# SPATIO-TEMPORAL EVOLUTION OF PERTURBATIONS IN ENSEMBLES INITIALIZED BY BRED, LYAPUNOV, AND SINGULAR VECTORS

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# ENSEMBLE FORECASTING

Leith (1974): Ensemble mean outperforms deterministic forecasting.

Hypothesis:
- Imperfect model (Canada)
- Perfect model: **dynamic vectors**
  - Bred vectors (USA)
  - Singular vectors (Europe)

- What do we know about the evolution of perturbations initialized along dynamic vectors in **simple** spatio-temporal systems?
- How do their spatial structures evolve?

# DYNAMIC VECTORS

| | Vector type | Notation | Magnitude | Computation interval | Eigenvector of | Control parameter(s) |
|---|---|---|---|---|---|---|
| Bred | Log-BV | $\mathbf{l}_{M_0}$ | Finite | $(-\infty, 0)$ | — | $M_0 = \ln(\epsilon_0)$ |
| Lyapunov | B-LV | $\mathbf{b}_n$ | Infinitesimal | $(-\infty, 0)$ | $\mathbf{A}(0, -\infty)\mathbf{A}^*(0, -\infty)$ | $n$ ($\leftrightarrow$ $n$-th LE) |
| Lyapunov | F-LV | $\mathbf{f}_n$ | Infinitesimal | $(0, \infty)$ | $\mathbf{A}^*(\infty, 0)\mathbf{A}(\infty, 0)$ | $n$ ($\leftrightarrow$ $n$-th LE) |
| Lyapunov | C-LV | $\mathbf{g}_n$ | Infinitesimal | $(-\infty, 0) \cup (0, \infty)$ | — | $n$ ($\leftrightarrow$ $n$-th LE) |
| Singular | SV | $\mathbf{s}_\tau$ | Infinitesimal | $(0, \tau)$ | $\mathbf{A}^*(\tau, 0)\mathbf{A}(\tau, 0)$ | $\tau$ (here $n = 1$ only) |

Infinitesimal perturbations evolve linearly: $\boldsymbol{\delta}\,\mathbf{u}(t+\tau) = \mathbf{A}(t+\tau, t)\,\boldsymbol{\delta}\,\mathbf{u}(t)$

Characteristic LVs are covariant with the *linear* dynamics: $\mathbf{g}_n(t+\tau) = \mathbf{A}(t+\tau, t)\,\mathbf{g}_n(t)$

Logarithmic bred vectors

$$t_m = mT$$

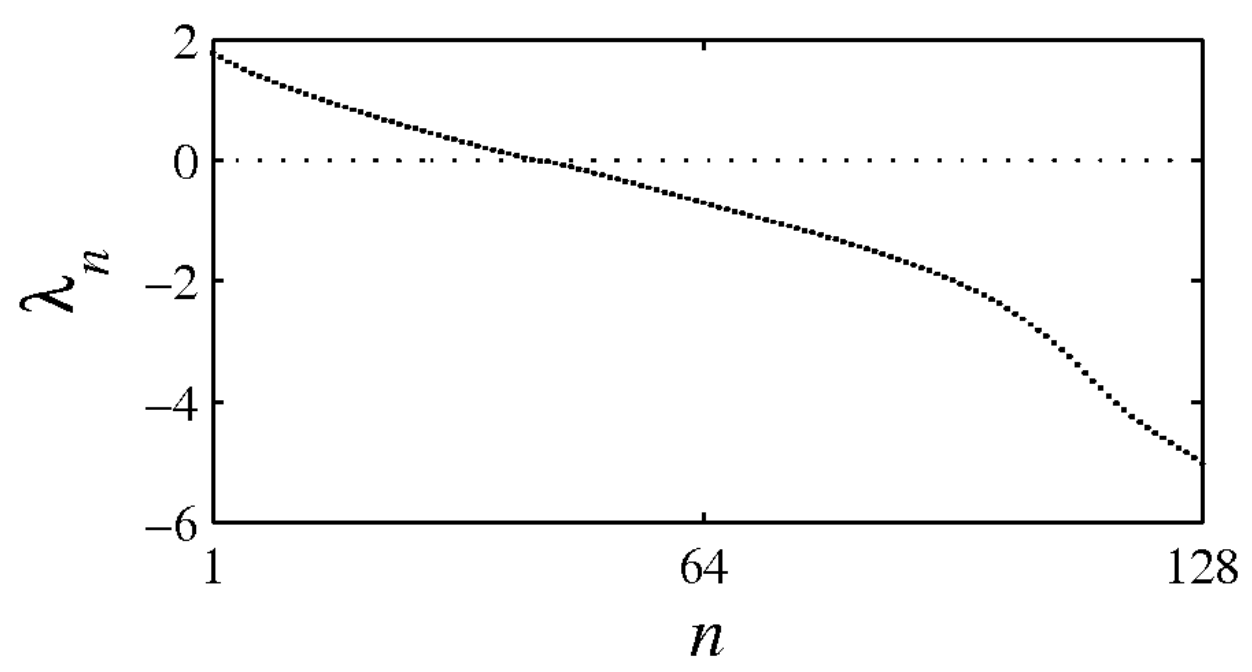
$$\boldsymbol{l}(t_m) = \boldsymbol{y}'(t_m) - \boldsymbol{y}(t_m)$$

$$\boldsymbol{y}'(t_m) = \boldsymbol{y}(t_m) + \epsilon_0 \frac{\boldsymbol{l}(t_m)}{\|\boldsymbol{l}(t_m)\|} \quad ; \quad \|\boldsymbol{l}\| = \left| \prod_{x=1}^{L} l_x \right|^{1/L}$$

