Testing Theories:
Finding Functional Fixed Points for Pinned Manifolds

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Organization

- Reverse historical approach.
- “Experimental” talk.
- See cond-mat/0606160.
- [Reminded of ancient Greek theater festivals.]
This is a glass talk, so we need this diagram

\[ F(\vec{x}) \]
However, we will mostly see this

\[ F(x) \]
How Computer Scientists Taught Physicists to Be Lazy

Physicists want: low $E$, long $t$, large $\lambda$ behavior of complex, heterogeneous systems, e.g., random magnets, superconductors with dirt.

- The ground state (or even partition fn. $Z$) can often be computed very quickly, even when the physical system has many local minima and extremely slow dynamics.

- This speed can be exploited in models with quenched disorder
  - to precisely study phase transitions
  - to study the effects of perturbations
  - to answer qualitative questions (e.g., # of states)

- Warning: some reasonable physical systems have no known fast algorithms for all cases. These correspond to NP-hard problems.
To study materials, learn computer science

Rather informally:

- A **decision problem** is one for which one replies yes/no for a given input.
- The set **P** consists of decision problems that can be solved in time bounded by a polynomial $N^k$ in the problem length $N$. “Tractable”.
- The set **NP** (“nondeterministic polynomial”) consists of decision problems for which “yes” answers can be **verified** in time polynomial in $N$. 
P and NP

Example decision problem instance:

Can you find a train itinerary from Trieste to Dresden that takes less than 15 hours?
[Shortest path problem is in \( P \).]

\( P \subset NP \), but we don’t know if \( P = NP \).
Which problems are tractable?

- Domain walls, random bond ferromagnet
- Partition fn., 2D EA spin glass
- Shortest paths
- Random field Ising magnet g.s. (any dim.)
- Many intersecting lines in random potential (any dim.)
- Coulomb glass ground state
- Random field Ising magnet partition fn. or highest state
- 3D spin glass ground state
- Barrier to motion of loop in plane

$P$
How accurate for $P$?

AS EXACT AS YOUR INPUT:

* The algorithms expand the configuration space.
* The “rough landscape” is smooth and downhill* in this space.
  * At the “bottom”, translate back to a physical solution, . . . which is guaranteed to be the exact g.s.

- The combinatorial math and particular rep’ns are often unfamiliar to physicists.
- But we are used to imaginary time for QM, e.g.
The Cover to the Program

Ferromagnets
S. Lemerle et al.
PRL 80 849 (98)

Ferroelectrics
P. Paruch et al. cond-mat/0411178

Contact line, wetting
S. Moulinet, E. Rolley

[Collection “courtesy” of T. Giamarchi]
Inspiration

- Statics of surfaces pinned by disorder
  - Domain walls in random magnets, contact lines on a rough surface, vortex lines in superconductor, electron world lines in a space AND time dependent potential, periodic scalar fields, e.g., vortex-free superconductors.

- “Simplest” finite-\(d\) glassy phases (?)
  - Elastic, no plastic rearrangements.
  - At low \(T\), disorder is irrelevant . . .
    * Theme of dramatic tension: elasticity v. disorder

- Characterize by roughness, \(w \sim L^\zeta\), energy fluctuations \(\sim L^\theta\). Statics are preliminary to
  - barriers to equilibration
  - dynamics (creep or sliding) in disordered background.
Plot Summary

The effective long wavelength pinning potential for $d < 4$ interface is universal (depends on symmetries of pinning potential).

⇒ Find fixed points for force-force correlation functions $\Delta(u)$.
⇒ Quantitatively confirm shape of $\Delta(u)$.

• First evidence for cusp at zero $u$ (20 yrs)
• “Chaos” (sensitivity to disorder)
• Universal amplitudes.
Production Crew

P. Le Doussal, K. Wiese, AAM, and 100 1GHz processors.
⇒ C++ code to find **exact ground state** for discrete interfaces $u(x)$ in dimensions $d = 1, 2, 3, 4, \ldots$ with

- User-defined lattices.
- Choice of disorder correlations, corresponding to
  - Random field (RB): $\langle [U(u', x') - U(u, x)]^2 \rangle = |u - u'| \delta(x - x')$
  - Random bond (RF): $\langle [U(u', x') - U(u, x)]^2 \rangle = e^{-|u-u'|} \delta(x - x')$
  - Periodic pinning (RP): $\langle [U(u', x') - U(u, x)]^2 \rangle = \sin\left[\frac{2\pi(u-u')}{P}\right] \delta(x - x')$
- Add in a moving harmonic well to the disorder [P. Le Doussal].

$$U_{\text{harmonic}}[u(x)] = \frac{m^2}{2} (u - v)^2$$

Simulation uses rolling disorder and can incrementally find $v \rightarrow v + \delta v$. 
The Play

**Act 1**: Random field pinning, $D = 2+1$ interface, $m^2 = 0.1$, $L \times W = 20 \times 20$, $\delta v = 0.04$, 100 steps.

**Act 2**: Same interface, but $m^2 = 0.01$

**Act 3**: Back to scene 1, but highlights: **avalanches/droplets**.

**Act 4**: The **shocking** events from scene 2.
Critics: quantify? context?

L=8, RF, single sample

\[ \langle \nu \rangle \]

\[ m^2 = 0.02 \]
Theory - Functional Renormalization Group

FRG seems to be a controlled verifiable approach to manifolds in a disordered potential.

• Below $d = 4$, $\infty$ number of relevant operators and metastability.

