Stochastic versus Uncertainty Modeling

Petra Friederichs, Michael Weniger, Sabrina Bentzien, Andreas Hense

Meteorological Institute, University of Bonn

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Uncertainty Modeling

- **Initial conditions** – aleatoric uncertainty

  Ensemble with perturbed initial conditions

- **Model error** – epistemic uncertainty

  Perturbed physic and/or multi model ensembles

Simulations solving deterministic model equations!

Stochastic Modeling

- **Stochastic parameterization** – aleatoric uncertainty

  Stochastic model ensemble

- **Initial conditions** – aleatoric uncertainty

  Ensemble with perturbed initial conditions

- **Model error** – epistemic uncertainty

  Perturbed physic and/or multi model ensembles

Simulations solving stochastic model equations!

Outline

Contrast both concepts

- Analytical solutions of simple damping equation

$$dv(t) = -\mu v(t)dt$$

- Simplified, 1-dimensional, time-dependent cloud model
- Problems

Equation of Motion

Small particle of mass m with velocity $v(t)$

Subject to damping force $F(t) = -\mu m \cdot v(t)$

$$dv(t) = -\mu v(t) dt$$

Solution of deterministic problem

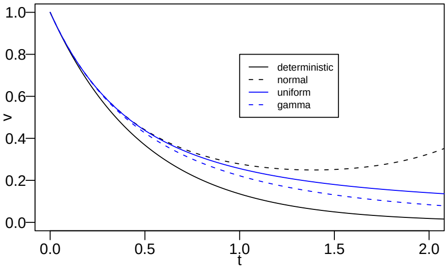
$$v(t) = v_0 \exp(-\mu t)$$

'Perturbed Physics'

γ is random variable

$$dv(t) = -\gamma v(t) dt.$$

Solutions for different γ



$$\gamma_{\mathcal{N}} \sim \mathcal{N}(\mu, \sigma^2) \qquad E[v_{\mathcal{N}}(t)] = v_0 \exp\left(-\mu t + \frac{\sigma^2}{2} t^2\right)$$

$$\gamma_{\mathcal{U}} \sim \mathcal{U}(\mu \pm \sqrt{3}\sigma) \qquad E[v_{\mathcal{U}}(t)] = v_0 \exp(-\mu t) \frac{\sinh(\sqrt{3}\sigma t)}{\sqrt{3}\sigma t}$$

$$\gamma_{\Gamma} \sim \Gamma\left(\frac{\mu^2}{\sigma^2}, \frac{\sigma^2}{\mu}\right) \qquad E[v_{\Gamma}(t)] = v_0 \left(1 - \frac{-\mu t}{\mu^2/\sigma^2}\right)^{-\mu^2/\sigma^2}$$

Stochastic Model

Damping from small-scale stochastic process

$$dv(t) = -\mu v(t)dt + v(t)\sigma\xi dt,$$

ξ_t: Langevin force with

$E[\xi_t] = 0$ and $\mathrm{Cov}[\xi_s, \xi_t] = \delta(t-s)$

Stratonovich calculus

$$dv_t = -\mu v_t dt + \sigma v_t \circ dW_t$$

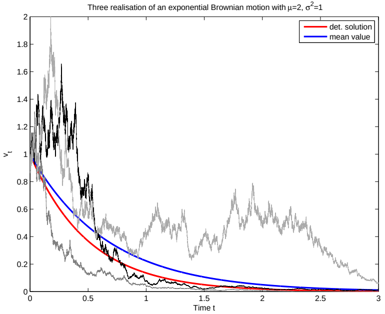
Solution

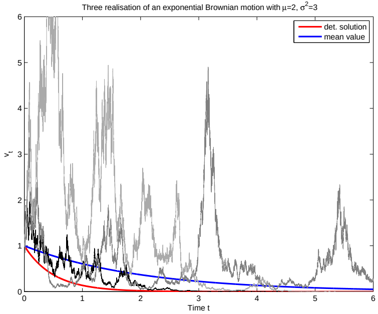
$$v_t = v_0 \exp\big(-\mu t + \sigma W_t\big) \qquad \text{(exponential Brownian motion)}$$

Exponential Brownian motion

$$E[v_t] = v_0 \exp\Big(-(\mu - \frac{\sigma^2}{2})t\Big)$$

$$Var(v_t) = v_0^2\Big[\exp\Big(-2(\mu-\sigma^2)t\Big) - \exp\Big(-2(\mu - \frac{\sigma^2}{2})t\Big)\Big]$$