# SPATIO-TEMPORAL CHAOTIC SYSTEM: LORENZ '96 MODEL

$$\frac{d\,y_x}{dt} = -y_x - y_{x-1}\left(y_{x-2} - y_{x+1}\right) + F \qquad x = 1,\ldots,128$$

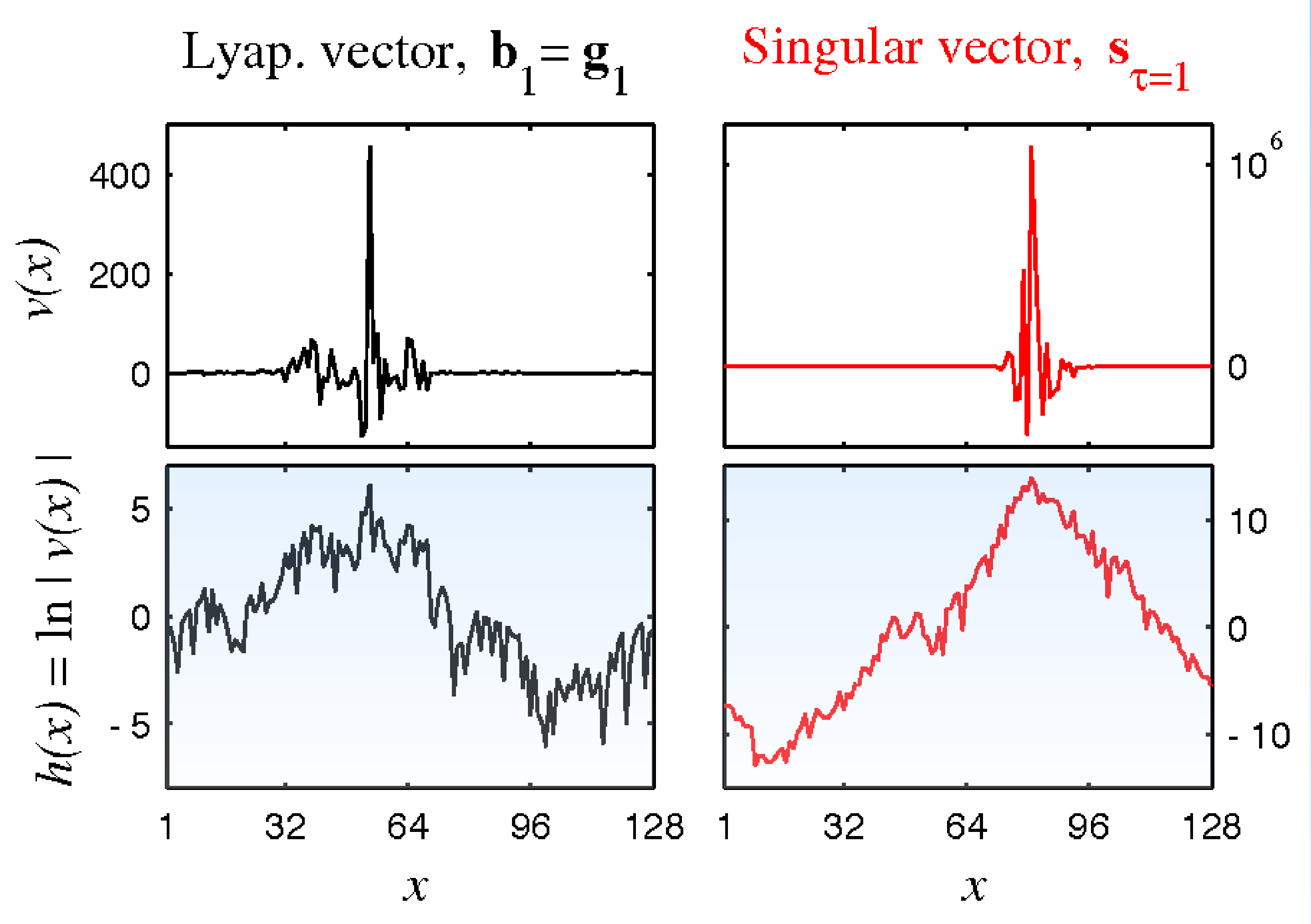
Lyapunov spectrum (*F=8*):



