Spin transport

Lecture given at the International School on Quantum Information

Cord Müller, MPIPKS Dresden, September 11, 2005

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Outline

Classical information processing uses charge encoding, represented for instance by states |0⟩ and |1⟩ of a supplementary electron present or absent in a quantum dot. Since they interact via long-range Coulomb forces, charge states suffer strongly from decoherence. A natural degree of freedom to encode information is the spin. The spin \( \frac{1}{2} \) states, typically noted |±1⟩ or |↑, ↓⟩ are the natural realization of a qubit. Information needs physical carriers. Candidates here are electrons (massive particle of spin \( s = \frac{1}{2} \)) and photons (massless, spin \( s = 1 \)).

**Spintronics:** Spin-based information transport and processing with electrons and photons.

The Leitmotiv of this lecture: Spin is a geometrical quantity. Use irreducible representations of the rotation group to take advantage of the symmetries of the system.

Recommended reading for a rapid introduction: Chap. 1 in the book by Chaichian and Hagedorn [1], chaps. 4 and 8 in the book by Blum [2], the recent review on “Spintronics” by Žutić, Fabian, and Das Sarma [3].

1. Irreducible spin superoperators

1.1. Spin

General references on spin: good angular momentum theory books, for example [1, 4]

1.1.1. Historical background

“Spin” is internal angular momentum. The name was coined by Uhlenbeck and Goudsmith [5] following W. Pauli [6] who postulated the existence of a
fourth quantum number to explain fine-structure features of atomic spectra. The clearest experimental manifestation of quantized spin is arguably the Stern-Gerlach experiment \cite{7} where silver atoms are deviated by an inhomogeneous magnetic field into two distinct spots on a detector screen.

P.A.M. Dirac showed that bispinors (vectors of four components, a spinor and an anti-spinor) appear naturally when one looks for a Schrödinger-type wave equation in a relativistic kinetic framework (Poincaré group of translations, rotations and Lorentz transformation of the four-dimensional Minkowski space-time). E. Wigner \cite{8} showed that spin is one of the fundamental quantum numbers that permits to identify an elementary particle in the first place: it characterises the particle’s properties under rotations in its proper rest frame.

1.1.2. Rotation group

Physical objects are described by coordinates $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with respect to a reference frame in configuration space. Rotations (called “active” when the object is turned and “passive” when the reference frame is turned) are represented by $3 \times 3$ matrices: $x' = Rx$. Proper rotations conserve the euclidean scalar product $x \cdot y = \sum_i x_i y_i$ and the orientation of the frame. The rotation matrices are therefore members of SO(3), the set of orthogonal matrices $RR^t = R^t R = 1_3$ of unit determinant $\det R = +1$. With the usual matrix multiplication as an internal composition law, these matrices form a group, satisfying the group axioms:

1. Internal composition: $\forall R_{1,2} \in \text{SO}(3): R_{21} = R_2 R_1 \in \text{SO}(3)$;
2. Existence of the identity: $\exists E: RE = ER = R \forall R \in \text{SO}(3)$ with $E = 1_3$;
3. Existence of the inverse: $\forall R \exists R^{-1}: RR^{-1} = R^{-1} R = E$.

The group is nonabelian because the matrices do not commute: $R_2 R_1 \neq R_1 R_2$. An exception are rotations of the plane around one and the same axis, forming the group SO(2).

A possible parametrisation of a rotation is the polar description $(\hat{n}, \theta) =: \theta$ with $\hat{n}$ the unit vector along the rotation axis and $\theta \in [0, \pi]$ the rotation angle. The following two rotations of configuration space are identical:

$$R(\hat{n}, \theta = \pi) = R(-\hat{n}, \theta = \pi). \quad (1)$$

These opposite points must be identified such that there are closed parameter curves that can not be reduced to a single point. This means that SO(3) is a doubly connected manifold. Instead of studying this “projective group”, one may also turn to its “universal covering group” SU(2), the group of all unitary $2 \times 2$ matrices over $\mathbb{C}$ with unit determinant. SU(2) is simply connected, and there is a homomorphism (a mapping preserving the group structure) from every element $U \in \text{SU}(2)$ to the rotation $R \in \text{SO}(3)$: the rotation $x' = R(\theta)x$ is described by

$$x' \cdot \sigma = U(\theta)x \cdot \sigma U(\theta)^\dagger. \quad (2)$$
Here, \( \mathbf{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) are the Pauli matrices. The unitary rotation matrix acting from the left and from the right is given by

\[
U(\theta) = \mathbb{1}_2 \cos \frac{\theta}{2} - i \mathbf{n} \cdot \mathbf{\sigma} \sin \frac{\theta}{2} = \exp(-i \frac{\mathbf{\theta} \cdot \mathbf{\sigma}}{2}).
\]  

(3)

Note that this mapping is two-to-one (\(-U \) and \(U\) yield the same rotation). Note that it takes a \(4\pi\)-angle to recover the identity transformation: \(U(\theta = 2\pi) = -\mathbb{1}_2\), but \(U(\theta = 4\pi) = +\mathbb{1}_2\). This is not a mysterious quantum property as is sometimes stated, but reflects the non-connectedness of \(\text{SO}(3)\)-rotations of reference frames. Dirac’s construction of a solid body connected by strings to a reference frame is supposed to convey an “experimental” idea of this property\footnote{4}. 

1.1.3. Representations

A group \(G\) can act in many different disguises that share the same abstract group structure, as defined by the multiplication law or group table. These different transformations are called representations: they are homomorphisms \(D_1 : G \to \text{GL}(V_1)\) into (linear) regular transformations \(D : V_i \to V_i\) of a vector space \(V_i\) into itself (\(D\) standing for the German word “Darstellung”, in anglo-saxon literature, often the symbol \(\Gamma\) is used). “Homomorphism” means that the representation has the same group structure as \(G\). Notably, for all elements \(g_1, g_2 \in G\) (with the product \(g_2g_1 \in G\)) one has \(D(g_2g_1) = D(g_2)D(g_1)\).

\(\dim V_i\) is called the dimension of the representation \(D_i\). Finite-dimensional linear representations (all that we will see in this lecture) are given in terms of quadratic matrices of size \(\dim V_i\). A representation is called reducible if there is a basis of \(V = V_1 \oplus V_2\) such that all transformations \(D \in D\) are written

\[
D = \begin{pmatrix} D_1 & * \\ 0 & D_2 \end{pmatrix}.
\]

(4)

In other words: the transformations \(D_1 : V_1 \to V_1\) are already a representation \(D_1\) of its own. If the matrix is block-diagonal (\(* = 0\), the representation is completely reducible. If it is not reducible, it is called irreducible, meaning that one has achieved to work in the smallest possible subspace.

