Interpreting Stone's model of Berry phases

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- **Magnetoelectricity of the (large-μ) ground state of Stone's model.**

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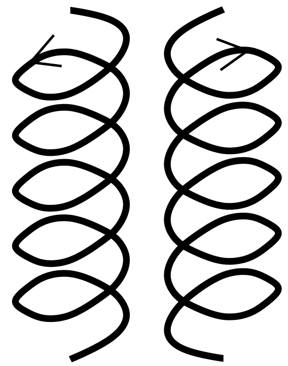
as shown by recent theoretical work on **x-ray dichroism and resonant scattering in noncentrosymmetric crystals** (Carra *et al.* 2003, Marri and Carra 2004).
($[\,,]^{(k)} \rightarrow$ Clebsch-Gordan coupling of irreducible tensors; $\boldsymbol{S} = \frac{1}{2}\boldsymbol{\sigma}$;
$\boldsymbol{\Omega}_L \equiv \frac{1}{2}(\boldsymbol{n} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{n})$, orbital anapole; $\mathcal{Q}^{(2)} \equiv [\boldsymbol{L}, \boldsymbol{L}]^{(2)}$, orbital quadrupole.)



Artist's view of a magnetic quadrupole

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[Order parameters: definition (2nd quant.)

$$(\boldsymbol{O}_L^{(k)})_q = \sum_{\substack{l,l'=l\pm1 \\ m,m',\sigma,\sigma'}} \frac{1}{2}\left[\langle\sigma'|\langle l'm'|(\boldsymbol{O}_L^{(k)})_q|lm\rangle|\sigma\rangle c^{\dagger}_{l'm'\sigma'}c_{lm\sigma} + \text{c.c.}\right]\,,$$

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with $c^\dagger_{lm\sigma}$ and $c_{lm\sigma}$ fermionic operators.]

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The basis set $|j \pm \frac{1}{2}, jm\rangle$ provides a convenient framework for describing **parity-breaking electron hybridisation** (e.g. $\boldsymbol{p}d$ mixing in transition-metal oxides), in the jj coupling scheme. $(\boldsymbol{n} \cdot \boldsymbol{\sigma}/2 = \boldsymbol{n} \cdot \boldsymbol{J}$, as $\boldsymbol{n} \cdot \boldsymbol{L} = 0.)$

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- **Determine form of order parameters in jj coupling**
- **LS$\rightarrow$ jj transformations** (Edmonds 1974)

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Coupled double Tensors (Judd, 1967)

$$w_\zeta^{(xy)z}(l',l) = \sum_{\xi,\eta,\lambda,\lambda',\sigma,\sigma'} C^{z\zeta}_{x\xi;y\eta} C^{y\eta}_{\frac{1}{2}\sigma';\frac{1}{2}\sigma} C^{x\xi}_{l'\lambda';l\lambda} c^\dagger_{l'\lambda'\sigma'} \tilde{c}_{l\lambda\sigma} + \text{ h.c.}\,, \qquad (\boldsymbol{LS})$$

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$$v_\zeta^{(j'j)z}(l',l) = \sum_{m,m'} C^{z\zeta}_{j'm';jm} c^\dagger_{l',j'm'} \tilde{c}_{l,jm} + \text{ h.c.}\,, \qquad (\boldsymbol{jj})$$

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where $\tilde{c}_{l\lambda\sigma} = (-1)^{l-\lambda+\frac{1}{2}-\sigma} c_{l-\lambda-\sigma}$ and $\tilde{c}_{l,jm} = (-1)^{j-m} c_{l,j-m}$ (irreducibility).

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$$w^{(20)2}(l',l) = -\frac{\sqrt{2}}{\sqrt{l(l+1)}C^{l'0}_{l0;10}\left\{\begin{smallmatrix} 1 & 1 & 2 \\ l' & l & l \end{smallmatrix}\right\}}\mathcal{M}^{(2)}_{L}(l',l)\,.$$

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LS → jj transformation

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with $[a,...,b] = (2a+1)\cdots(2b+1)$.

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$$\widetilde{\mathcal{M}}_J^{(2)}(l',l) = \frac{1}{5}\left(\frac{l+l'-1}{2}\right)\left(\frac{l+l'+3}{2}\right)\mathcal{M}_S^{(2)}(l',l) + \frac{2}{3}\mathcal{M}_T^{(2)}(l',l)$$
$$+\frac{1}{5}\mathcal{M}_F^{(2)}(l',l) - \frac{1}{2}\mathcal{M}_L^{(2)}(l',l)$$
$$= -\frac{3}{2}(2l+1)(2l'+1)\left\{[\boldsymbol{n},\boldsymbol{J}]^{(l'+\frac{1}{2},l-\frac{1}{2})2}\,\delta_{l',l-1} + [\boldsymbol{n},\boldsymbol{J}]^{(l'-\frac{1}{2},l+\frac{1}{2})2}\,\delta_{l',l+1}\right\}.$$

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$$\widetilde{\boldsymbol{P}}_J(l',l) = \boldsymbol{n}(l',l) + \boldsymbol{P}_S(l',l) - 2\boldsymbol{P}_T(l',l)$$
$$= -\frac{3(l+l'+1)}{2}\left[\boldsymbol{n}_J^{l'+\frac{1}{2},l-\frac{1}{2}}\,\delta_{l',l-1} + \boldsymbol{n}_J^{l'-\frac{1}{2},l+\frac{1}{2}}\,\delta_{l',l+1}\right]$$

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Our conclusion is further supported by what follows.

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Symmetry: Rotation group - SU(n)

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