# LOCALIZATION & HOPF-COLE TRANSFORMATION

**Surface picture**

h(x,t)=ln|v(x,t)|



# THEORETICAL BASIS OF THE SURFACE PICTURE FOR SPATIO-TEMPORAL CHAOTIC SYSTEMS

- Pikovsky & Kurths (1994), Pikovsky & Politi (1998): An infinitesimal perturbation **in log scale** belongs to the universality class of the Kardar-Parisi-Zhang (KPZ) equation [$\partial_t h = \xi + \nabla^2 h + (\nabla h)^2$]

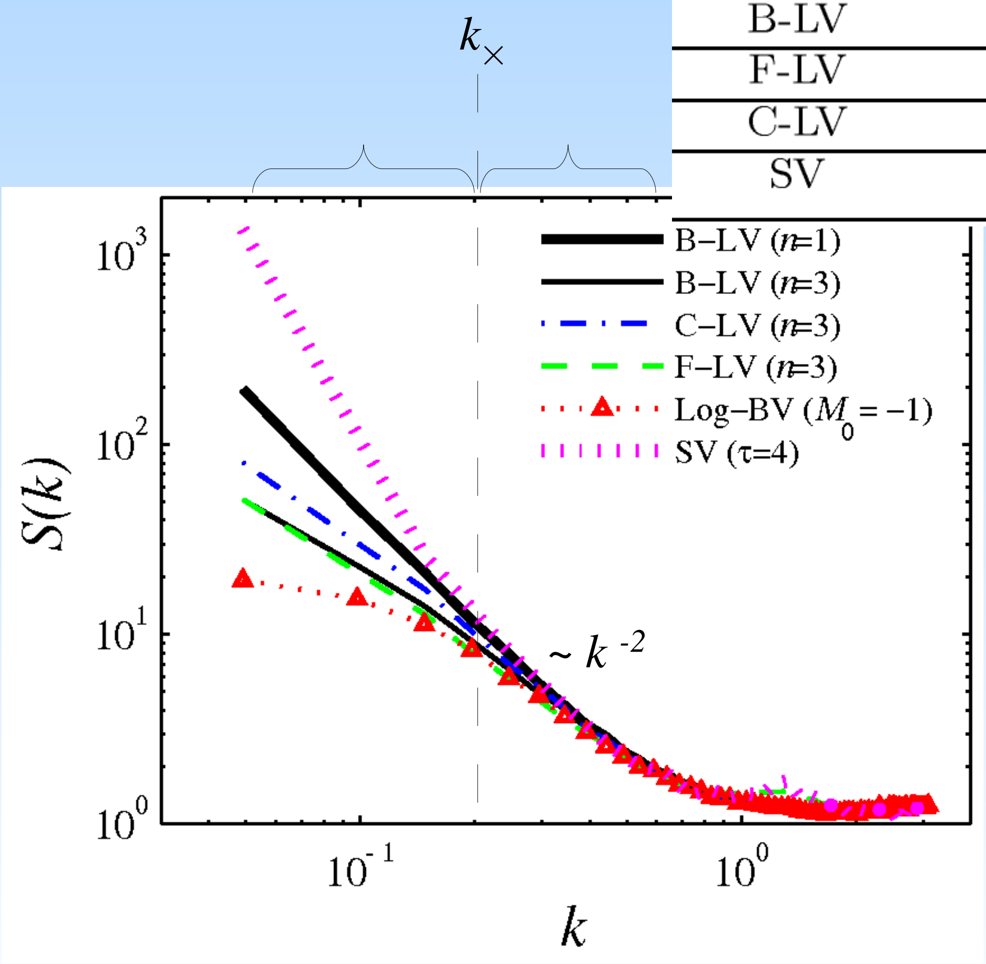
- López et al (2004), Primo et al. (2005,2006): Finite perturbations and bred vectors (uncorrelation at long scales).

- Szendro et al. (2007), Pazó et al. (2008): Lyapunov vectors beyond the first one are “piecewise” KPZ.

- Pazó et al. (2009): Singular vectors are exponentially localized and scale as KPZ with periodic noise.

# SPATIAL POWER SPECTRAL DENSITY

| Vector type | Cross. length $l_\times = \frac{2\pi}{k_\times}$ | $S(k < k_\times)$ | $S(k > k_\times)$ |
|---|---|---|---|
| Log-BV | $\sim M_0^2$ | Constant | $k^{-2}$ |
| B-LV | $\approx (L/n)^\theta \; (\theta \approx 1)$ | $k^{-1}$ | |
| F-LV | $\approx (L/n)^\theta \; (\theta \approx 1)$ | $k^{-1}$ | |
| C-LV | $\approx (L/n)^\theta \; (\theta \approx 1)$ | $k^{-1.15}$ | |
| SV | $\sim \tau^{\gamma/2} \; (\gamma \simeq 0.78)$ | $k^{-4}$ | |