Non-delta Correlated Gaussian Noise

$$\mathrm{d}v(t) = -\mu \mathrm{d}t + v(t)\epsilon_t v(t)\mathrm{d}t$$

Solution

$$v_t = v_0 \exp\left(-\mu t + \int_0^t \epsilon_s \mathrm{d}s \right)$$

Noise ϵ_t with

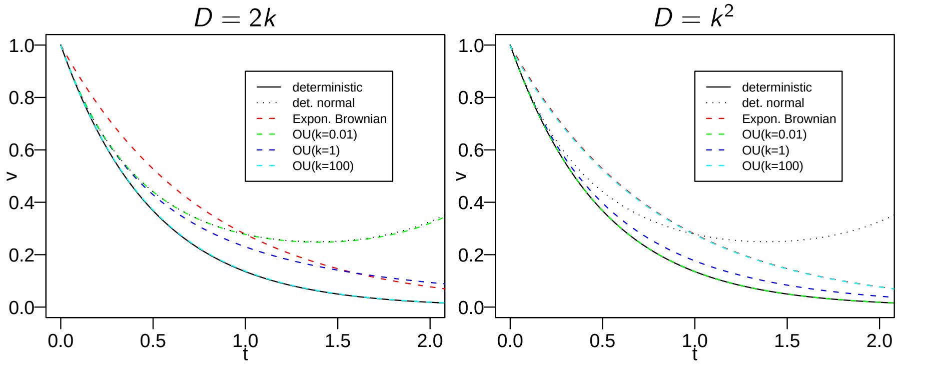
$$\mathrm{E}[\epsilon_t] = 0, \quad \mathrm{Cov}[\epsilon_s, \epsilon_t] = \frac{D}{2k}\exp(-k|t-s|).$$

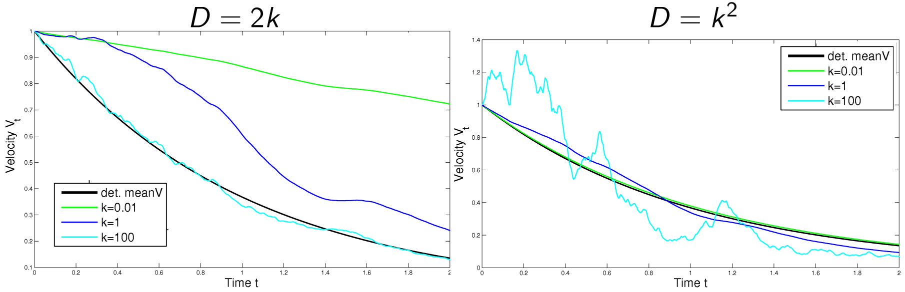
is an Ornstein-Uhlenbeck process solving

$$\mathrm{d}\epsilon_t = -k\epsilon_t \mathrm{d}t + \sqrt{D}\mathrm{d}W_t$$

Non-delta Correlated Gaussian Noise

$$\mathrm{E}[v_t] = v_0 \exp\left(-\mu t + \frac{D}{2k^2} \left(t - \frac{1-e^{-kt}}{k}\right)\right)$$




$D = 2k$
Velocity V_t
Time t
det. meanV
k=0.01
k=1
k=100
$D = k^2$
Velocity V_t
Time t
det. meanV
k=0.01
k=1
k=100

'Perturbed physics'

$E[v(t)] \to \infty$ for $t \to \infty$ for all parameter distributions with non-positive support

Stochastic model

Exponential Brownian motion:

- Non-stable solutions ($\sigma^2 > 2\mu$), increasing variance ($\sigma^2 > \mu$)

OU noise:

- Noise induced drift increases with $1/k$ ($D = 2k$)
- 'Perturbed physics' solution for $k \to 0$ ($D = 2k$)
- Exponential Brownian motion for $k \to \infty$ ($D = k^2$)

1-dimensional Cloud Model

- Cylinder - constant radius
- Horizontally homogen in constant environment
- Prognostic equations for $w(z)$, $T(z)$, $q_v(z)$, $q_c(z)$, $q_r(z)$