• Writing $\langle [V_\ell(u, \vec{x}) - V_\ell(0, \vec{0})]^2 \rangle = -2R_\ell(u)\delta(\vec{x})$, D. S. Fisher (1986) derived flow equations, using $\Delta(u) = -R''(u)$,

$$\frac{d\Delta(u)}{d\ell} = (\epsilon - 4\zeta)\Delta(u) + \zeta u\Delta'(u) + \frac{1}{2} [\Delta''(u)]^2 - \Delta''(u)\Delta''(0)$$

• *Non-analytic fixed points:* $\Delta(u)$, force-force correlations, have a **cusp** at $u = 0$. 
Relevance

$R(u)$ and its derivatives $\Rightarrow$ the physical picture of pinned interfaces:

- Fisher, Narayan, Balents; Balents, Bouchaud, Mezard (1986-1996): sequence of scalloped potentials $[\text{singularity in } R(u)]$ due to hopping between metastable states, suggestive connections to Burgers equation.

- Le Doussal, recently: scallops derived from harmonic well + disorder; precise connection to Burgers equation.

- Fixed points for flow of $R(u)$ gives exponent $\zeta$ for roughness, etc.

- Finite drive, changing disorder $[\text{"chaos"}]$, and temperature round out the singularity at different scales $[\text{zero pinning force } \Delta''(0)]$. 
Measured correlations vs. 1-loop predictions

- Compute fixed point: large enough $L$, small enough $m$, so that

$$\tilde{\Delta}[m(v - v')]\zeta = m^{\epsilon-4\zeta-d[v' - \langle u(v')\rangle][v - \langle u(v)\rangle]}$$

is converged.

- Rescale to $Y(u) = \tilde{\Delta}(u)/\tilde{\Delta}(0)$ and scale $z = um\zeta$ to get $\int Y = 1$ (RF), $\int Y^2 = 1$ (RB).
Measured correlations vs. 1-loop predictions

-0.2
0
0.2
0.4
0.6
0.8
1

Y(z)
0 1 2 3 4

z (z/4 for RB)

Y_{RF}

Y_{RB}

Y(z), 1-loop RF

RF, d=3, L=16

Y(z), 1-loop RB

RB, d=2, L=32
Where one form of the 2-loop prediction is $Y(z) = Y_1(z) + (4 - d)Y_2(z)$
Residuals, RB

Where one form of the 2-loop prediction is $Y(z) = Y_1(z) + (4 - d)Y_2(z)$
General prediction: $Y(z)$ is a parabola with zero mean (i.e., $6(z - \frac{1}{2})^2 - \frac{1}{2}$).
“Chaos” (sensitivity to disorder)

Recent predictions by P. Le Doussal [PRL 96, 235702 (2006)] for correlations

$$\Delta_{12}(y) = \langle [v + y - u_1(v + y)][v - u_2(v)] \rangle$$

between samples with disorders $U_1$ and $U_2$, with difference measured by $\delta$.
We can check this - shapes of curves (1 adjustable parameter).

![Graph showing the correlation function $\Delta_{12}(y)$ with different values of $\delta$.](image)

**RF, $D=3+1$, $L=16$, $m^2=0.02$**
Chaos

Normalized $\Delta_{12}(y)$, fixed perturbation $\delta$

$\Delta_{12}(0)/\Delta_{11}(0)$, varying $\delta$ [parameter free ratio]

\[ Y_s(z) = (1+d^2)^{-1/2} \]

- $Y_s(z)$, $\delta=0.8$ chaos, 1-loop RF
- RF, $d=2$, $L=16$, $\delta=0.8$, $Y_s(z)$

1-loop expansion
- $D = 0+1$
- $D = 2+1$, $L = 16$, $m^2 = 0.02$
- $D = 2+1$, $L = 32$, $m^2 = 0.005$
- $D = 3+1$, $L = 8$, $m^2 = 0.02$
**Functional Burgers Equation**

\[ d = 0: \text{particle in a single } V(u) \text{ given by a random walk } + \frac{m^2}{2}(u - v)^2. \]

*Exact correspondence* between \( v - \langle u \rangle \) and velocity in Burgers equation, given \( t \to m^{-2}, V \to v - \langle u \rangle, \nu \to t \): jumps in \( \langle u \rangle \) are shocks in 1D decaying Burgers equation.

\[ \partial_t V + V \partial_x V = \nu \partial_x^2 V \]

Functional equation: formally similar.
Consequences:
In a single sample, see coalescence of jumps as decrease \( m^2 \).
Sequence of $m^2$ in a single sample

$L=8$, RF, single sample

$v-\langle u \rangle$

$\langle u \rangle = 0.02$
Sequence of $m^2$ in a single sample

$L=8$, RF, single sample

$v - \langle u \rangle$

-2
-1
0
1
2

$v$

8 10 12 14 16

$v - \langle u \rangle$

-2
-1
0
1
2

$v$

8 10 12 14 16

$red m^2 = 0.02$

$blue m^2 = 0.016$
Sequence of $m^2$ in a single sample

$L=8$, RF, single sample

$v - \langle u \rangle$

$v$

$m^2 = 0.02$
$m^2 = 0.016$
$m^2 = 0.01$
Highlights & Sequels

- Can precisely study disordered systems.
- Confirmed prediction for nonanalytic form for pinned manifolds: *linear cusps* in *force-force correlator* \( \Delta(u) \) [20 years ago].
- One-loop calculation appears to be unreasonably good, but *not the full story* for RF, RB; RP shows expected exact parabola.
- Supports exponent values, validates approach, *physical picture*.
- Functional *decaying Burgers eqn.* for \( v - u(x) \).