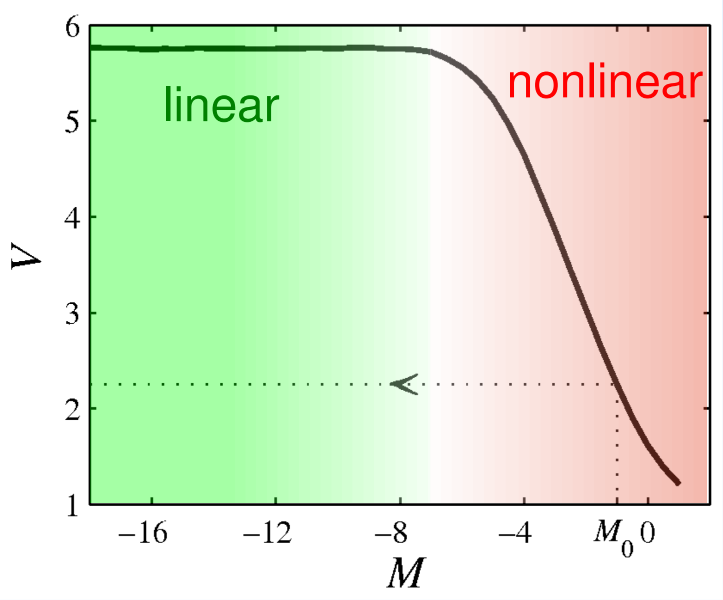
## CHARACTERIZATION OF SURFACES: mean ($M$) and variance ($V$)

$$M(t) = \frac{1}{L}\sum_{x=1}^{L} h(x,t)$$ ⟺ logarithm of the geometric **norm**

$$V(t) = \frac{1}{L}\sum_{x=1}^{L} \left[h(x,t) - M(t)\right]^2$$

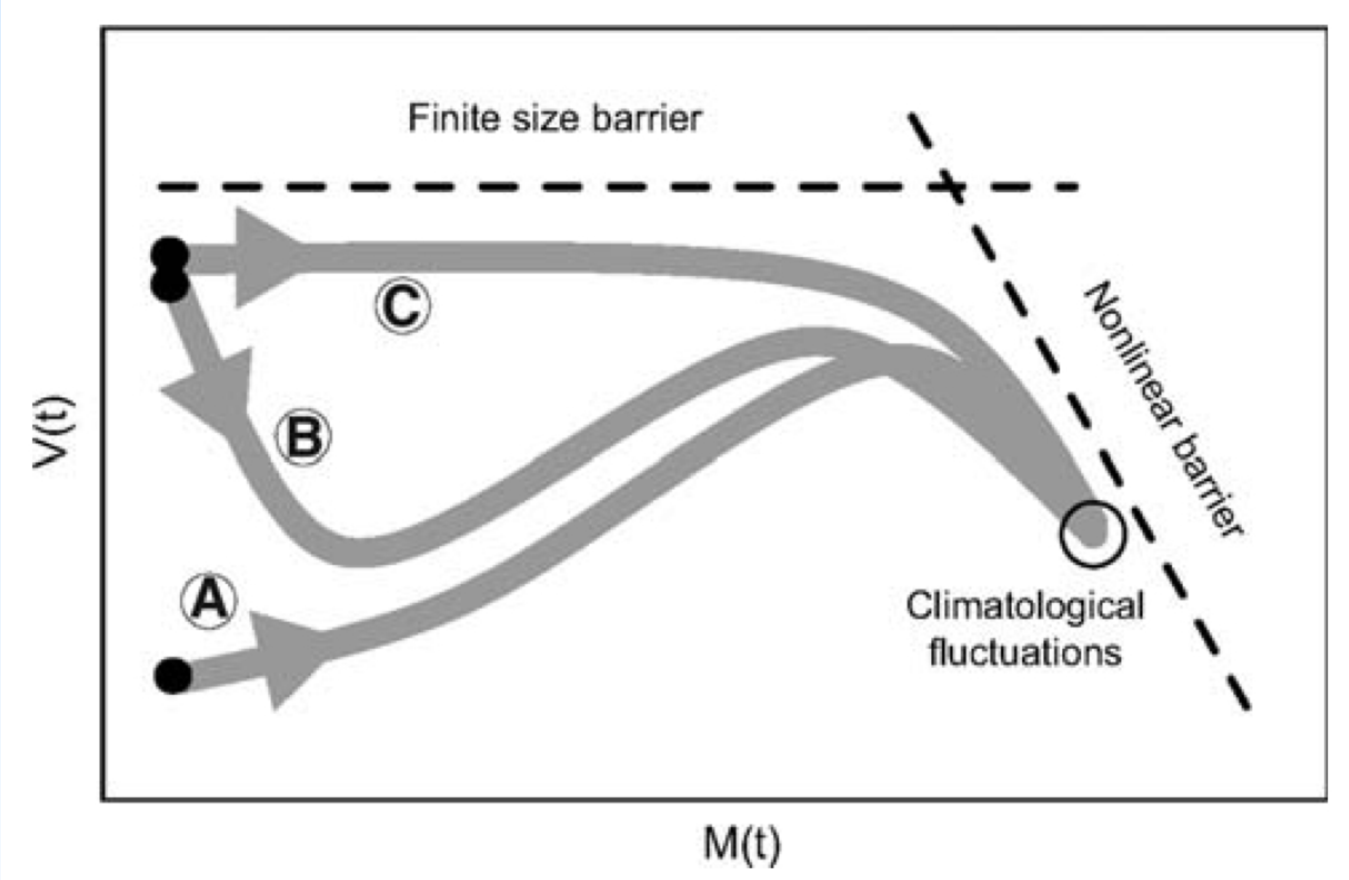
Imposing a perturbation to have a norm, say $M = M_0$, one gets the (logarithmic) bred vectors