The rotation group \(\text{SU}(2)\) is a manifold that depends on a continuous set of parameters \(\mathbf{\theta} = (\theta_1, \theta_2, \theta_3)\) with respect to which it is infinitely differentiable. This structure is called a Lie group. The group multiplication law is completely determined by the commutation relation of its generators:

\[
[S_j, S_k] := S_j S_k - S_k S_j = i\hbar \epsilon_{jkl} S_l.
\]

(5)

These generators are said to form a Lie algebra. In the so-called natural representation of \(\text{SU}(2)\) by itself [eq. (3)], the generators are \(S = \frac{\hbar}{2} \mathbf{\sigma}\). Other representations will feature different generators, but all group representations share the same commutation relation! Finite rotations are generated by exponentiation: \(U(\mathbf{\theta}) = \exp(-i \mathbf{\theta} \cdot \mathbf{S}/\hbar)\). Topologically speaking, \(\text{SU}(2)\) is compact.
According to a general theorem, all representations of a compact Lie group are completely reducible to finite-dimensional irreducible representations.

Including the orbital part $L$ of angular momentum, one defines the total angular momentum $J = L \otimes I_S + I_L \otimes S =: L + S$. For the remainder of this section, we will consider only the spin part.

What is quantum about spin?

As pointed out above, reference frames for solid bodies introduces half-integer spin (E. Cartan already in 1913 [9]). Half-integer spin does not require special relativity, it arises also in Galilean relativity. But there are genuine quantum features:

1. If the Hamiltonian is invariant under infinitesimal transformations, the generators $T$ of rotations $U = \exp(-i\mathbf{T} \cdot \mathbf{T})$ are conserved quantities (Noether theorem):

$$H' = UHU^\dagger \iff [H, U] = 0 \iff [H, S] = 0 \iff \dot{S} = 0.$$  \hspace{1cm} (6)

In quantum mechanics, these generators are actually observables with dimension of angular momentum ($\bar{S} = \hbar \mathbf{T}$).

2. The observables $S$ generate irreducible representations $D^{(s)}$ of dimension $d_s = 2s+1$ with $s = 0, \frac{1}{2}, \ldots$ and discrete magnetic quantum numbers $m = -s, -s+1, \ldots, s$. States are noted $|sm\rangle$. The so-called Casimir operator $S^2$ tells us which irreducible representation: $S^2|sm\rangle = \hbar^2 s(s+1)|sm\rangle$, whereas the magnetic quantum number gives the projection of the spin onto the quantization axis (usually called the $z$-axis): $S_z|sm\rangle = \hbar m|sm\rangle$.

3. States can be classified regarding their transformation properties (cf. the multipole expansion for charge distributions in classical electrodynamics), but here with a more general importance due to the superposition principle. Example: an atomic $s$-orbital (“superposition of all possible Kepler orbits”) is invariant under all rotations, i.e., transforms under $D^{(0)}$.

1.1.4. Irreducible tensor operators

An irreducible tensor operator by definition is a set of $2K+1$ components $T_q^{(K)}$, $q = -K, -K + 1, \ldots, K$, that transform under irreducible representations of rank $K$ (i.e., whose transformation does not mix different $K$):

$$(T_q^{(K)})' = UT_q^{(K)}U^\dagger = \sum_{q=-K}^{K} D_{qq'}^{(K)} T_q^{K}$$  \hspace{1cm} (7)

Equivalently, one specifies the infinitesimal rotation properties by requiring

$$[J_\pm, T_q^{(K)}] = \hbar \sqrt{K(K+1)} - q(q \pm 1) T_q^{(K)}$$  \hspace{1cm} (8)

$$[J_0, T_q^{(K)}] = \hbar T_q^{(K)}$$  \hspace{1cm} (9)
Here, the angular momentum raising and lowering operators are $J_{\pm} = J_x \pm i J_y$.

Examples:

(i) Scalar operator: a single quantity $T^{(0)}_0$ that commutes with all components of the total angular momentum $J$. For instance $J^2$, the Casimir operator indexing the irreducible representations.

(ii) Vector operator: Its components $A = (A_1, A_2, A_3)$ satisfy

$$[J_j, A_k] = i\hbar \epsilon_{jkl} A_l$$

(iii) Tensor operator of rank 2: direct product of 2 vectors $T^{ij} = A_i B_j$ can be decomposed into irreducible representations of rank $K = 0, 1, 2$.

Exercise 1: Is the Hamiltonian $H = \frac{p^2}{2m} - \mu S \cdot B$ of a free massive particle coupled via the magnetic moment $\mu = \mu S$ of its spin $S$ to an external magnetic field $B$ an irreducible tensor operator? If so, determine its rank.

1.2. Spin superoperators

1.2.1. Unitary spin dynamics

As a simple mode for spin dynamics, we shall study the Hamiltonian

$$H = -\frac{\mu}{\hbar} S \cdot B$$

It describes the coupling of the magnetic moment $\mu = \mu(S/\hbar)$ to a magnetic field $B$. For electrons, $\mu = -g \mu_B$ in terms of the Bohr magneton $\mu_B = |e|\hbar/(2mc)$ and the gyromagnetic ratio $g = 2.003 \ldots$ in vacuum. The density matrix or statistical operator $\rho$ of a spin $S$ is a positive linear operator of trace unity on the state Hilbert space $\mathcal{H}_s = \mathbb{C}^{d_s}$ of dimension $\dim \mathcal{H}_s = d_s = 2s + 1$.

According to one of the fundamental axioms of quantum theory, any closed system shows unitary dynamics governed by the Liouville-von Neumann equation $i\hbar \partial_t \rho = [H, \rho]$.

1.2.2. Superoperators

Rewriting the LvN-equation as $i\hbar \partial_t \rho = \mathcal{L} \rho$ defines the Liouvillian

$$\mathcal{L} = \frac{1}{\hbar}[H, \cdot]$$

The linear operators on $\mathcal{H}_s$ are themselves elements of a linear vector space (we can add operators and multiply them by complex numbers). This vector space is called Liouville space $L(\mathcal{H}_s)$ and is spanned, for example, by the basis of dyadics

$$|m\rangle\langle n| =: |mn\rangle, \quad n, m = 1, \ldots, d_s$$

induced by the basis vectors $|n\rangle$ of $\mathcal{H}_s$. As a operator between operators, the Liouvillian $\mathcal{L} : L(\mathcal{H}_s) \rightarrow L(\mathcal{H}_s)$ is called a superoperator [10] [11] [12] [13].

The matrix elements of the Liouvillian [12] in the dyadic basis are

$$(mn|\mathcal{L}|m'n') = \mathcal{L}_{mn,m'n'} = H_{m'n'} \delta_{nn'} - H_{n'n} \delta_{mm'}$$
Exercise 2:  Show that the eigenvalues of the Liouvillian in the basis \( \{|mn\} \) induced by the energy basis \( H|m\rangle = \varepsilon_m |m\rangle \) are the possible transition frequencies \( \omega_{mn} = (\varepsilon_m - \varepsilon_n)/\hbar \). These are experimentally accessible quantities, in contrast to the absolute energy eigenvalues of the Hamiltonian \( H \).


