Data assimilation by means of synchronization

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Introduction

The model and synchronization scheme

Surface growth picture and mean-variance diagram

Perfect model limit: Noiseless case

Non perfect model: Assimilation problem

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in collaboration with

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Assimilation and synchronization

- Yang et. al. (2006) J. Atmos. Sci.: “Lorenz 3-d model”
- Duane et. al., (2006) Nonlin. Proc. Geophys.: “Lorenz 3-d model, geostrophic model”
- Cohen et. al. (2008) Phys. Rev. Lett.: “High-dimensional chaos optical experiment”

Assimilation by means of synchronization: the truth

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where ξ_{obv}, ξ_{model} represent noisy observations and model errors, respectively

Our "reality": Lorenz '96 model

This toy model mimics some aspects of the dynamics of the atmosphere such as advection, constant forcing, and linear damping.

$$\partial_t u_n = -u_{n-1}(u_{n-2} - u_{n+1}) - u_n + F + \eta_n(t)$$

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Euler algorithm with time step $\Delta t = 10^{-4}$ and periodic boundary conditions.

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- Prediction time horizon $\sim \lambda_{\max}^{-1}$.

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- For a generic perturbation the surface exhibits scale-invariant correlations below the correlation length $\ell_c(t) \sim 1/q_c(t)$

$$\mathcal{S}(q,t) \sim \begin{cases} q^{-(2\alpha+1)} & \text{if} \quad q \gg q_c(t) \\ a(t) & \text{if} \quad q \ll q_c(t) \end{cases} . \qquad (1)$$

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- For dissipative chaos: $\partial_t w(x,t) = \partial_{xx} w + \xi(x,t) w$
- $h \equiv \ln |w|$ leads to

$$\partial_t h(x,t) = \partial_{xx} h + (\partial_x h)^2 + \xi(x,t)$$

The KPZ equation for surface growth.

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- This allows us to obtain the extent of the spatial correlations from the simple measurement of the surface width!!

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- We have $\varepsilon(t) \sim \exp(\lambda_{\max} t)$, where $\lambda_{\max}$ is just the average surface velocity ($\overline{h} = \lambda_{\max} t$) and also corresponds to the first Lyapunov exponent.

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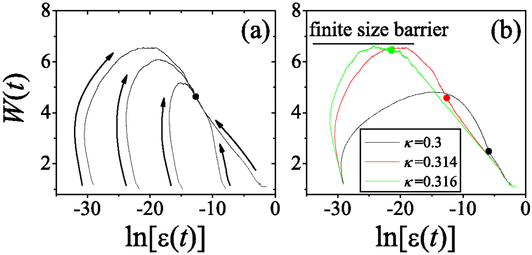
- The mean-variance of the logarithm of perturbations (MVL) diagram: $W^2(t)$ *vs.* $\ln \varepsilon(t)$



Perfect model limit: Noiseless case

- The model describes reality "perfectly"

  Reality:

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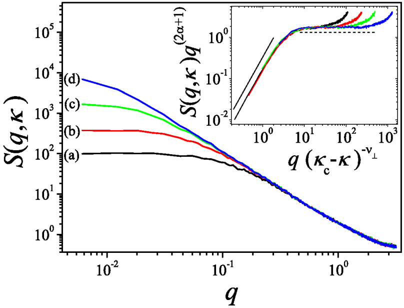
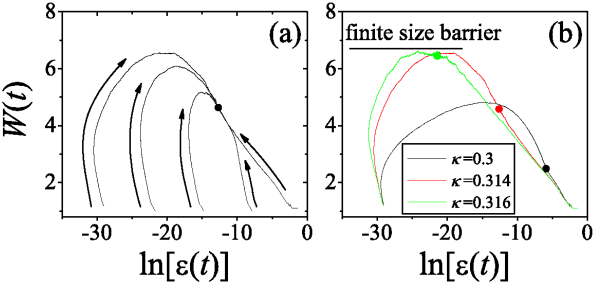
- No time scales are poorly described: Noise intensity $D = 0$
- Perfect synchronization ($||w|| \to 0$) for large enough coupling, $\kappa > \kappa_c$,
- Partial synchronization ($||w||$ finite) below the threshold, $\kappa < \kappa_c$, with a typical lenght scale $\xi_\times$

Spatial correlations of errors: Power spectral density

- Error surface $h = \ln |w|$
- roughness exponent is KPZ: $\alpha = 1/2$
- $q_\times \sim \xi_\times^{-1} \sim |\kappa - \kappa_c|^\nu$, as $\kappa \to \kappa_c^-$.
- the exponent $\nu \approx 0.85$ is also universal.



MVL diagram: Fixed point



- The asymptotic state of the synchronization error is a fixed point (W_a, ε_a)
- $W_a^2(\kappa) \equiv \lim_{t\to\infty} W(t)^2$ and $\varepsilon_a(\kappa) \equiv \lim_{t\to\infty} \varepsilon(t)$.