$$M_0 = -\infty$$
$$\mathrm{l}_{M_0} = \mathrm{b}_1 = \mathrm{g}_1$$



# MVL DIAGRAM

Introduced by Primo et al. in Phys. Rev. E (2005) and Phys. Rev. Lett. (2007)

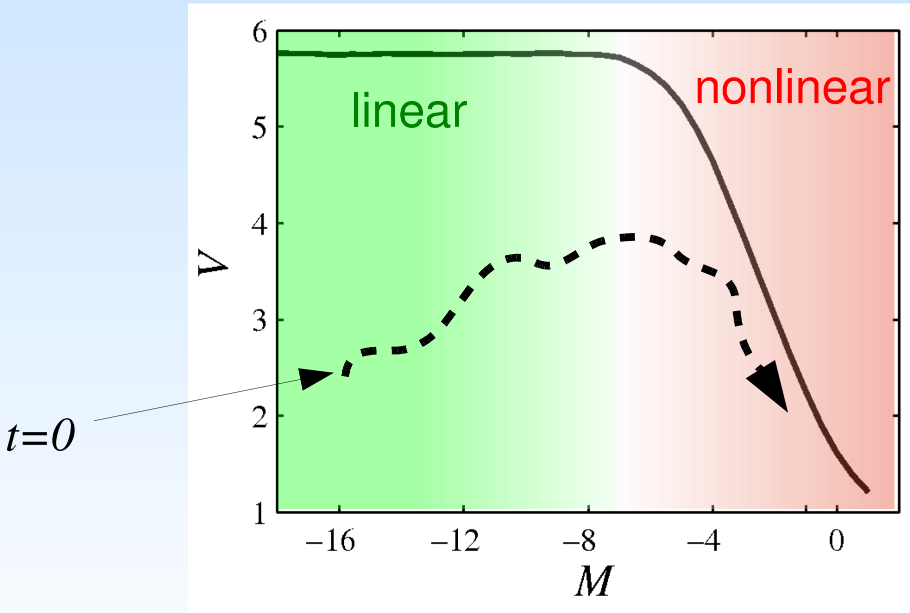



- J.M. Gutiérrez et al., Spatiotemporal characterization of ensemble prediction systems: the mean-variance of logarithms (MVL) diagram, Nonlin. Processes Geophys. (2008).
- J. Fernández et al., MVL spatiotemporal analysis for model intercomparison in EPS: application to the DEMETER multimodel ensemble. Clim. Dyn. (2009).

# SPATIO-TEMPORAL EVOLUTION OF PERTURBATIONS

Initialization
$$\begin{cases} \boldsymbol{y}'(t=0)=\boldsymbol{y}(t=0)+\boldsymbol{v} \\ \boldsymbol{v} \propto \{\boldsymbol{l}_{M_0}, \boldsymbol{b}_n, \boldsymbol{f}_n, \boldsymbol{g}_n, \boldsymbol{s}_\tau\} \\ \ln\|v\| = M(t=0) = -16 \\ V(t=0) \text{ is intrinsic to the selected vector.} \end{cases}$$

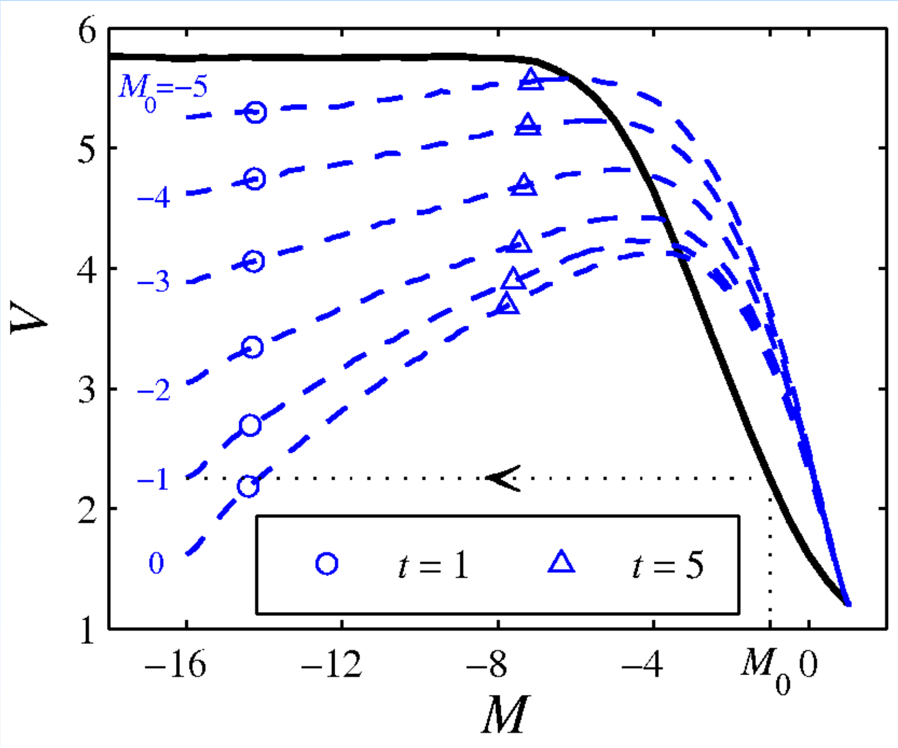
We track $\boldsymbol{v}(t)=\boldsymbol{y}'(t)-\boldsymbol{y}(t)$ in $M,V$ coordinates