1.2.3. Non-unitary spin dynamics

“Relaxation, damping, dephasing, decoherence,...” have in common irreversible dynamics (“arrow of time” \( [16] \)) due to the irrevocable loss of information into inobservable degrees of freedom (“bath, environment”).

As an introductory model, we consider \( H = -\mu S \cdot B(t) \) with a fluctuating classical magnetic field. Predictions about the spin will involve an average over the uncontrolled \( B \)-field which we will consider as a classical noise (in principle, its dynamics can of course be traced back to a quantum description of moving charges somewhere, but for the present purpose this is not necessary). As a stochastic process \( [17] \), the field is completely specified by its correlation functions

\[
C_{i_1,i_2,...,i_n}(t_1,t_2,...,t_n) := B_{i_1}(t_1)B_{i_2}(t_2)...B_{i_n}(t_n).
\]

The fluctuating field is taken to be Gaussian such that only its first two moments need to be specified, \( B_i(t) = 0 \) (no static field) and \( B_i(t_1)B_j(t_2) =: B^2c_{ij}(t_1,t_2) \).

We assume a stationary process that depends only on the time difference \( t_1 - t_2 \) and has a very short internal correlation time \( \tau_c \) such that \( c_{ij}(t) = c_{ij}\tau_c\delta(t) \). This last assumption of white noise assumes that the noise correlation time \( \tau_c \) is much shorter than the relevant time scale on which we want to study the system dynamics. Lastly, we assume that the fluctuations are isotropic, \( c_{ij} = \frac{1}{3}\delta_{ij} \).

To obtain the effective dynamics, we develop the time-propagated density matrix \( \rho(t + \Delta t) = U(t + \Delta t)\rho(t)U(t + \Delta t)^\dagger \) to second order in the interacting Hamiltonian \( [11] \)

\[
\rho(t + \Delta t) = \rho(t) - i \int_{t}^{t+\Delta t} [H(t'), \rho(t')]dt' \\
- \frac{1}{2} \mathcal{T} \int_{t}^{t+\Delta t} (H(t_1)H(t_2)\rho(t) + \rho(t)H(t_1)H(t_2))dt_1dt_2 \\
+ \int_{t}^{t+\Delta t} H(t_1)\rho(t)H(t_2)dt_1dt_2
\]

Now we perform the average over the field fluctuations. The term linear in \( H \) disappears because \( B \) is zero on average. In the second order terms, one of the time integrations is contracted by the correlation function \( \delta(t_1 - t_2) \); the remaining integrand is time independent such that the integrals gives just a

\[\text{In 2.2.2, we will formulate an equivalent way of obtaining an effective master equation}\]
factor $\Delta t$. The effective equation of motion for the averaged density matrix on a coarse-grained time scale $\Delta t \gg \tau_c$ is then given by
\[
\lim_{\Delta t \to 0} \frac{\rho(t + \Delta t) - \rho(t)}{\Delta t} =: \dot{\rho}(t) =: \mathcal{L}\rho(t)
\]
with the *Lindbladian* \[\mathcal{L}\]

\[
\mathcal{L}\rho = -\frac{1}{2\tau_s} \sum_i [S_i, [S_i, \rho]]
\]

(18)

Its prefactor, setting the new time scale of evolution, is the spin relaxation rate $1/\tau_s := \omega_0^2/\tau_c$ given in terms of the effective Larmor frequency $\omega_0 = \mu \sqrt{B^2/3}/\hbar$. The Lindbladian is a pure relaxation superoperator in the Lindblad form $\mathcal{L}\rho = \sum_i [S_i, \rho S_i] + [S_i, \rho S_i]$ which assures the complete positivity of the time-evolved density matrix.

**Exercise 3:** Show that the superoperator matrix elements of the Lindbladian are given by
\[
\mathcal{L}_{mn,m'n'} = \frac{1}{\tau_s} \left( S_{mm'} \cdot S_{n' n} - s(s + 1) \delta_{mm'} \delta_{n'n'} \right)
\]
and verify the trace-preserving property $\sum_{m} \mathcal{L}_{mm,m'n'} = 0$ from this expression.

From now on, we omit the overline and write the Lindbladian $\mathcal{L}$ as before.

### 1.3. Irreducible spin superoperators

Formally, the solution of (17) is very simple: $\rho(t) = \exp[\mathcal{L}t] \rho(t)$. In Liouville space, this a matrix equation $|\rho(t)\rangle = \exp[\mathcal{L}t] |\rho\rangle$ of dimension $d_s^2 \times d_s^2$. For spin $\frac{1}{2}$, diagonalizing a $4 \times 4$ matrix is elementary, but already for spin 1 with $d_s^2 = 9$ this becomes cumbersome. The task therefore is to diagonalize $\mathcal{L}$ intelligently, i.e., using the rotational symmetries.

#### 1.3.1. Scalar relaxation process: what results we should expect

The Lindbladian was obtained by an isotropic average and is thus a *scalar* object, i.e., invariant under rotations (“transforms under the trivial representation $D^{(0)}$”). We may anticipate that the statistical operator can be decomposed $\rho = \sum_K \rho^{(K)}$ into parts that transform under the irreducible representations $D^{(K)}$ of the rotation group. In a adapted basis of Liouville space, the Lindbladian as a scalar object can only connect subspaces of equal rank $K$. Furthermore, inside each subspace it can not distinguish between different orientations and must be diagonal as well. Thus, it will be written as a purely diagonal matrix
\[
\mathcal{L} = \begin{pmatrix}
\lambda_0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_{d_s^2}
\end{pmatrix}
\]

(20)
How many different eigenvalues may we expect? The total number is the dimension of the Liouville space $d_s^2$. Each subspace of rank $K$ will have dimension $d_K = 2K + 1$. Therefore, we will have to find only $2s + 1$ different eigenvalues $\lambda_K$, each of which has degeneracy $2^K + 1$:

<table>
<thead>
<tr>
<th>particle</th>
<th>$s$</th>
<th>$d_s^2$</th>
<th>eigenvalues</th>
<th>degeneracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>electron</td>
<td>$\frac{1}{2}$</td>
<td>2</td>
<td>$\lambda_0, \lambda_1$</td>
<td>1, 3</td>
</tr>
<tr>
<td>photon</td>
<td>1</td>
<td>3</td>
<td>$\lambda_0, \lambda_1, \lambda_2$</td>
<td>1, 3, 5</td>
</tr>
</tbody>
</table>

1.3.2. State multipoles

In the usual Hilbert space basis, the statistical operator reads

$$\rho = \sum_{m,m'} \rho_{mm'} |sm\rangle \langle sm'|$$  \hspace{1cm} (21)

Here, the ket $|sm\rangle$ transforms under the irreducible representation $D^{(s)}$, whereas the bra $\langle sm'|$, as its complex conjugate, transforms under $(D^{(s)})^\ast$, the contragredient representation. The ket-bra $|sm\rangle\langle sm'|$ transforms under the direct product $D^{(s)} \otimes (D^{(s)})^\ast$ which is reducible. One introduces therefore an ensemble of elements that do transform under the irreducible representation $D(K)$,

$$T_q^{(K)} := T_q^{(K)}(s,s) := \sum_{m,m'} (-1)^{r-m} \langle ssm' - m|Kq|sm'| \rangle |sm\rangle$$  \hspace{1cm} (22)

with $K = 0, 1, \ldots, 2s$ and $q = -K, -K + 1, \ldots, K$. The Hermitian conjugate is $(T_q^{(K)})^\ast = (-)^q T_{-q}^{(K)}$.