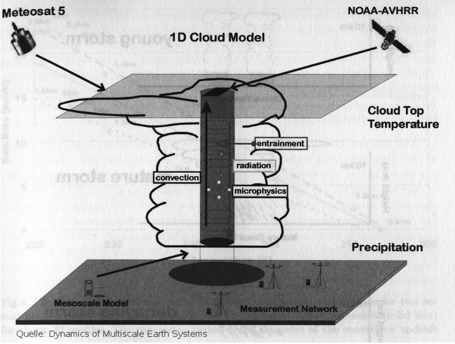
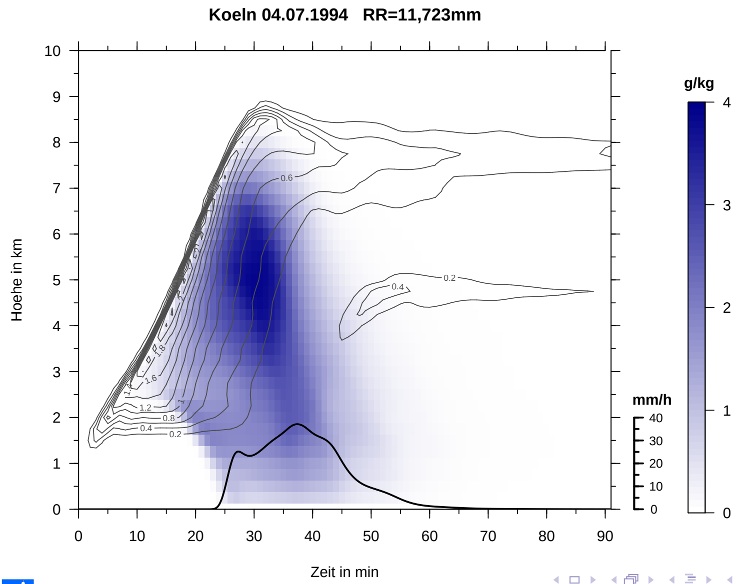


Fig: Dynamics of Multiscale Earth Systems, C.Simmer(Eds.)


Koeln 04.07.1994 RR=11,723mm
Hoehe in km
Zeit in min
g/kg
mm/h
0.2
0.4
0.6
0.8
1.2
1.6
1.8
0
10
20
30
40
50
60
70
80
90
1
2
3
4
5
6
7
8
9

$$0 = -\frac{1}{\rho}\frac{\partial(\rho w)}{\partial z} - \frac{2}{R}\tilde{u}$$

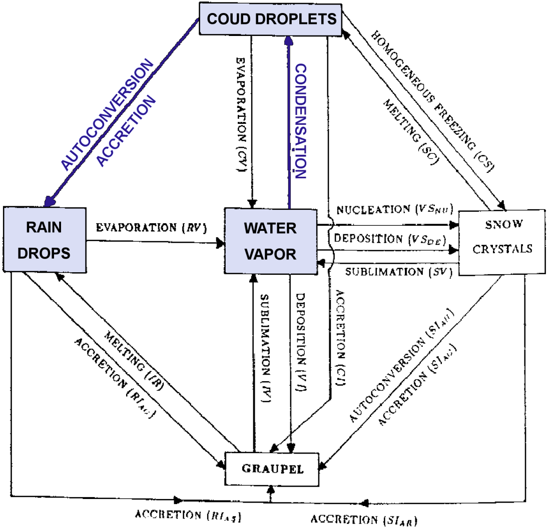
$$\frac{\partial w}{\partial t} = -w\frac{\partial w}{\partial z} + \frac{2}{R}\alpha^2|w_e - w|(w_e - w) + \frac{2}{R}(w - \tilde{w})\tilde{u} + \left(\frac{T_v - T_{v,e}}{T_{v,e}}\right)g - (q_c + q_r)g$$

$$\frac{\partial T}{\partial t} = -w\frac{\partial T}{\partial z} + \frac{2}{R}\alpha^2|w_e - w|(T_e - T) + \frac{2}{R}(T - \tilde{T})\tilde{u} - \Gamma_d w + SS_T$$

$$\frac{\partial q_v}{\partial t} = -w\frac{\partial q_v}{\partial z} + \frac{2}{R}\alpha^2|w_e - w|(q_{v,e} - q_v) + \frac{2}{R}(q_v - \tilde{q}_v)\tilde{u} - \frac{dq_v}{dt}$$

$$\frac{\partial q_c}{\partial t} = -w\frac{\partial q_c}{\partial z} + \frac{2}{R}\alpha^2|w_e - w|(q_{c,e} - q_c) + \frac{2}{R}(q_c - \tilde{q}_c)\tilde{u} - \frac{dq_c}{dt}$$

$$\frac{\partial q_r}{\partial t} = \underbrace{-w\frac{\partial q_r}{\partial z}}_{\text{advection}} + \underbrace{\frac{2}{R}\alpha^2|w_e - w|(q_{r,e} q_r)}_{\text{turbulent entrainment}} + \underbrace{\frac{2}{R}(q_r - \tilde{q}_r)\tilde{u}}_{\text{dynamical entr.}} - \underbrace{\frac{dq_r}{dt} + V_R\frac{\partial_r}{\partial z} + \frac{q_r}{\rho}\frac{\partial(\rho V_R)}{\partial z}}_{\text{sources}}$$



Micro-physical processes after Kessler (1969)

Fig: von der Emde, Kahlig (1989)

Condensation:

$q_v \rightarrow q_c$

$$-\left.\frac{dq_v}{dt}\right|_{VC} = \frac{q_v - q_{vws}}{1 + \frac{\varepsilon L_c^2 q_{vws}}{c_p R_a T^2}} \cdot \frac{1}{\Delta t} \quad \text{for} \quad q_v > q_{vws}$$