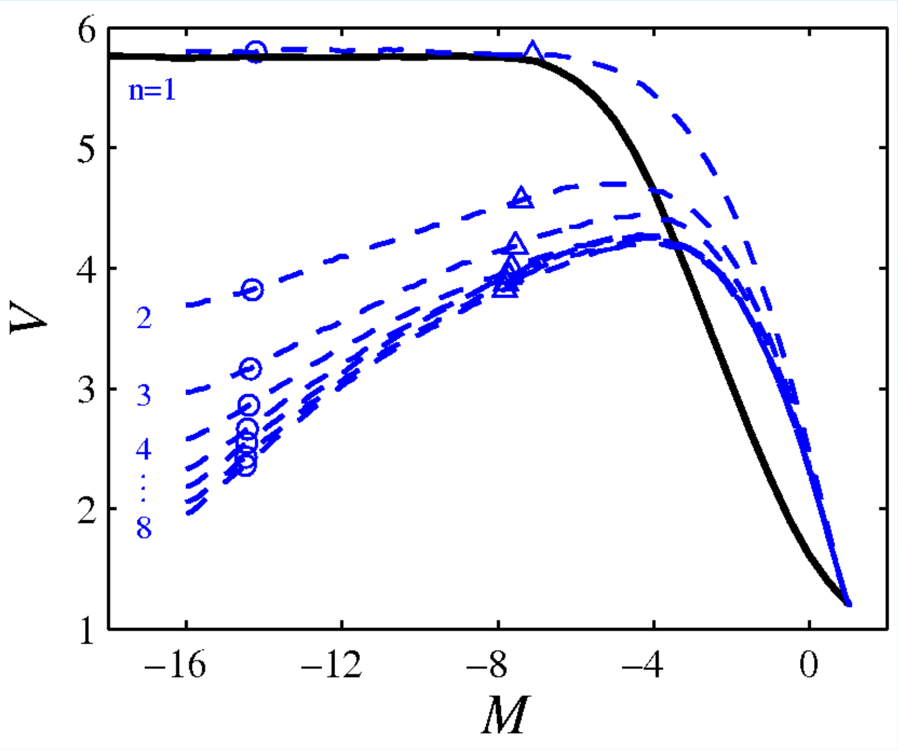
# RESULTS

## 1. Vectors generated from the past are well adapted (and exhibit similar behaviour). Exponential growth rate ≈ $\lambda_1$