The Clebsch-Gordan coefficients $\langle s_1 s_2 m_1 m_2 |Kq\rangle$ are the coefficients of the unitary basis change from the direct product $H_{s_1} \otimes H_{s_2}$ towards the Hilbert subspace $H_K$ of spin $K$ that should be familiar from the addition of two spins. The CG coefficients are non-zero only if two selection rules are satisfied: (i) the two magnetic quantum numbers on the left add up to the one on the right, $m_1 + m_2 = q$; and (ii) if the angular momentum on the right satisfies the triangle inequality $|s_1 - s_2| \leq K \leq s_1 + s_2$.

In our case, we do not couple two spins, but a spin and its complex conjugate. Since there is a contragredient representation in the game, the CG coefficients in (22) feature a characteristic minus sign in front of $-m$. The triangle selection rule implies for us $0 \leq K \leq 2s$.

The irreducible tensor operators $T_q^{(K)} = |Kq\rangle$ form a basis of Liouville space that is properly orthonormal with respect to the trace scalar product of matrices:

$$\langle Kq|K'q'\rangle := \text{tr}\{(T_q^{(K)})^\dagger T_{q'}^{(K')}\} = \delta_{KK'} \delta_{qq'}$$  \hspace{1cm} (23)

Any linear operator $A$ can be decomposed in this basis,

$$A = \sum_{Kq} A_{Kq} T_q^{(K)} \quad \text{with} \quad A_{Kq} := \langle Kq|A| = \text{tr}\{(T_q^{(K)})^\dagger A\}.$$  \hspace{1cm} (24)
The irreducible components

$$\rho_{Kq} = \text{tr}\{(T_q^{(K)})^\dagger \rho\} = \left\langle (T_q^{(K)})^\dagger \right\rangle$$

of the density matrix are called state multipoles or statistical tensors, and have been introduced already in the 1950’s by Fano and Racah [2, 18].

**Exercise 4:**

1. Show that all $T_q^{(K)}$ except $T_0^{(0)}$ have zero trace and calculate the state monopole moment $\rho_{00}$ (use $\sqrt{2s+1} (ssm' - m|00) = (-)^{s-m} \delta_{mm'}$).

2. Show that the irreducible vector operator is proportional to the spin operator, $T_0^{(1)} q = c^{-1/2} s S q$. Hint: consider the action of $T_0^{(1)} (\sqrt{c} s (ssm' - m|10) = (-)^{s-m} m)$, argue with rotational invariance and fix the proportionality constant $c_s$ by computing $\text{tr}[(T^{(1)})^2]$.

### 1.3.3. Irreducible spin superoperators

By inserting the decomposition (24) on the left and right side of an arbitrary superoperator acting like $L A$, one obtains

$$L A = \sum_{KqK'q'} |K'q'\rangle \langle K'q'|L|Kq\rangle \langle Kq|A$$

such that with the notation $L_{K'q',Kq} := \langle K'q'|L|Kq\rangle$ the superoperator reads

$$L = \sum_{KqK'q'} L_{K'q',Kq}|K'q'\rangle$$

where the tetradsics on the right hand side transform under $D^{(K')} \otimes (D^{(K)})^*$, in an analogous manner to (21). Following the same strategy as previously, we recouple the elements again using the appropriate CG-coefficients to get irreducible superoperators [19]

$$T^L_{lm}(K, K') := \sum_{q,q'} (-)^{K-q} \langle K'Kq'q'\rangle |Kq\rangle (28)$$

Finally, any superoperator in completely decomposed form reads

$$L = \sum_{Lm} \sum_{K,K'} L_{Lm}(K, K') T^L_{lm}(K, K')$$

with coefficients

$$L_{Lm}(K, K') = \sum_{q,q'} (-)^{K-q} \langle K'Kq'q'\rangle |Kq\rangle \text{C}_{K'q',Kq}$$

This decomposition is completely general and applies to arbitrary superoperators. It is only worth the effort, however, if the superoperator has rotational symmetries. The greatest gain in computational speed and conceptual clarity
is obtained if the superoperator is a scalar such that only its only nonzero component is \( L = 0, m = 0 \). In that case, which applies to our Lindbladian (18), one finds by virtue of the triangle rule that \( K = K' \); as promised, \( \mathcal{L} \) indeed connects subspaces of equal rank. We can choose the decomposition

\[
\mathcal{L} = \sum_{K=0}^{2s} \lambda_K T^{(K)}. \tag{31}
\]

In this form, the \( T^{(K)} \) are orthogonal projectors onto the subspaces \( L(\mathcal{H}_s)^{(K)} \) of irreducible tensor operators of rank \( K \):

\[
T^{(K)} = \sqrt{2K + 1} T_0^{(K)}(K, K) = \sum_q |Kq\rangle(Kq|. \tag{32}
\]

They are indeed orthogonal,

\[
T^{(K)} T^{(K')} = \delta_{KK'} T^{(K)} \tag{33}
\]

by virtue of (23) and resolve the identity in Liouville space, \( \sum_K T^{(K)} = 1 \), by virtue of a completeness relation of CG-coefficients.

