an invertible skew pairing of A and H. Thus the generalized quantum double $\mathcal{D}(A, H; \overline{\sigma}_{21})$ is defined as well. Actually, the linear map

$$\theta : \mathcal{D}(H, A; \sigma) \longrightarrow \mathcal{D}(A, H; \overline{\sigma}_{21})$$

given by

$$\theta(a \otimes h) = \sum \sigma(h_{(1)}, a_{(1)})\, h_{(2)} \otimes a_{(2)}\, \overline{\sigma}(h_{(3)}, a_{(3)})$$

is an isomorphism of bialgebras.

Theorem

Consider two *finite–dimensional* bialgebras H and A and a *non–degenerate* invertible skew pairing $\sigma : H \times A \longrightarrow \mathbb{K}$. Choose a basis $(h_i)_{i \in I}$ of H, and let $(a_i)_{i \in I}$ be the corresponding dual basis of A, i.e.,

$$\sigma(h_i, a_j) = \delta_{ij} \ .$$

Then

$$R = \sum_{i \in I} (a_i \otimes 1) \otimes (1 \otimes h_i)$$

is a universal R–matrix of $\mathcal{D}(H, A; \sigma)$.

For more details, see, for example: A. Klimyk, K. Schmüdgen, Quantum groups and their representations, Springer–Verlag, 1997.

Example

Consider the Hopf algebra B discussed earlier, constructed with $p = q^2$, where q is a non–zero element of $\mathbb{K}$. Later on we shall assume that $q \neq \pm 1$. Recall that B is the universal algebra with generators g, g^{-1} and X and relations

$$gg^{-1} = g^{-1}g = 1$$
$$gX = q^2 Xg ,$$

and that the coproduct is given by

$$\begin{aligned} \Delta(g) &= g \otimes g \\ \Delta(X) &= X \otimes 1 + g \otimes X . \end{aligned}$$

Recall also that we have constructed a non-degenerate symmetric dual pairing φ of B with itself. We choose

$$H = B^{\mathrm{cop}} \quad , \quad A = B .$$

Regarded as a map of $B^{\mathrm{cop}} \times B$ into $\mathbb{K}$, we shall denote φ by σ,

$$\sigma : B^{\mathrm{cop}} \times B \longrightarrow \mathbb{K} .$$

Then σ is an invertible skew pairing of B^{cop} and B, and hence the generalized quantum double $\mathcal{D}(B^{\mathrm{cop}}, B; \sigma)$ can be constructed. To simplify the notation, we shall use the canonical injections to identify B^{cop} and B with sub–Hopf–algebras of the double.

As before, the generators of B will be denoted by $g,\ X$, whereas those of B^{cop} will be denoted by $\overline{g},\ \overline{X}$. Using the definition of the product in $\mathcal{D}(B^{\mathrm{cop}}, B; \sigma)$, we obtain the following cross relations for the generators of B^{cop} and B:

$$X\overline{X} - \overline{X}X = -\alpha(g - \overline{g})$$

$$\overline{g}X = q^{-2}X\overline{g} \quad , \quad \overline{X}g = q^2 g\overline{X}$$

$$\overline{g}g = g\overline{g} \ .$$

It is easy to see that $g\overline{g}$ is a central element of the double, moreover, it is grouplike, i.e., we have

$$\Delta(g\overline{g}) = g\overline{g} \otimes g\overline{g} \quad , \quad \varepsilon(g\overline{g}) = 1 \ .$$

The latter implies that the ideal J of $\mathcal{D}$ generated by $g\overline{g} - 1$ is, in fact, a so–called Hopf ideal. This implies that $\mathcal{D}/J$ is a Hopf algebra as well. Let

$$\pi : \mathcal{D} \longrightarrow \mathcal{D}/J$$

be the canonical projection. Setting

$$\pi(X) = E \quad , \quad \pi(\overline{X}) = F \quad , \quad \pi(g) = K \ ,$$

it follows that

$$\pi(\overline{g}) = K^{-1} \ ,$$

and it is then easy to see that $\mathcal{D}/J$ is isomorphic to the Hopf algebra $\tilde{U}_q(sl(2))$ defined earlier, provided we set

$$\alpha = -(q - q^{-1})^{-1} \ .$$

Actually, the value of α is arbitrary, as long as it is different from zero: A change of α simply amounts to a rescaling of E and/or F.