Conversion:

$q_c \rightarrow q_r$

$$-\left.\frac{dq_c}{dt}\right|_{CR_{AU}} = k_c (q_c - q_{c,0}) \quad \text{for} \quad q_c > q_{c,0}$$

$$q_{c,0} = \frac{1}{\rho}, \; k_c = 10^{-3}$$

Accretion:

$q_c \rightarrow q_r$

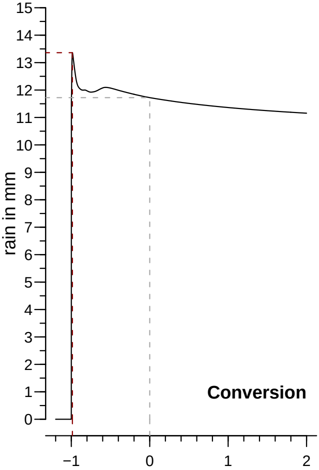
$$-\left.\frac{dq_c}{dt}\right|_{CR_{AC}} = \frac{\pi}{4} E \alpha_r n_{0,r} q_c \cdot \lambda^{-(3+\beta_r)} \cdot \Gamma(3+\beta_r)$$

$E = 1$ collection efficiency, α_r, β_r;$n_{0,r}$: Marshall-Palmer

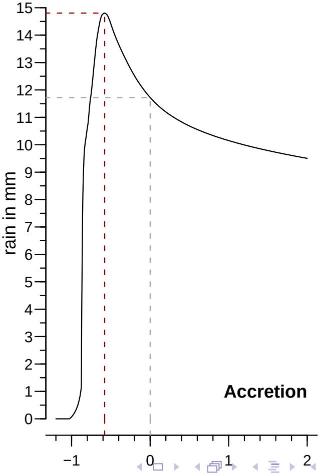
$$\lambda^{-(3+\beta_r)} = \left(\frac{\rho q_r}{\pi \rho_{ps} n_{0,r}}\right)^{\frac{(3+\beta_r)}{4}}$$

Perturbed Physics

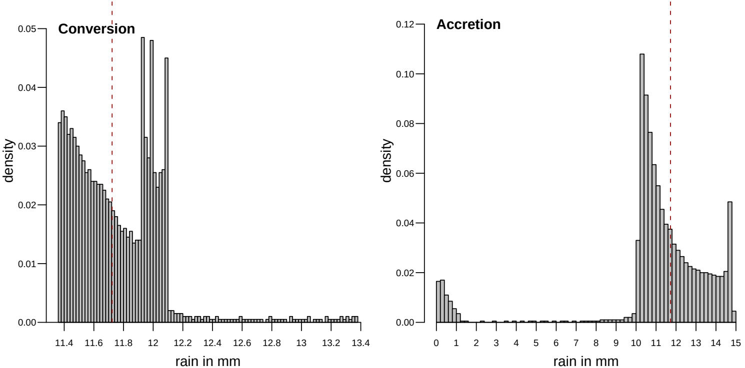
$$k_c \rightarrow k_c\ (1+\eta)$$



$$E \rightarrow E\ (1+\eta)$$



Perturbed Physics



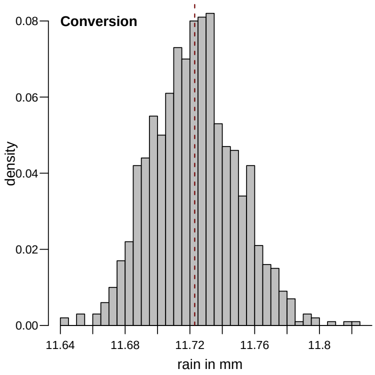
Introduction
Analytical Solutions
The Cloud Model
Conclusions

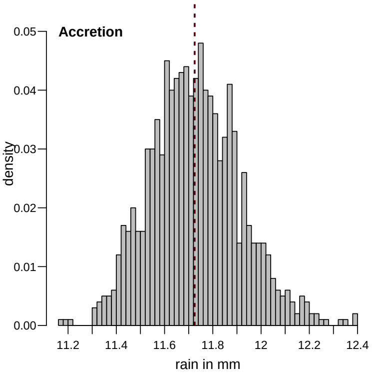
Stochastic Parameterization

(solved with Milstein Scheme)

$$k_c \longrightarrow k_c \, (1 + \xi_t)$$

$$E \longrightarrow E \, (1 + \xi_t)$$





Conclusions and Outlook

- 'Perturbed physics' ensembles do not represent realizations from real process
- Stochastic parameterization may provide realistic distributions
- Solutions strongly depend on covariance function of noise (in time and in space)
- Stochastic parameterizations should be derived from microphysical processes