### 1.3.4. Calculation of scalar eigenvalues

Once the invariant subspaces are known, the eigenvalues are obtained by projecting the superoperator onto an arbitrary basis element:

\[
\lambda_K = \langle Kq|\mathcal{L}|Kq\rangle = \text{tr}\{(T_q^{(K)})^\dagger \mathcal{L} T_q^{(K)}\}. \tag{34}
\]

For the spin relaxation Lindbladian (18), one must calculate commutators of the form \([S_p, [S_p, T_q^{(K)}]]\). In order to use the commutation relations (8) and (9), one writes the scalar product of spin operators

\[
\sum_i S_i S_i = \sum_{p=0,\pm 1} (S_p)^\dagger S_p = \sum_{p=0,\pm 1} (-)^p S_p S_{-p} \tag{35}
\]

in terms of the standard components \( S_0 = S_z, S_{\pm 1} = \mp \frac{1}{\sqrt{2}}(S_x \pm i S_y) \) such that

\[
\lambda_K = -\frac{1}{2\tau_s} \sum_{p=0,\pm 1} (-)^p \text{tr}\{(T_q^{(K)})^\dagger [S_{-p}, [S_p, T_q^{(K)}]]\} \tag{36}
\]

It is now a simple exercise to show with the help of (8) and (9) that the double commutator gives back the tensor operator itself, \([S_{-p}, [S_p, T_q^{(K)}]] = c(p, q, K) T_q^{(K)}\). Summing all three terms gives remarkably simple eigenvalues,

\[
\lambda_K = -\frac{K(K + 1)}{2\tau_s}, \quad K = 0, 1, \ldots, 2s. \tag{37}
\]
1.3.5. Isotropic spin relaxation

We can now solve the equation of motion

\[ \dot{\rho}(t) = \sum_{Kq} \dot{\rho}_{Kq}(t) T_{q}^{(K)} = \sum_{Kq} \rho_{Kq}(t) \mathcal{L} T_{q}^{(K)} = \sum_{Kq} \rho_{Kq}(t) \lambda_{K} T_{q}^{(K)} \]  

(38)

in each irreducible mode, \( \dot{\rho}_{Kq}(t) = \lambda_{K} \rho_{Kq}(t) \), to obtain the decay of state multipoles

\[ \rho_{Kq}(t) = e^{-t/\tau_{K}} \rho_{Kq}(0) \]  

(39)

with their relaxation rates \( \tau_{K}^{-1} = |\lambda_{K}| = K(K + 1)/2\tau_{s} \).

The scalar mode relaxation rate \( \lambda_{0} = 0 \) assures the trace preservation of \( \rho(t) \).

The vector mode relaxation rate \( |\lambda_{1}| = \tau_{s}^{-1} \) describes the relaxation of the orientation or average spin vector:

\[ \langle S(t) \rangle = \text{tr}(\rho(t) S) = e^{-t/\tau_{s}} \langle S(0) \rangle \]  

(40)

since \( \text{tr}(T_{q}^{(K)} S_{q}') \) projects onto \( K = 1 \) (remember the exercise 4.(1) and the orthogonality relation (23)).

2. Mesoscopic spin transport

2.1. General remarks

"Transport" is a movement from a point \( r \) to a point \( r' \) induced by an external cause. In a regular medium, propagation is ballistic: the average square of the distance covered after a time \( t \) scales like \( \langle x^{2}(t) \rangle \sim v^{2}t^{2} \) where \( v \) is the particle’s velocity. A disordered medium contains impurities that interrupt the ballistic movement. Quenched disorder is fixed for each realization of an experiment, but varies from experiment to experiment when samples are changed. Predictions about observables will involve an average \( \langle \cdot \rangle \) that may be done over a classical disorder distribution or by tracing out uncontrolled quantum degrees of freedom. Generically, the averaged expectation value \( \langle \cdot \rangle \) of travelled distance behaves as a diffusive quantity: \( \langle x^{2}(t) \rangle \sim Dt \) with \( D \) the diffusion constant.

In a classical transport setting (think of a soccer ball shot into a forest) one propagates probabilities. In quantum transport, one propagates amplitudes, allowing for interference phenomena if phase coherence is preserved. This requires small system sizes, low temperatures, etc. Examples are

1. Aharonov-Bohm effect: the resistivity of a metallic ring as function of a magnetic flux oscillates with period of one flux quantum \( \Phi_{0} = h/e \).

2. Weak localization (WL): The resistance of weakly disordered metallic samples show negative magnetoresistance \( \frac{d\rho}{dB} > 0 \) at small field. Explanation: The quantum return probability to a point includes the constructive interference of counterpropagating amplitudes and is therefore larger...
than the classical probability. This interference effect is destroyed by a large enough magnetic field since loops of different sizes pick up different Aharonov-Bohm phases.

2.2. Diffusive spin transport

Imagine that we can inject spin-polarized particles \( p_+(0) = 1 \) on one end of a diffusive medium of length \( L \). What is the probability \( p_+(L) \) of retaining the spin polarization at the other end (assuming that we have spin-sensitive detection)? Need two quantities: the momentum diffusion constant \( D \) that determines the transmission time \( t \sim L^2/2D \) and the spin relaxation time \( \tau_d \) as function of the microscopic parameters of the scattering mechanisms for momentum and spin.

2.2.1. Classical diffusion

In a hydrodynamic description, diffusion is a direct consequence of linear response \( \mathbf{j} = -D \nabla n \) and a local conservation law \( \partial_t n + \nabla \cdot \mathbf{j} = \sigma \). Solving the diffusion equation \( (\partial_t - D \nabla^2)n = \sigma \) in Fourier components \( (q, \omega) \), we see that a small fluctuation \( \delta n(x, t) \) from a homogenous equilibrium density decays exponentially as \( \delta n(q, t) = \exp(-Dq^2t)\delta n(q) \). A kinetic description in terms of collisions of carrier particles with obstacles permits to derive the diffusion constant \( D \) associated with this process as function of microscopic parameters.

2.2.2. Master equation approach to diffusion

A microscopic quantum derivation of diffusive behaviour typically uses a system Hamiltonian

\[
H = H_0 + V.
\]

The one-particle Hamiltonian \( H_0 \) describes ballistic propagation in momentum and spin eigenstates \( |k\sigma\rangle := |k\rangle \otimes |\sigma\rangle \) with eigenenergies \( \varepsilon_0(k, \sigma) \). The impurity potential \( V \) describes momentum scattering, spin-flip scattering, spin-orbit coupling, and the like, by randomly distributed scatterers. Deriving observable quantities will involve averages over all possible realizations of the disorder. Typically, averages over the random positions \( r_i \) of impurities will be done assuming a uniform distribution over the entire volume \( L^d \),

\[
F(\{r_i\}) = \int (\Pi_j \frac{d^d r_j}{L^d}) F(\{r_i\})
\]

Starting from the Liouville equation \( \dot{\rho}(t) = \frac{i}{\hbar} [\hat{V}(t), \rho(t)] \) in the interaction representation \( \hat{A}(t) = U_0(t) \hat{A} U_0(t) \) and developing to second order in the interaction leads to the master equation for the effective density matrix \( \rho_{\text{eff}} = \mathcal{P} \),

\[
\dot{\rho}_{\text{eff}}(t) = -\int_0^t [\hat{V}(t), [\hat{V}(t-t'), \rho_{\text{eff}}(t-t')] dt'.
\]

\footnote{Remember the perturbation theory we carried out in [1.2.3]
Here, one uses the fact that $\overline{V} = 0$ (always achievable by redefining the zero of energies in $H_0$). Furthermore, one assumes a Gaussian model of disorder such that only impurity pair-correlations $\overline{VV}$ need to be retained.