Logarithmic bred vectors



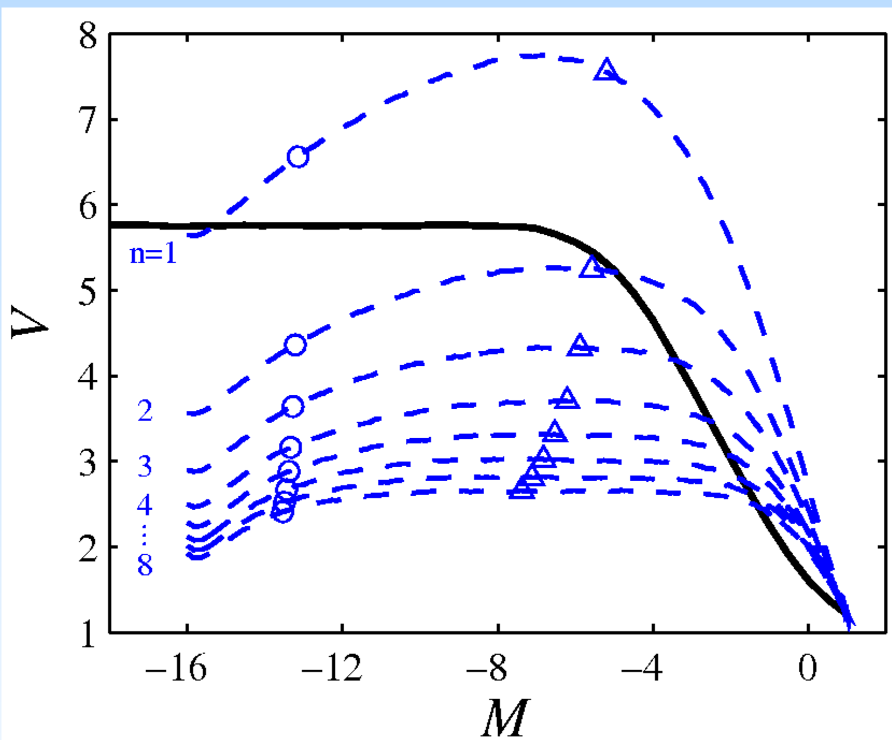

Backward Lyapunov vectors



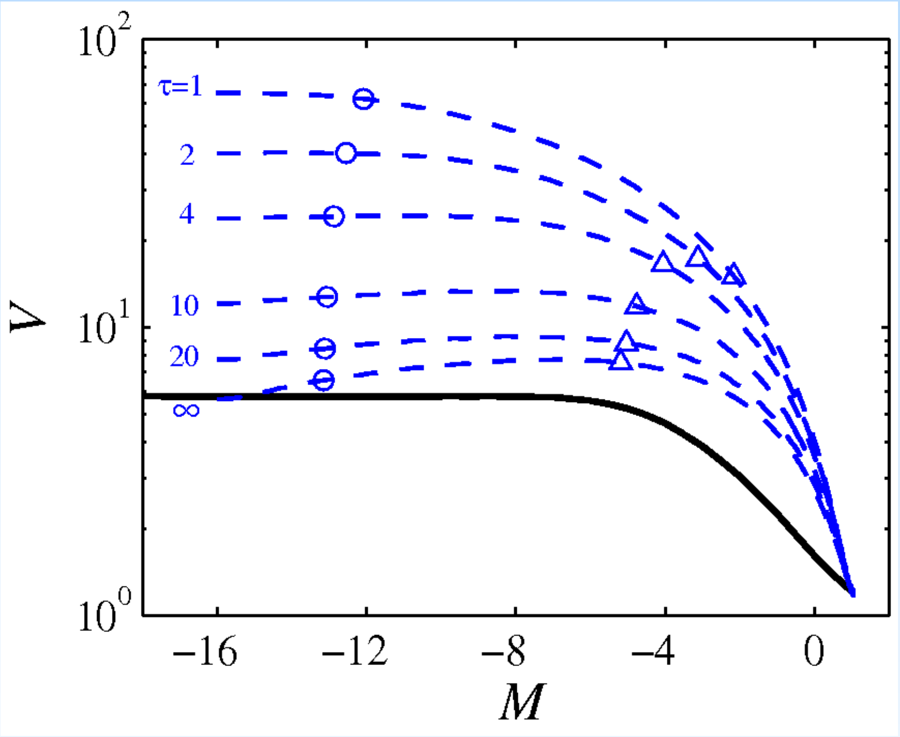
# RESULTS

## 2. Vectors generated from the future are not in the attractor (and exhibit severe transients). Exponential growth rate ≠ $\lambda_1$