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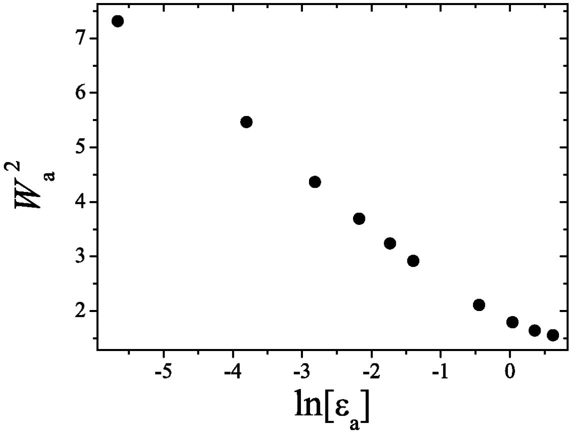

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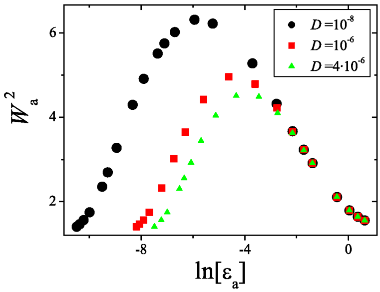
Asymptotic fixed point for $D \neq 0$



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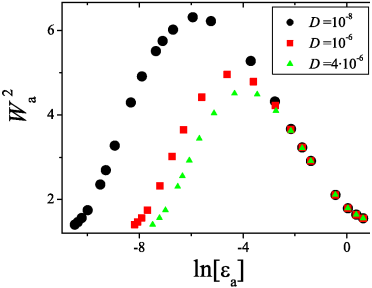
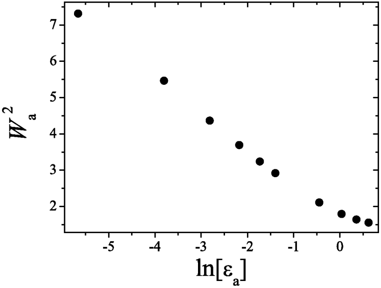
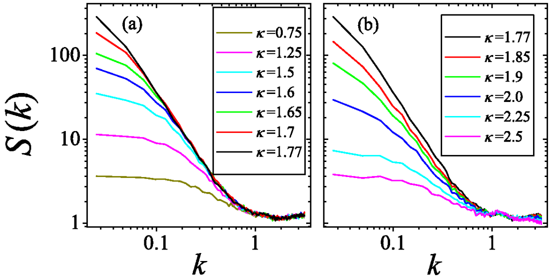




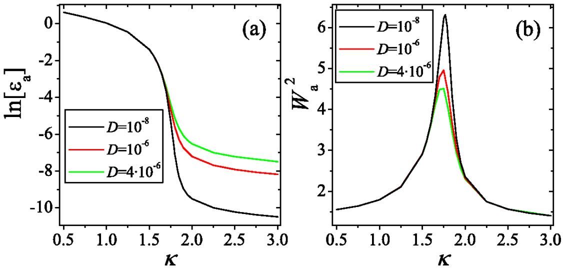
$D = 0$

Spatial correlations of model-reality errors

$\kappa < \kappa_{\text{opt}}$ $\kappa > \kappa_{\text{opt}}$

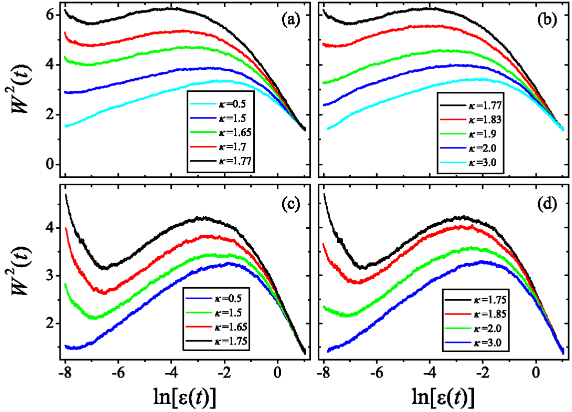


Optimal coupling



Free evolution of assimilated trayectories

$\kappa < \kappa_{\rm opt}$ $\kappa > \kappa_{\rm opt}$



a,b: $D = 10^{-8}$ and c,d: $D = 10^{-6}$

Workshop announcement

Exploring Complex Dynamics in High-Dimensional Chaotic Systems: From Weather Forecasting to Oceanic Flows

MPIPKS Dresden, 25 - 29 January 2010

Scientific Coordinators: Juan M. López, Arkady Pikovsky, and Antonio Politi

Organisation: Claudia Pöenisch (MPIPKS Dresden, Germany)

For further information please visit:
http://www.mpipks-dresden.mpg.de/~ecodyc10/