By selecting a typical term of type $\tilde{V}(t)\tilde{\rho}_{\text{eff}}(t-t')\tilde{V}(t-t')$ under the integral in (43), and inserting the identity (write $|k_1\sigma_1\rangle = :1:$ for brevity), we naturally find the so-called four-point scattering vertex,

$$V_{23,14} := \langle 2|V_{\text{imp}}|1\rangle\langle 4|V_{\text{imp}}|3\rangle =$$

The arrows on the propagation lines indicate whether the state appears as a ket $|\cdot\rangle$ or a bra $\langle\cdot|$. one speaks of “incident” momentum and spin for $|\cdot\rangle$ and “scattered” momentum and spin for $\langle\cdot|$. The standard way of proceeding would now be to evaluate the average, use a Markov approximation for coarse-grained evolution on time scales longer than the bath correlation time. In the end, it should be possible to derive a diffusion equation for the probability density in real space leading to the hydrodynamic description of 2.2.1, thus determining the diffusion constant from the microscopic scattering parameters. The derivation is very involved, and we take a short-cut.

2.2.3. Elastic scattering mean free time

Let’s look at a momentum scattering impurity potential of the type

$$V_{\text{el}} = \sum_{i=1}^{N_{\text{el}}} v_0(r - r_i)$$

with $v_0(r)$ centered at random locations $r_i$ without action on the spin:

$$\langle k_2\sigma_2|V_{\text{el}}|k_1\sigma_1\rangle = \delta_{\sigma_1\sigma_2} \sum_{i=1}^{N_{\text{el}}} \int \frac{d^d r}{L^d} e^{i(k_1-k_2) \cdot r} v_0(r - r_i)$$

An individual scattering process $|k_1\sigma_1\rangle = :1: \mapsto |k_2\sigma_2\rangle = :2:$ induced by $V_{\text{el}}$ has the amplitude $\langle 2|V_{\text{el}}|1\rangle$ (to first order in $V$: Born approximation). According to Fermi’s Golden Rule, the total scattering rate out of state $|1\rangle$ is

$$\frac{1}{\tau_{\text{el}}} = 2\pi \sum_{k_2\sigma_2} \frac{|\langle 2|V_{\text{el}}|1\rangle|^2}{|2\varepsilon_1 - \varepsilon_2|}$$

which actually can be seen to correspond to (the imaginary part of) the partially contracted scattering vertex $\sum_{2} V_{22,11}$ defined in (14). The elastic scattering mean free time is an experimentally accessible parameter (for instance through
conductivity measurements for electrons) and will be taken as given in the following. It describes the average lifetime on a momentum eigenstate in the presence of scattering impurities.

The weak disorder limit in which the Born approximation and Fermi’s golden rule are valid corresponds to \( \tau^{-1}_{\text{el}} \ll \varepsilon \). For massive particles with kinetic energy \( \varepsilon = k^2/2m \) and velocity \( v = k/m \), one introduces the elastic scattering mean free path \( \ell_{\text{el}} = v\tau_{\text{el}} \). The weak disorder limit can thus also be stated as \( 1/(k\ell_{\text{el}}) \ll 1 \) which indicates that successive scatterers are placed in the far field of scattered waves (distance \( \ell_{\text{el}} \gg \) wavelength \( \lambda \)). This is essentially a low-density argument: In terms of the density \( n_{\text{el}} \) of elastic scatterers and their total scattering cross section \( \sigma_{\text{el}} \) one has \( \ell_{\text{el}} = 1/(n_{\text{el}}\sigma_{\text{el}}) \).

The relevant momentum diffusion constant is then \( D = v\ell_{\text{el}}/d \). The numerical factor of the dimension \( d \) in the denominator is conventional. The scaling in \( v \) and \( \ell_{\text{el}} \) stems for the diffusion equation in Fourier space \( i\omega + Dq^2 \) where the relevant scales are \( \omega \sim \tau^{-1}_{\text{el}} \) and \( q \sim \ell^{-1}_{\text{el}} \).

### 2.2.4. Spin-flip scattering vertex

Consider now a spin-flip interaction with a collection of freely orientable magnetic impurities centered at sites \( r_i \),

\[
V_{\text{sf}} = \sum_{i=1}^{N_{\text{sf}}} g(r - r_i) S \cdot J_i \quad (48)
\]

The relevant spin-flip vertex is \( V_{14,23} = (2\pi\rho\tau_{\text{sf}})^{-1}\hat{V}_{\sigma_1\sigma_4,\sigma_2\sigma_3} \) in terms of the normalized spin-flip vertex (or spin-flip superoperator)

\[
\hat{V}_{\sigma_1\sigma_4,\sigma_2\sigma_3} = \frac{S_{\sigma_2\sigma_1} \cdot S_{\sigma_4\sigma_3}}{s(s+1)} \quad (49)
\]

The scalar product of spin operators originates from the isotropic average \( \langle J_i \rangle_\alpha J_j \rangle_\beta = \delta_{ij} \frac{1}{2} \delta_{\alpha\beta} J^2 \) over all possible orientations of the impurity spin. The position-dependent potential part \( g(r) \) induces momentum scattering such that the total momentum scattering rate reads

\[
\frac{1}{\tau} = \frac{1}{\tau_{\text{el}}} + \frac{1}{\tau_{\text{sf}}} \quad (50)
\]

In order to determine the spin relaxation rates, we need to diagonalize the spin superoperator \( \hat{V}_{\sigma_1\sigma_4,\sigma_2\sigma_3} \). But here, we can make use of our results for the Lindbladian \( \mathcal{L} \) treated in 1.2.3 by comparing the scattering approach to the master equation formulation, we can identify \( \mathcal{L} = \tau^{-1}_{\text{sf}}(\hat{V} - I) \). Thus, the spin decay rates of the state multipoles in the present case read

\[
\lambda_K = -\frac{K(K+1)}{2s(s+1)\tau_{\text{sf}}} \quad (51)
\]

We have of course \( \lambda_0 = 0 \) (the trace is preserved), and the spin decay rate of the spin orientation \( K = 1 \) is \( \left| \lambda_1 \right| = 1/(s(s+1)\tau_{\text{sf}}) =: 1/\tau_d \).
2.2.5. Diffusive spin relaxation

During a propagation time $t$, the spin density matrix will therefore relax as

$$\rho(t) = \sum_{Kq} e^{i\lambda K t} \rho_{Kq}(0) T_q^{(K)} \tag{52}$$

Starting from an initial polarized state $|+\rangle := |s, m = +s\rangle$ with $p_+(0) = \langle +|\rho_0|+\rangle = 1$, we find (exercise !)

$$p_+(t) = \frac{1}{2s+1} + e^{-t/\tau_d} \frac{3s^2}{s(s+1)(2s+1)} + \ldots \tag{53}$$

with the last line valid for electrons ($s = \frac{1}{2}$). Naturally, the probability relaxes towards its equilibrium value $p^\text{eq}_m = \frac{1}{2s+1}$ to have any magnetic quantum number $m = -s, \ldots, s$.