Forward Lyapunov vectors




Singular vectors

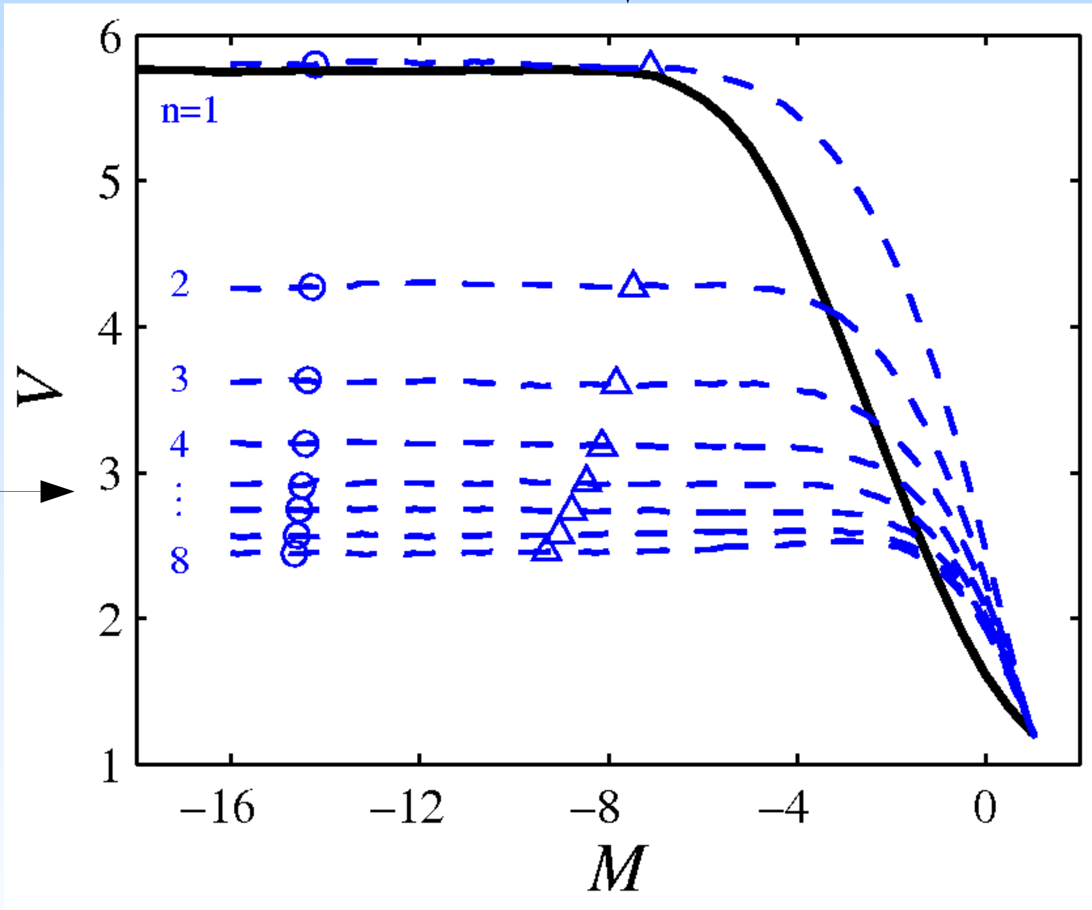


## RESULTS

### 3. Characteristic Lyapunov vectors allow to control growth and structure.

different growth rates

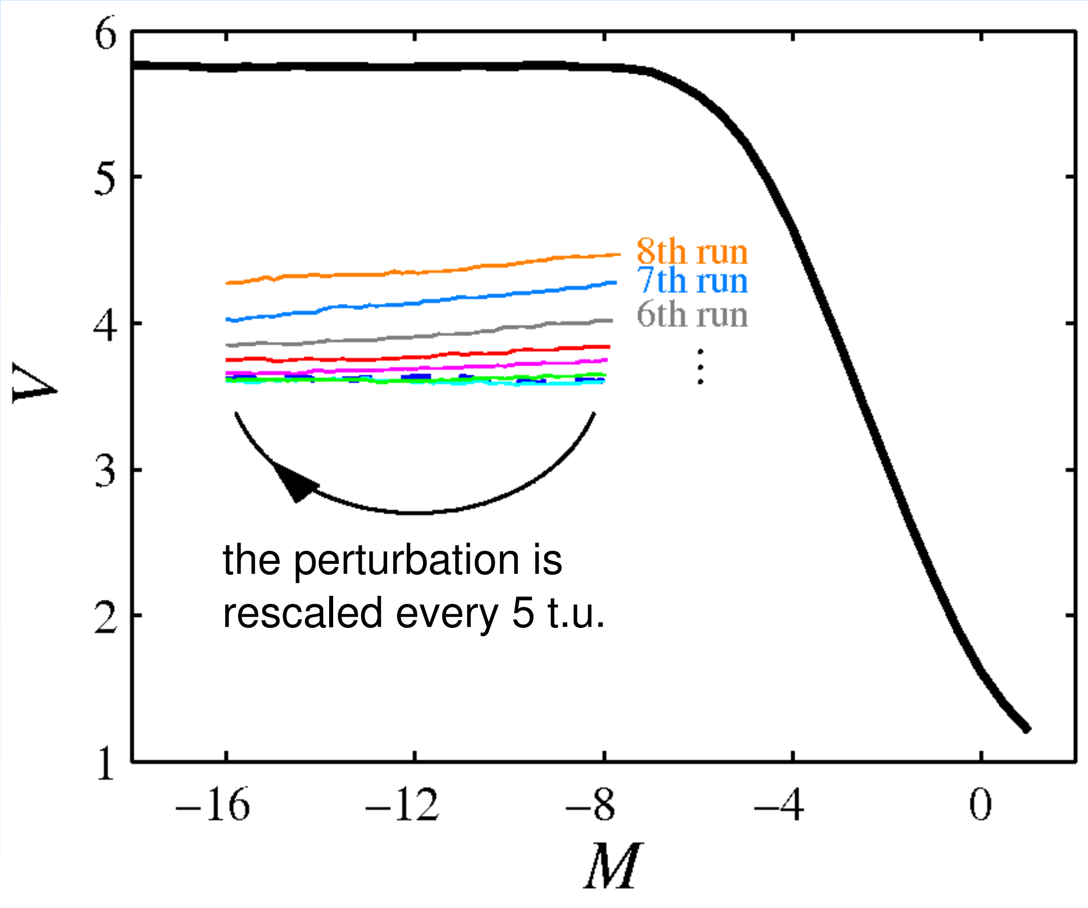
constant structure



# RESULTS

## 4. Characteristic Lyapunov vectors. Robustness!

3-rd Characteristic Lyapunov vector



# CONCLUSIONS

- Vectors evolved from the past: well adapted, common growth rate $\approx\lambda_1$
- Vectors with information from the future: controllable growth but badly adpated.
- Characteristic LV:
  - Well adapted and different growth rates $\{\lambda_n\}$.
  - Very robust!

[The results in this talk have been submitted to Tellus A]