2.3. Dephasing of weak localization

Weak localization is observed as a decrease of diffusivity or conductivity. The reason is the constructive interference of counterpropagating amplitudes along identical scattering paths. The interference correction to the classical diffusion constant $D_0$ can be written by integrating over all closed amplitude loops,

$$\frac{\delta D(\omega)}{D_0} \propto \int \frac{d^d Q}{D_0 Q^2 + i\omega} \tag{54}$$

Including spin-flip scattering into the picture can be done by diagonalizing the spin-flip vertex appropriately. The irreducible subspaces turn out to be the singlet and triplet state subspaces $|Kq\rangle$ (not operators $|Kq\rangle$ as for the propagated intensity: the difference is due to the fact that the two amplitudes in weak localization loops propagate in opposite directions). To sketch the result, the WL correction is written

$$\frac{\delta D}{D_0} \propto \int d^d Q \sum_K w_K \frac{w_K}{D_0 Q^2 + \tau_c(K)^{-1}} \tag{55}$$

where each spin channel comes with a certain weight

$$w_K = (-)^{2s+K} \frac{2K+1}{2s+1} \tag{56}$$

For electrons, $w_0 = -\frac{1}{2}$ and $w_1 = \frac{3}{2}$. More importantly, each spin channel is damped with its coherence time $\tau_c(K)$. They are given in terms of the spin relaxation eigenvalues $\lambda$(51):

$$\frac{1}{\tau_c(K)} = \frac{2}{\tau_{sf}} + \lambda_K = \frac{2}{\tau_{sf}} \left( 1 - \frac{K(K+1)}{4s(s+1)} \right) \tag{57}$$
such that for electrons $\tau_c(0) = \tau_{sf}/2$ and $\tau_c(1) = 3\tau_{sf}/2$.

Magnetic impurities are a very efficient source of decoherence/dephasing for interference phenomena of spin-carrying particles!

2.4. Spin-orbit scattering

Spin-orbit interaction is a coupling between the momentum degree of freedom of itinerant spins and their spin orientation. This mechanism is of high relevance for spintronics applications [3] and mesoscopic physics.

2.4.1. Elliott-Yafet mechanism

A coupling between orbital motion and spin arises already when an itinerant electron is scattered by a scalar impurity potential $V(r)$: the electrostatic field $E = -e\nabla V(r)$ deflecting the electron gives rise to a magnetic field in the electron’s frame of reference. Transforming back to the impurity frame gives the potential

$$V_{so} = \frac{\hbar}{4mc^2} \sigma \cdot (\nabla V(r) \times \mathbf{k}) .$$

This potential is still a linear coupling to the spin $\mathbf{S} \cdot \mathbf{B}_{\text{eff}}$ with a $\mathbf{k}$-dependent effective magnetic field. As far as spin transformations are concerned, we can therefore use all the machinery developed in the previous sections. Notably, the spin-orbit scattering mean free time for the momentum relaxation is $1/\tau_{so} = 2\pi \rho N_{so} \langle v^2_0 \rangle$ where $\langle \cdot \rangle$ is an angular average over possible wavevector orientations $\mathbf{k}$ with modulus fixed by the energy conservation $\delta(\varepsilon_k - \varepsilon_{k'})$.

Furthermore, the spin relaxation time is given by (51): $\tau_d = \frac{3}{4} \tau_{so}$.

In principle, all processes with momentum change will induce a spin-orbit coupling of this type, e.g., scattering by impurities and boundaries (dominant at low temperatures) or phonons (dominant at higher temperatures).

2.4.2. Dyakonov-Perel mechanism

If $H_0$ is time-reversal invariant, the energy levels are two-fold degenerate, $\varepsilon(\mathbf{k}, \sigma) = \varepsilon(-\mathbf{k}, -\sigma)$ (Kramers degeneracy). If $H_0$ is space-inversion invariant (under parity operation $P r = -r, P \sigma = \sigma$), then one has $\varepsilon(\mathbf{k}, \sigma) = \varepsilon(-\mathbf{k}, \sigma)$.

A necessary ingredient for the DP spin-coupling is the absence of inversion symmetry which implies $\varepsilon(\mathbf{k}, \sigma) = \varepsilon(-\mathbf{k}, -\sigma) \neq \varepsilon(\mathbf{k}, -\sigma)$: different spin states are no longer degenerate and can be coupled, which induces a precession of the electronic spin around a $\mathbf{k}$-dependent effective vector $\mathbf{\Omega}(\mathbf{k})$ during ballistic motion: $V_{so} = \frac{1}{2} \sigma \cdot \mathbf{\Omega}(\mathbf{k})$. The effective field $\mathbf{\Omega}$ depends on the crystal structure and the band structure of the conduction band.

The Dyakonov-Perel spin-orbit precession can be pictured as a diffusion process or random walk on the Bloch sphere. Between scattering events separated in time by the total momentum scattering mean free time $\tau$, the spin will on average precess by an angle $\delta \phi \sim \mathbf{\Omega} \tau$. For small precession frequencies such that $\mathbf{\Omega} \tau \ll 1$, the spin will cover only a small angle before the field vectors change.
its direction after the next momentum scattering event. Then, a new precession sets in, and so forth. After a time $t$ (corresponding to $t/\tau$ such steps) the total angle covered will be $\phi^2 \sim \delta \phi \cdot t/\tau$ (remember the diffusion relation $L^2 \sim Dt$ with $D = \ell^2/\tau$; now, the step-length $\ell$ is $\delta \phi$). The relevant spin coherence time $\tau_s$ may be estimated by the time $t$ it takes to arrive at $\phi = \pi$ (or $2\pi$, numerical factors are of no relevance in this simple argument), which gives $1/\tau_s \sim \Omega^2$. In contrast to the Elliott-Yafet mechanism, the spin decoherence rate gets smaller with smaller $\tau$ ("motional narrowing") because the spin does not have the time to precess a lot if the momentum scattering events follow too fast.

2.5. Spintronics - some characters of the play

2.5.1. Spin-resolved density of states

The spin-up and spin-down components $\nu_{\uparrow\downarrow}(\omega)$ of the total density of states metals or semiconductors are degenerate at thermal equilibrium and in the absence of polarizing fields. For ferromagnetic metals (F) however, the spontaneous alignment of spins leads to a finite spin polarization (in the direction opposite to the magnetization) even at equilibrium. If the net spin polarization

$$n_s = \frac{N_{\uparrow} - N_{\downarrow}}{N_{\uparrow} + N_{\downarrow}}$$

is positive, the $\uparrow$-spins are called majority spins, the $\downarrow$-spins are minority spins. After spin injection from a ferromagnet, the spin polarization in a normal metal can be of order 15%.

2.5.2. Spin field-effect transistor

This prototypical spintronics device was proposed in 1990 by Datta and Das [23]. It consists of a ferromagnet-normal-ferromagnet (F/N/F) junction of conductors. The electrons in the normal metal between the two ferromagnets (with, say, both $\uparrow$ spin orientation) are subjected to a spin-orbit precession that can be precisely controlled with an external gate voltage. If the spin $\uparrow$ does not precess, then conduction will be normal. For a precession such that spin $\uparrow$ has rotated to $\downarrow$, the entrance to the second ferromagnet is prohibited, and no current will flow.

This very nice device has a small disadvantage: it does not exist yet. The purity required for the normal metal (of linear size of order 1 $\mu$m) as well as the control fields are not yet met by actual experimental techniques. But other interesting things are under study.

2.5.3. Tunneling magneto resistance

A rather old scheme of charge current control through spin dates back to the 70’s [24]: a ferromagnet–insulator–ferromagnet junction (F/I/F) where spin-polarized electrons can tunnel between the two contacts. A simple model assumes that the tunneling matrix elements do not involve spin-flip scattering and
are the same for all spins. Then, the conductance \( G = 1/R \) of the junction with parallel or anti-parallel orientation will be given by

\[
G_{\uparrow\uparrow} \sim M_1 M_2 + m_1 m_2 \\
G_{\uparrow\downarrow} \sim M_1 m_2 + m_1 M_2
\]

(60)

where \( M_i \) is the density of states (at the Fermi energy) of majority spins in the ferromagnet \( i \), and \( m_i \) the minority density, since \( \uparrow \)-spins on the left will tunnel into the \( \uparrow \)-subband on the right and vice versa for \( \downarrow \)-spins. A simple exercise permits to verify that the tunneling magnetoresistance ratio is given by

\[
TMR = \frac{R_{\uparrow\uparrow} - R_{\uparrow\downarrow}}{R_{\uparrow\uparrow} + R_{\uparrow\downarrow}} = \frac{2P_1 P_2}{1 - P_1 P_2}
\]

(61)

and depends therefore on the spin polarisation \( P_i = (M_i - m_i)/(M_i + m_i) \) of the left and right ferromagnets. The technological advantage of magnetoresistance junctions is that the magnetization persists even after the driving currents are switched off (non-volatile memory). This is used in many current data storing devices.

2.5.4. Mesoscopic spin filter

A spin polarizer-analyzer involving a quantum dot and magnetic fields has been presented experimentally by Folk et al., [25]:

![Figure taken from [3].](image)

A spin polarization of 70% is reached

2.5.5. Spin Hall effect

After an early theoretical proposition by Dyakonov and Perel [26] that spin-orbit coupling in 2D-semiconductors can lead to a spatial separation of spin densities, the spin Hall effect has been recently experimentally realized by Kato et al. [27]:

![Figure taken from [3].](image)
3. Quantum dot spin computation

3.1. Interacting spins

The original proposal by D. Loss and D. DiVincenzo [29] (see also [30]) suggested two excess electron spins $S_1$ and $S_2$ in neighboring quantum dots with a time-dependent exchange coupling $J(t)$ that can be controlled through an external gate voltage:

$$H = J(t)S_1 \cdot S_2$$

(62)

Also, individual spin control $g_i(t)B_i(t) \cdot S_i$ via a local effective field should be possible. This allows the realization of the universal $\sqrt{\text{SWAP}}$ gate on realistic time scales taking into account the decohering effect of the environment. Loss and DiVincenzo set up a master equation approach using spin superoperators but do not make explicit use of irreducible representations.

3.2. Interaction with nuclear spins

Today’s major limitation to a coherent functioning of spin quantum dots comes from the hyperfine interaction with nuclear spins from the semiconductor matrix, the so-called “central spin problem” [31]. Master equation treatments of this problem have been developed, showing a rich variety of non-markovian dynamics [32].
3.3. Spin-relaxation measurements

Recent experiments by Johnson et al. [33] with two coupled spin quantum dots allowed to determine the spin dephasing time from all-electrical high frequency measurements. The inferred time is of order 10 ns, much too short for gate operations, but naturally, new schemes and propositions are devised.

![Diagram showing spin-relaxation measurements](image)

Figure 2 | Dependence of the occupancy of the (1,1) state on measurement time, $t_{dd}$, and external field, $B$. a. Charge sensor conductance $g_{\text{ext}}$ as a function of $V_L$ and $V_R$ with short pulses ($t_{dd} = 8 \mu s$) at $B = 100 \text{ mT}$. Large
4. Closing remarks on a sunny Friday afternoon

18 Circuit Theory for the Electrically Declined

J. F. Gregg and M. J. Thornton

Clarendon Laboratory, Oxford University, Parks Road, Oxford OX1 3PU, U.K.

18.1 The Soldering Iron and the Spin Electronician

Such is the sophistication of many contemporary University Physics Courses that their followers are at ease with the finer details of the Dirac equation and have no difficulty in thinking in a many-dimensional Hilbert-space; however they are often less confident when faced with knowing which end of a soldering iron gets hot. Spin Electronics is above all a practical science which ultimately promises to implement a new and revolutionary technology in a form which will ultimately impact everyday existence. Card-carrying theoretical physicists doubtless have their part to play in this new and exciting field, but for the rapid and successful development of this science, the importance of practical knowledge and experimental dexterity is paramount. Those who would claim proficiency as Spin Electronicians must, above all, be capable of the simple, basic skills with which every TV repair engineer is acquainted. To those devotees of Spin Electronics whose degree courses have left you electrically deprived, this chapter

Taken from: M. Ziese and M.J. Thornton, “Spin Electronics” (Springer 2001)

... leads to ...

![Circuit diagram](image-url)
On the other hand, respectable theoreticians (card-carrying?) arrive at 34.

Fig. 4. First (a), second (b) and third (c) order contributions to the self-energy \( \Sigma(p) \). The diagrams which will be taken into account in the self-consistent Born approximation are highlighted by the dashed frames. Cross-sections of the diagrams which produce the imaginary part of the self-energy are denoted by the dotted lines.

or even

Fig. 5. Fourth-order contributions to the self-energy \( \Sigma(p) \). The diagrams which will be taken into account in the self-consistent Born approximation are highlighted by the dashed frames. Cross-sections of the diagrams which produce the imaginary part of the self-energy are denoted by the dotted lines and denoted by \( c1 - c4 \). The connection of implies cutting three lines at the same time.
So here you are, the Spin TV repairman, and you have to judge the worthiness of different curved manifolds.
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